Rigidity Operators and the Flexibility of Infinite Bar-joint Frameworks

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Abstract

In this thesis we bring together various techniques from functional analysis and operator theory to develop the linear infinitesimal theory of crystal frameworks in the Euclidean space $\mathbb{R}^d$. In this mathematical theory we obtain sufficient conditions for the boundedness of the rigidity matrix $R(\mathcal{G})$, viewed as a Hilbert space operator, for certain infinite tree frameworks.

Also, we provide an analysis of the vector subspace of strictly periodic flexes implied by the translational symmetry of crystal frameworks and we prove a relation in which the space of supercell n-fold periodic flexes can be written as a direct sum of the relevant vector subspaces of phase periodic flexes.

A main result in the thesis is the development of the almost periodic rigidity theory for crystal frameworks in $\mathbb{R}^d$. We prove that a crystal framework is almost periodically infinitesimally rigid if and only if it is periodically infinitesimally rigid and the corresponding RUM spectrum is the minimal set $\{(1, 1, \ldots, 1)\}$.

Finally, we conclude the thesis by defining and identifying crystal flex bases for the real vector space of all infinitesimal flexes of a crystal framework $\mathcal{C}$. In particular, we determine a crystal flex basis for the crystal framework $\mathcal{C}_{\mathrm{Oct}}$

formed by pairwise vertex connected regular octahedra. This bar-joint framework (for the mineral perovskite) features in early papers on the investigation of rigid unit modes in material crystals.

Mum and Dad, this is to you.

Declaration

I hereby declare, that with my supervisor's guidance, this thesis is my own work and whenever the work of others had been consulted then it was appropriately acknowledged.

Chapter 7 contains published joint work with Stephen C. Power and Derek Kitson [5].

Chapter 8 is also part of joint work with Stephen C. Power and Derek Kitson (in preparation).

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Chapter 1

Introduction

A *bar-joint framework* is a structure consisting of stiff bars connected by flexible joints and the question of when such a structure is "rigid" has attracted the interests of researchers from the early beginnings of the 19th century.

Early contributions towards rigidity theory date back to 1776, when Euler conjectured that "A closed spatial figure allows no changes, as long as it is not ripped apart" [31]. In other words, a closed figure is a model whose faces are made of rigid plates and these plates are hinged together along the edges where they meet such that adjacent faces are allowed to rotate about their common hinge. For convex polyhedra, this conjecture was later answered by Cauchy (1813) who proved that "If there is an isometry between the faces of two strictly convex polyhedra which is an isometry on each of the faces, then the two polyhedra are congruent". A corollary of Cauchy's Theorem is that all such convex polyhedra are in fact rigid.

Maxwell, 1864, proved Maxwell's counting rule [51]: a necessary condition for three dimensional rigidity is that the number of bars is at least 3 times the number of points minus 6.

In 1958, Alexandrov [1] extended Cauchy's Theorem to include all frameworks given by convex polyhedra with faces triangulated by edges between additional vertices on the original edges. The resulting structure in this case, with triangles as faces and including the triangulating edges, is rigid. These theorems determine the rigidity of a special class of frameworks, convex polyhedra, regardless of their geometric positions. Not only interesting on their own, convex polyhedra can be connected to form three dimensional crystal frameworks with special flexing properties.

The first major contribution to the rigidity of mathematical bar-joint frameworks is due to Laman [45], 1970, who proved that generic planar rigidity is in fact a property of the underlying graph, regardless of its geometry. This can simply be determined by counting the vertices and edges of all subgraphs of the graph of the framework. But despite this celebrated result, it still remains an open problem to find a combinatorial characterization for three dimensional frameworks.

Gluck, 1975, proved that "almost all" polyhedra are rigid, and this settled Euler's conjecture for polyhedra in generic position. Finally, Euler's conjecture was proved to be false when Connelly [14], [15], [16], constructed an example of a polyhedron that is not rigid.

Although progress was slow in the early beginnings of rigidity theory, it now provides an increasingly growing area of mathematical research with contributions bringing together techniques from graph theory, linear algebra, topology, representation theory, and in the case of infinite frameworks, various techniques from functional analysis, operator theory and limit algebras are now being used.

Apart from being an exciting field of study as it stands alone, rigidity theory has found its way to a wide range of applications, beyond engineering and structural mechanics, but moving towards robotics [71], [72], formation control [27], material sciences [25], [33], [24] and biochemistry [38], to say the least.

In particular, the mathematical analysis of frameworks with translational symmetry known as *crystal frameworks* makes use of functional analysis and operator theory to precisely determine the spaces of "strictly periodic and super cell periodic" flexes related to crystal frameworks. The determination of these periodic flexes depends on the analysis of a *matrix function.* This matrix function leads to the identification of "RUM spectrum", where material scientists mostly rely on laboratory experiments [25] or computer analysis [33] to identify.

Furthermore, a new area where the analysis of crystal frameworks is used is in the study of biomolecules, the functional biological unit [28], [13].

1.1 Thesis overview

In this thesis we aim to gradually develop an understanding of mathematical bar-joint frameworks and various forms of infinitesimal flexibility. We start with basic definitions and build up the theory for finite frameworks and infinite frameworks until we reach the main focus of the thesis which is the infinitesimal flex properties of crystal frameworks. A main result is the characterization of almost periodic rigidity for arbitrary crystal frameworks in $\mathbb{R}^d$. We make use of techniques from functional analysis and operator theory and we study special classes of infinitesimal flexes implied by the translational symmetry of crystal frameworks such as "phase periodic" flexes. Also we illustrate various forms of flexibility through a range of contrasting examples.

We now offer a brief summary for the following chapters:

Chapter 2: Preliminaries. In this chapter we put together a range of background material related to various topics throughout the thesis. Mainly, this is related to rigidity matrices of infinite frameworks and specific classes of infinitesimal flexes such as vanishing, square summable, etc. This is followed by the introduction of basic graph theory terminology related to the mathematical definitions for bar-joint frameworks.

Chapter 3: Finite Bar-joint Frameworks. This chapter aims to ease the "upgrade" towards the class of infinite bar-joint frameworks by analysing simple finite frameworks and their forms of flexibility. Moreover, some flexing properties of infinite framework can be determined from their finite sub-frameworks as in the case of *sequential rigidity*. In the final section of this chapter we calculate the flexibility dimensions for some examples by identifying a base for the space of all infinitesimal flexes, an idea that will later be generalized for crystal frameworks.

Chapter 4: Infinite Bar-joint Frameworks. In this chapter, a framework is viewed as an infinite structure in $\mathbb{R}^d$ and we see how the infinitesimal flex condition and the rigidity matrix can be generalized for such frameworks. General countably infinite bar-joint frameworks were first considered in Owen and Power [52]. In this case the rigidity matrix is viewed as a linear transformation between infinite dimensional spaces and for certain infinite frameworks this rigidity transformation is in fact bounded. We also introduce an infinite "strip" framework that can be "tailored" to admit specific infinitesimal flex spaces as the infinitesimal flexibility is completely determined by its geometry.

Chapter 5: Crystal Frameworks. The first section of this chapter can be considered as a background for the mathematical identification of *crystal bar-joint frameworks*. After the definitions and a range of planar, and spatial examples we move on to more specific classes of infinitesimal flexes that are exclusive to crystal frameworks due to their high symmetry. We use the *matrix function* introduced by Power, [58], to identify the class of *strictly periodic* infinitesimal flexes. Furthermore, this matrix function leads to the determination of the "RUM" spectrum and the class of *phase periodic* infinitesimal flexes introduced by Owen and Power [54], [58]. Finally, we prove a direct sum relation between the spaces of strictly periodic, and phase periodic infinitesimal flexes.

Chapter 6: Almost Periodic Functions. This chapter develops the necessary mathematical framework for multi-variable almost periodic functions. We generalize the existing single variable theory and prove the property of "Approximation by Trigonometric Polynomials" for almost periodic functions in $\mathbb{R}^2$. We make close use of the convenient approach of Partington [55] for the single variable theory.

Chapter 7: Almost Periodic Rigidity. Here we bring together our understanding of periodic rigidity and almost periodic sequences to develop *almost periodic rigidity* theory. It is shown that a crystal framework is almost periodically infinitesimally rigid if and only if it is strictly periodically rigid and the RUM spectrum is *trivial.* After that we give examples of crystal frameworks with different almost periodic rigidity properties. This chapter is joint work with S.C. Power and D. Kitson [5].

Chapter 8: Bases For The Flexes Of Crystal Frameworks. Here we consider the infinite linear decomposition of infinitesimal flexes of general infinite frameworks in terms of a countable basis. Also we define and identify *crystal flex basis* in the case of crystal frameworks. We try to focus on infinitesimal flexes with some form of periodicity and see how they can give an idea for the crystal bases or spanning sets for the real vector space $\mathcal{H}_{\mathrm{fl}}(\mathcal{C})$. The results of this chapter are built on an understanding of $\mathcal{H}_{\mathrm{fl}}(\mathcal{C})$ and are independent of the chapters 6 and 7. This chapter is part of joint work with S.C. Power and D. Kitson [4].

Chapter 9: Further Developments And Related Work. In this chapter we suggest further developments related to some of the areas developed in the thesis.

This thesis is very much about the linear infinitesimal theory of crystal frameworks, although on occasion there are comments on continuous flexibility and continuous rigidity. For example, in Chapter 3 we comment on the continuous rigidity of the "double square" finite framework and this is later used to determine the continuous rigidity of the "double square" crystal framework and the infinite "cobweb" for example.

The application that is most relevant to this thesis is the identification of the "rigid unit mode spectrum" which leads also to the determination of strictly periodic and phase periodic flexes. In fact there is a good amount of experimental data on rigid unit modes obtained by researchers in chemistry and crystalline materials. Also, there are ongoing developments of computer programs that provide quantified analysis of the flexibility of crystals. An example of such programs is CRUSH, introduced by Giddy et al [29] and

Hammonds et al [35] (and has undergone many improvements). The mathematical identification of the RUM spectrum is implied by the *matrix function* introduced by Power [58]. Using this function one can deduce the strictly periodic, phase periodic and almost periodic flexing properties of the crystal framework as we shall see in the following chapters.

Chapter 2

Preliminaries

In this chapter we gather some background material in functional analysis related to sequence spaces, Hilbert spaces and linear operators together with some basic graph theory. These definitions and results will be used throughout the thesis and can be found in many text books, for example, [70], [17], [34], [39], [44], [26], and [69]. For more about graph theory we refer the reader to [23], [9], [31].

2.1 Hilbert Spaces

In this section we state some basic Hilbert space definitions and theorems. We will refer to these later on as we develop the analysis of mathematical frameworks and related flex spaces.

Definition 2.1.1. A non-empty set E is called a *complex vector space* if E

is an additive abelian group and for every vector $x \in E$ and scalar $\lambda \in \mathbb{C}$ there is a vector $\lambda x \in E$ in such a way that for all vectors x, y and scalars α, β we have:

1. $\alpha(\beta x) = (\alpha\beta)x$
2. $1x = x$
3. $\alpha(x + y) = \alpha x + \alpha y$
4. $(\alpha + \beta)x = \alpha x + \beta y.$

Definition 2.1.2. A *norm* on a complex vector space is a real valued function:

$$\|.\| : E \to \mathbb{R}$$

which satisfies:

1. $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$
2. $\|\lambda x\| = |\lambda|\|x\|$
3. $\|x + y\| \leq \|x\| + \|y\|$

If $\|.\|$ is a norm on E then E is called a *normed space.*

Theorem 2.1.3. *In any normed space E the function $d : E \times E \to \mathbb{R}$ defined by:*

$$d(x, y) = \|x - y\|$$

is a translation-invariant metric.

To say that d is *translation-invariant* means that translation of a pair of points by the same vector leaves their distance unchanged; i.e.

$$d(x+z, y+z) = d(x, y) \text{ for all } x, y, z \in E.$$

Definition 2.1.4. Let E be a normed space. A sequence $(x_n)_{n=1}^{\infty}$ in E is *convergent* in E if:

$$\text{there exists } x \in E \text{ such that } \lim_{n\to\infty} \|x_n - x\| = 0$$

and we write $x_n \to x$.

Definition 2.1.5. Let E be a normed space. A sequence $(x_n)_{n=1}^{\infty}$ is called a *Cauchy sequence* if:

$$\text{for all } \varepsilon > 0 \text{ there exists } n_0 \in \mathbb{N} \text{ such that } n, m \geq n_0 \text{ implies that}$$
$$\|x_n - x_m\| < \varepsilon.$$

Definition 2.1.6. Let E be a normed space and $(x_n)_{n=1}^{\infty}$ be a sequence in E. We say that the series $\sum_{n=1}^{\infty} x_n$ *converges* and has *sum* s in E, if the sequence $(s_n)_{n=1}^{\infty}$ of partial sums converges to s where $s_n = \sum_{k=1}^{n} x_k$.

Definition 2.1.7. A normed space E is *complete* if it is complete in the metric defined by the norm. That is, if every Cauchy sequence in E converges to a limit in E.

Definition 2.1.8. A *Banach space* is a complete normed space.

Example 2.1.9. The complex vector space ℓ^∞ of all bounded sequences $(x_n)_{n=0}^\infty$ of complex numbers with component-wise addition and scalar multiplication and with the norm defined by

$$\|x\|_\infty = \sup_{n\in\mathbb{N}}|x_n|$$

is a Banach space.

One can similarly define the Banach space $\ell^\infty(\mathbb{Z}, \mathbb{R}^d)$ of $\mathbb{R}^d$-valued sequences which are two-way infinite. In Chapter 7, we shall consider the multi-sequence version of this space, $\ell^\infty(\mathbb{Z}^2, \mathbb{R}^2)$ as well as the subspace of almost periodic sequences.

Theorem 2.1.10 (Completion[44])**.** *Let E be a normed space. Then there is a Banach space $\hat{E}$ and an isometry A from E onto a subspace W of $\hat{E}$ which is dense in $\hat{E}$. The space $\hat{E}$ is unique, except for isometries.*

Proving the completeness theorem, roughly speaking, requires the assignment of suitable limits to Cauchy sequences in E that do not converge keeping in mind that some sequences may want to converge to the same limit and this idea can be expressed by defining a suitable equivalence relation. The completion of E can be constructed as the space of equivalence classes of Cauchy sequences of elements of E and therefore we need to make $\hat{E}$ into a vector space by defining two algebraic operations from which it follows that on W the vector space operations induced from $\hat{E}$ agree with those induced from E by means of A. Furthermore, A induces a norm on W which can be extended to $\hat{E}$.

Definition 2.1.11. If E is a vector space over $\mathbb{C}$, an *inner product* on E is a function

$$\langle .,. \rangle : E \times E \to \mathbb{C}$$

such that for all $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in E$, the following are satisfied:

1.$\langle y, x \rangle = \overline{\langle x, y \rangle}$

2. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

3.$\langle x, x \rangle \geq 0; \langle x, x \rangle = 0 \Leftrightarrow x = 0.$

Theorem 2.1.12. *For any x, y, z in an inner product space E and λ in $\mathbb{C}$,*

1. $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$

2. $\langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle$

3. $\langle x, 0 \rangle = 0 = \langle 0, x \rangle$

4. if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in E$ then $x = y$.

Definition 2.1.13. If x is a vector in an inner product space E, then the *norm* of x associated with the inner product is defined by

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}.$$

This norm makes E a normed space and the metric on E associated with the inner product is defined by

$$d(x, y) = \|x - y\| = \langle x - y, x - y\rangle^{\frac{1}{2}}.$$

Definition 2.1.14. If x, y are vectors in the inner product space E, then x, y are *orthogonal*, denoted $x \perp y$, if $\langle x, y\rangle = 0$. If A is a subset of E then $x \perp A$ if $x \perp y$ for all $y \in A$ and the *orthogonal complement* of A is defined by:

$$A^{\perp} = \{x \in E : \langle x, y\rangle = 0 \, \text{for all} \, y \in A\}.$$

Definition 2.1.15. A family $(e_n)_{n=1}^{\infty}$ in $E \setminus \{0\}$ is called an *orthogonal sequence* if $e_n \perp e_m$ whenever $n \neq m$. If, further, $\|e_n\| = 1$ for each $n \in \mathbb{N}$, then the family $(e_n)_{n=1}^{\infty}$ is called an *orthonormal sequence.*

Theorem 2.1.16 (The Cauchy-Schwarz Inequality)**.** *Let E be an inner product space and $x, y \in E$, then*

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

with equality if and only if x and y are linearly dependant.

Lemma 2.1.17 (Continuity of The Inner Product)**.** *If in an inner product space, $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.*

Theorem 2.1.18 (The Pythagorean Theorem)**.** *If $x_1, x_2, ..., x_n$ are pairwise orthogonal vectors in an inner product space E, then*

$$\|\textstyle\sum_{i=1}^{n} x_i\|^2 = \sum_{i=1}^{n} \|x_i\|^2.$$

Theorem 2.1.19 (The Parallelogram Law)**.** *If E is an inner product space and $x, y \in E$, then*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Example 2.1.20. Consider the space ℓ^∞; It is impossible to define an inner product on ℓ^∞ such that $\langle x, x\rangle = \|x\|_\infty^2$ for all $x \in \ell^\infty$. This follows from the failure of the parallelogram law in ℓ^∞, for if $x = (1, 1, 0, 0, ...)$ and $y = (1, -1, 0, 0, ...)$, then $\|x\| = 1 = \|y\|$ and $\|x + y\| = 2, \|x - y\| = 2$.

Theorem 2.1.21 (The Polarization Identity)**.** *For any x, y in an inner product space E,*

$$\langle x, y\rangle = \tfrac{1}{2}\textstyle\sum_{n=1}^{4} i^n\|x + i^n y\|^2.$$

Definition 2.1.22. A *Hilbert space* is a complete inner product space. That is, complete in the metric induced by the norm.

Proposition 2.1.23. *If E is a vector space and $\langle ., .\rangle_E$ is an inner product on E and if $\mathcal{H}$ is the completion of E with respect to the metric induced by the norm on E , then there is an inner product $\langle ., .\rangle_{\mathcal{H}}$ on $\mathcal{H}$ such that $\langle x, y\rangle_{\mathcal{H}} = \langle x, y\rangle_E$ for x and y in E and the metric on $\mathcal{H}$ is induced by this inner product. That is, the completion of E is a Hilbert space.*

Example 2.1.24. A concrete example of a Hilbert space is the space ℓ^2 of all complex sequences $(x_n)_{n=1}^\infty$ that are square summable, that is, satisfy

$$\textstyle\sum_{n=1}^{\infty}|x_n|^2 < \infty,$$

with component-wise addition and scalar multiplication and with inner product defined by

$$\langle x, y\rangle = \sum_{n=1}^{\infty} \xi_n \overline{\eta_n} \ , \ x = (\xi_n)_{n=1}^{\infty}, y = (\eta_n)_{n=1}^{\infty}.$$

And the norm derived from this inner product

$$\|x\| = [\ \sum_{n=1}^{\infty} |\xi_n|^2\ ]^{\frac{1}{2}}.$$

Definition 2.1.25. If (e_n) is an orthonormal sequence in a Hilbert space $\mathcal{H}$ then, for any $x \in \mathcal{H}$, $\langle x, e_n\rangle$ is the nth *Fourier coefficient* of x with respect to (e_n). The *Fourier series* with respect to (e_n) is the series $\sum_n \langle x, e_n\rangle e_n$.

Definition 2.1.26. An orthonormal sequence (e_n) in a Hilbert space $\mathcal{H}$ is *complete* if the only member of $\mathcal{H}$ which is orthogonal to every e_n is the zero vector. (e_n) is then said to be an *orthonormal basis*.

Definition 2.1.27. A Hilbert space is *separable* if it has an orthonormal basis.

Example 2.1.28. For the space ℓ^2, $(e_n)_{n=1}^{\infty}$ is the canonical orthonormal basis where e_n is a sequence with 1 at the nth position and zero otherwise.

Example 2.1.29 ([26])**.** Let L be the space of continuous functions on $\mathbb{R}$ such that

$$p(f) = (\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(x)|^2 dx)^{\frac{1}{2}}$$

is defined for L. Consider the quotient space L/N where $N = \ker p$. This is a normed space and p is a norm on it. We denote by H the completion of this space with respect to the norm $p(x)$. Let $H_0 = \text{clos}\{\text{span}_{\lambda\in\mathbb{R}}\{e^{i\lambda t}\}\}$. The corresponding inner product on H and H_0 is

$$\langle f, g\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{g(x)}dx.$$

If $\lambda_1 \neq \lambda_2$ then $e^{i\lambda_1 t} \perp e^{i\lambda_2 t}$. So H_0 has an uncountable set of pairwise orthonormal elements which implies the non-separability of H_0.

The example above gives a Hilbert space with connections with almost periodic functions. Such functions arise from the uniform norm closure ($\|.\|_\infty$-norm) of this span of the exponentials. We consider this space in detail in Chapter 6.

One of the aims of this thesis is to develop a good understanding of how infinite frameworks can differ in terms of their infinitesimal flexibility. Using familiar properties from Hilbert spaces and Banach sequence spaces, we can define different classes of velocity sequences and infinitesimal flexes and accordingly analyse their infinite frameworks. This can be seen, for example, in the construction of an infinite framework that admits a vanishing infinitesimal flex but is "ℓ^2-infinitesimally rigid" in Section 4.3 or in the development of the theory of "almost periodic rigidity".

2.2 Linear Operators

For every mathematical framework $\mathcal{G}$ one can form the associated "rigidity matrix $R(\mathcal{G})$" and ask whether $R(\mathcal{G})$ gives a bounded linear operator with respect to various norms. We shall consider this problem when obtaining a sufficient graph condition (Section 4.4). In this section we collect together some standard terminology and results about linear operators and their matrices.

Definition 2.2.1. If X and Y are vector spaces over a field $\mathbb{K}$, a *linear operator* from X to Y is a mapping $T : X \to Y$ such that

$$T(\lambda x + \mu y) = \lambda T x + \mu T y$$

for all x, y in X and λ, μ in $\mathbb{K}$. If X, Y are normed spaces, a linear operator is said to be *bounded* if there exists $M \geq 0$ such that

$$\|Tx\| \leq M\|x\| \text{ for all } x \in X.$$

The *norm*, or *operator norm* is the non-negative real number

$$\|T\| = \sup\{\|Tx\| : x \in X, \|x\| \leq 1\}$$

and for any $x \in X$,

$$\|Tx\| \leq \|T\|\|x\|.$$

Let $\mathcal{B}(X, Y)$ be the set of bounded linear operators from X to Y and for $X = Y$, $\mathcal{B}(X, X) \equiv \mathcal{B}(X)$.

Theorem 2.2.2. *A linear map from one normed space to another is continuous if and only if it is bounded.*

Theorem 2.2.3. *If X, Y are normed spaces then the space $\mathcal{B}(X, Y)$ is itself a normed space with respect to point-wise operations and operator norm. If, further, Y is a Banach space then so is $\mathcal{B}(X, Y)$.*

Theorem 2.2.4 (Extension by Continuity)**.** *Let X be a normed space, Y a Banach space and $T : X \to Y$ a linear operator defined on a dense subspace $D(T)$ of X. If T is bounded as an operator from $D(T)$ to Y, then it has a unique extension to a bounded operator from all of X into Y. Moreover, this extension has the same norm as T.*

Definition 2.2.5. The *rank* of an operator $T \in \mathcal{B}(X)$ is the dimension of its range. $T \in \mathcal{B}(X)$ has *finite rank* if $\dim(Image(T)) < \infty$.

Definition 2.2.6. Let $T \in \mathcal{B}(X)$. T is *invertible* if there exist an $S \in \mathcal{B}(X)$ such that

$$ST = TS = I.$$

Definition 2.2.7. A mapping $U : \mathcal{H} \to \mathcal{K}$ where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces is a *unitary operator* if it is linear and bijective and preserves inner products: that is, satisfies

$$\langle Ux, Uy \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathcal{H}.$$

Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are *isomorphic* if there is a unitary operator from $\mathcal{H}$ to $\mathcal{K}$.

Theorem 2.2.8. *Let $\mathcal{H}$ be a separable Hilbert space. Then $\mathcal{H}$ is isomorphic either to $\mathbb{C}^n$ for some $n \in \mathbb{N}$ or to ℓ^2.*

Definition 2.2.9. Let $X \neq \{0\}$ be a complex normed space and $T \in \mathcal{B}(X)$. The *spectrum* of T is the set

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ fails to be invertible}\}.$$

Definition 2.2.10. We say a complex number λ is an *eigenvalue* of T if there is a nonzero vector e such that $Te = \lambda e$, the vector e is the *eigenvector* of T associated with the eigenvalue λ.

Proposition 2.2.11. *If $T \in \mathcal{B}(X)$ and $\dim(X) < \infty$, then the set of eigenvalues of T is precisely the spectrum of T.*

Theorem 2.2.12. *$\sigma(T)$ is a compact set lying entirely in the closed disk $\{\lambda : |\lambda| \leq \|T\|\}$.*

Theorem 2.2.13. *$\sigma(T) \neq \phi$.*

Definition 2.2.14. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. An operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is said to be *Hilbert-Schmidt* if there exists a complete orthonormal sequence $(e_n)_{n=1}^{\infty}$ in $\mathcal{H}$ such that

$$\textstyle\sum_{n=1}^{\infty} \|Te_n\|^2 < \infty.$$

Definition 2.2.15. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. The *Hilbert-Schmidt norm* of T

$$\|T\|_{\mathrm{HS}} = [\textstyle\sum_{n=1}^{\infty} \|Te_n\|^2]^{\frac{1}{2}},$$

where $(e_n)_{n=1}^{\infty}$ is a complete orthonormal sequence in $\mathcal{H}$.

As we develop the analysis of infinite mathematical frameworks in the following chapters, it will become clear how the associated rigidity matrix commutes with transformation operators which are in fact unitary on Hilbert spaces. Furthermore, in the case of crystal frameworks, the rigidity matrix satisfies commutation relations with all isometric symmetries of the crystal framework. These properties of the rigidity matrix will be later used to prove theorems related to the almost periodic rigidity of crystal frameworks, for example.

Definition 2.2.16. Let $\mathcal{H}$ be a Hilbert space and T a bounded operator on $\mathcal{H}$. The *matrix of T* with respect to the orthonormal basis $(e_n)_{n\in\mathbb{N}}$ is the array $[\alpha_{ij}]_{i,j=1}^{\infty}$ given by

$$\alpha_{ij} = \langle Te_j, e_i\rangle.$$

If, as usual, the first index indicates rows and the second one columns, then the matrix is formed by writing the coefficients in the expansion of Ae_j as the j column. Operator multiplication corresponds to the matrix product defined by

$$\gamma_{ij} = \textstyle\sum_k \alpha_{ik}\beta_{kj}.$$

The following comments will be useful for the determination of the boundedness of the rigidity matrix for infinite frameworks ([34]).

The first significant way in which infinite matrix theory differs from the finite version: every operator corresponds to a matrix, but not every matrix corresponds to an operator. Necessary conditions for this are that each row and each column of the matrix is square summable, but these conditions are not sufficient. For example the diagonal matrix whose *nth* diagonal term is n. Also, even if there is an upper bound to the norm, of the row matrices and the column matrices, this is still not sufficient.

A sufficient condition for an infinite matrix to represent an operator is that the family of all entries has to be square summable but this condition is not necessary as in the unit matrix.

Proposition 2.2.17. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, the Hilbert-Schmidt operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has the matrix $[\alpha_{ij}]_{i,j=1}^{\infty}$ with respect to an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ where*

$$\|T\|_{HS} = [\textstyle\sum_{i,j=1}^{\infty} |\alpha_{ij}|^2]^{\frac{1}{2}}.$$

2.3 Basic Graph Theory

This section offers a brief summary of some basic definitions in finite graph theory. The terminology is later used to define mathematical frameworks and the matrices associated with a graph will be used for the factorization of the rigidity matrix obtained in Proposition 4.5.4.

Definition 2.3.1. A *graph* G is a pair (V, E) where V is a finite set whose elements are the *vertices* of the graph and $E \subseteq V \times V$ is a collection of pairs of vertices called the *edges* of the graph.

For example, the graph in Figure 2.1 has vertex set $V = \{1, 2, 3, 4, 5, 6, 7\}$ and edge set $E = \{(1, 2), (3, 4), (4, 5), (3, 5), (4, 6)\}$.

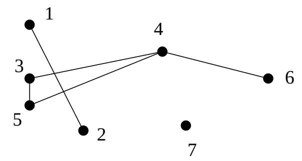

Figure 2.1: A graph (V, E)

If $p, q \in V$ and $e = (p, q) \in E$ then p and q are called the *endpoints* of the edge e, and the vertices p and q are said to be *adjacent*. A vertex p is *incident* with an edge e if p is an endpoint of e. The set of all edges in E incident to a vertex p is denoted by $E(p)$. A *simple* graph (V, E) is a graph with no loop edges (v, v).

One of the first applications of graph theory was to the structure of molecules in chemistry with vertices representing atoms and edges representing the chemical bonds. Chemical properties differ according to how the atoms are connected, which can be easily seen from the graph model. Graphs that allow multiple edges are called *multi-graphs*, where for example, double edges indicate a double chemical bond. Since multi-graphs are not common in Euclidean rigidity theory, all our graphs will be simple unless stated otherwise.

Two graphs are *isomorphic* if there is a correspondence between their vertex sets that preserves adjacency. Thus $G = (V, E)$ is isomorphic to

$G' = (V', E')$ if there is a bijection $\phi : V \to V'$ such that $(p, q) \in E$ if and only if $(\phi(p), \phi(q)) \in E'$.

Definition 2.3.2. A *subgraph* $G' = (V', D')$ of the graph $G = (V, E)$ is a graph consisting of some vertices and some edges of G ($V' \subseteq V$ and $E' \subseteq E$). If $G' \subsetneq G$ then G' is a *proper subgraph* of G.

Definition 2.3.3. If G' is a subgraph of G and G' contains all the edges $(p, q) \in E$ with $p, q \in V'$, then G' is an *induced subgraph* of G; we say that V' *induces* or *spans* G' in G. $G' \subsetneq G$ is a *spanning* subgraph of G if $V' = V$.

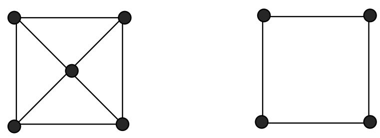

Figure 2.2: A graph G and a subgraph G'

Definition 2.3.4. The *complete graph* K_n is the graph (V, E) with $|V| = n$ and E consisting of all $\frac{n(n-1)}{2}$ pairs of vertices.

Definition 2.3.5. The *complement* $\overline{G}$ of G is the graph on V with edge set $V \times V \setminus E$.

Definition 2.3.6. Let $G = (V, E)$ be a graph and $p \in V$. The *degree of* p, $d(p)$, is the number $|E(p)|$ of edges of the graph with the vertex p as an endpoint. Vertices of degree zero are called *isolated vertices* and those of degree one are called *pendant vertices.* The number $\delta(G) := \min\{d(p), p \in$

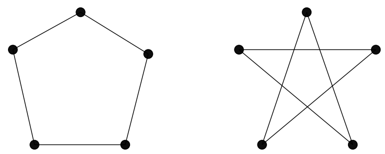

Figure 2.3: A graph G and its complement $\overline{G}$

$V\}$ is the *minimum degree* of G, and the number $\Delta(G) := \max\{d(p), p \in V\}$ is its *maximum degree*. If all the vertices have the same degree n, then $G = (V, E)$ is *n-regular*.

Definition 2.3.7. Let $G = (V, E)$ be a graph, a *path* in G is a finite sequence of distinct vertices $\{p_0, p_1, ..., p_n\}$ such that p_{i-1} and p_i are adjacent for $i = 1, \dots, k$.

The edges joining successive vertices in the sequence are called the *edges of the path*; the number of these edges is called the *length* of the path.

A *cycle* in G is a finite sequence of vertices $\{p_0, p_1, ..., p_n\}$ such that $p_0 = p_k$ and $p_1, \dots, p_k$ are distinct and each p_{i-1} and p_i are adjacent for $i = 1, \dots, k$. The *length* of a cycle is the number of its edges (or vertices); the cycle of length n is called an n-cycle. An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a *chord* of that cycle. Thus, an *induced cycle* in G (a cycle in G forming an induced subgraph) is one that has no chords.

The *distance*, $d_G(p, q)$, in G of two vertices p, q is the length of the shortest p-q path in G; if no such path exists, we set $d_G(p, q) := \infty$. The greatest distance between any two vertices in G is the *diameter* of G, denoted by

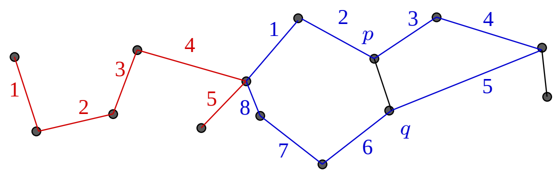

Figure 2.4: A path of length 5, a cycle of length 8 that is not an induced cycle and a chord p-q

diam$G$.

Definition 2.3.8. A graph $G = (V, E)$ is said to be *disconnected* if the vertex set can be partitioned into two non-empty sets A and B so that no edge has an end point in A and the other endpoint in B . We say that the graph is *connected* if no such partitioning exists. A maximal connected subgraph of G is a *component* of G. A vertex which separates two other vertices of the same component is a *cutvertex*, and an edge separating its ends is a *bridge*. Thus, the bridges in a graph are those that do not lie on any cycle.

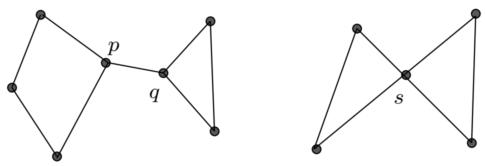

Figure 2.5: A graph with cutvertices p, q, s and a bridge (p, q)

A *forest* is a graph that has no cycles. A connected forest is called a *tree* and in this way a forest is a graph whose components are trees. The vertices of degree 1 in a tree are its *leaves* except that the *root* is not called a leaf, even if it has degree 1.

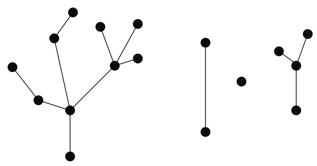

Figure 2.6: A forest

Definition 2.3.9. A graph $G = (V, E)$ is *bipartite* if it has a partition of its vertex set into two cells A and B:

$$V = A \cup B, \quad A \cap B = \emptyset,$$

so that every edge in E has one endpoint in each cell.

The following definition is particularly relevant to finite mathematical frameworks in two dimensions.

Definition 2.3.10. A graph $G = (V, E)$ is $(2,3)$-*sparse* if for every subgraph $G' = (V', E')$ with at least one edge, $|E'| \leq 2|V'| - 3$. G is called $(2,3)$-*tight* if G is $(2,3)$-sparse and $|E| = 2|V| - 3$.

The following matrices can be associated with a graph G. For some of these matrices one needs to consider an *orientation* of the edges, that is, to give each edge $e = [p, q]$ a direction from p to q or from q to p. When this is done, G becomes an *oriented graph*. Although an orientation of the graph is not needed to define a mathematical framework, and therefore its rigidity matrix, these matrices will be used later when factorizing the *rigidity matrix*.

Definition 2.3.11. Let $G = (V, E)$ be a graph. The *adjacency matrix* $A(G)$ is the $|V| \times |V|$ matrix such that the entry for row i, corresponding to vertex v_i, and column j, corresponding to vertex v_j, is 1 if $[v_i, v_j]$ is an edge in G and zero otherwise.

Definition 2.3.12. Let $G = (V, E)$ be an oriented graph. The *incidence matrix* $E(G)$ is the $|E| \times |V|$ matrix determined by an orientation of G such that the entry for row e_k and the column corresponding to vertex v_i is 1 if v is the source of the edge e_k and -1 if it is the range of the edge e_k and zero otherwise. i.e. The row of $E(G)$ corresponding to the edge $e = [v_i, v_j]$ is

$$\begin{array}{cc} & \begin{array}{ccccc} & v_i & & v_j & \end{array} \\ e_k = [v_i, v_j] & \begin{pmatrix} 0\ldots0 & 1 & 0\ldots0 & -1 & 0\ldots0 \end{pmatrix}. \end{array}$$

Definition 2.3.13. Let $G = (V, E)$ be an oriented graph. The *Laplacian matrix* $L(G)$ is the $|E| \times |E|$ matrix $E(G)E(G)^T$ where $E(G)$ is the incidence matrix determined by the same orientation.

Definition 2.3.14. Let $G = (V, E)$ be a graph. The *degree matrix* $D(G)$ is the $|V| \times |V|$ diagonal matrix such that the $v_i v_i$ entry is the degree of vertex v_i in G.

Theorem 2.3.15 ([9])**.** *Let $G = (V, E)$ be an oriented graph. Then*

$$L(G) = E(G)E(G)^T = D(G) - A(G).$$

Chapter 3

Finite Bar-joint Frameworks

Suppose that we have a graph $G = (V, E)$. Then a realization of this graph in d-space is an assignment $p = (p_1, \ldots, p_{|V|})$ of points $p_i \in \mathbb{R}^d$ to the vertex set V. A *mathematical bar and joint framework* is a graph with a realization in d-space where the vertices of the graph represent the framework's flexible joints and the edges correspond to the framework's stiff bars.

Formally, A *framework* $\mathcal{G}$ in $\mathbb{R}^d$ (or bar-joint framework) is a pair (G, p) where $G = (V, E)$ is a simple connected graph and $p = (p_1, p_2, ...)$ is a framework vector made up of framework points p_i in $\mathbb{R}^d$ associated with the vertices $v_1, v_2, ...$ of V with $p_i \neq p_j$ if (v_i, v_j) is an edge. The *framework edges* are the closed line segments $[p_i, p_j]$ associated with the edges of the graph $G = (V, E)$.

3.1 Forms of Rigidity

Consider a finite framework in $\mathbb{R}^2$, a continuous time dependent transformation of the framework points is a flexing of the structure if the edge lengths remain unchanged but the final configuration is not congruent to the original configuration. If no flexing exists, the structure is said to be continuously rigid.

A rigid body motion is the displacement of the framework while keeping the distances between all pairs of framework vertices unchanged whether those pairs form a framework edge or not. In $\mathbb{R}^2$ a rigid body motion results from translations in either coordinate directions or rotations (for example, about one of the framework's vertices). A transformation of this nature that changes the distance between at least one pair of vertices that are not connected by an edge is a continuous flexing of the structure. For example, the square is continuously flexible in $\mathbb{R}^2$ but if we connect one diagonal it becomes rigid (Figure 3.1). In $\mathbb{R}^3$, the same framework becomes flexible as one triangle can rotate relative to the other among the common edge. Although it might seem clear that the square admits a continuous flex in the plane, the formal proof involves solving systems of quadratic equations and even for "small" frameworks this is far from easy. This complication contributed towards defining the more convenient term of infinitesimal flexibility.

Definition 3.1.1. Let $\mathcal{G} = (G, p)$ be a finite framework in $\mathbb{R}^d$. A *continuous motion* of $\mathcal{G}$ is a continuous path, $P(t) : [0, 1] \to \mathbb{R}^{dv}$ such that:

Figure 3.1: The square with one diagonal is rigid in 2D but flexible in 3D

(i) $P(0) = p$,

(ii) $\|P_i(t) - P_j(t)\| = \|p_i - p_j\|$ for all $t \in [0, 1]$ and all edges $e = [p_i, p_j]$.

The motion P is a *rigid body motion* if condition (ii) is satisfied for all $t \in [0, 1]$ and all pairs of vertices p_i, p_j whether they form an edge or not. If there exists at least one pair of vertices, p_k and p_l, such that $\|P_k(t) - P_l(t)\| \neq \|p_k - p_l\|$ for all $t \in [0, 1]$ and $[p_k, p_l]$ is not an edge then P is a *continuous flex*, or sometimes called a *finite flex*, *mechanism* or *deformation.* A framework $\mathcal{G}$ is said to be *continuously rigid* if all of its motions are rigid body motions and *continuously flexible* otherwise.

We will denote the vector space of all velocity vectors assigned to the framework's vertices by $\mathcal{H}_v(\mathcal{G})$ (or simply, $\mathcal{H}_v$ when the framework in question is understood).

Definition 3.1.2. Let $\mathcal{G} = (G, p)$ be a finite framework in $\mathbb{R}^d$ with $|V| = n$. An *infinitesimal flex* is a vector $u = (u_1, \ldots, u_n)$ in the vector space $\mathcal{H}_v(\mathcal{G}) = \mathbb{R}^d \oplus \cdots \oplus \mathbb{R}^d$ such that the orthogonality relation

$$\langle p_i - p_j, u_i - u_j \rangle = 0,$$

holds for each edge $e = [p_i, p_j]$.

Regarding each u_i as a velocity vector this means that for each edge the components of the endpoints velocities in the edge direction are in agreement.

If the above condition is satisfied by all pairs of framework vertices, not just those that form edges, then u is said to be a *trivial infinitesimal flex*, or an *infinitesimal rigid body motion.*

A framework $\mathcal{G}$ is *infinitesimally rigid* if every infinitesimal flex of $\mathcal{G}$ is trivial and *infinitesimally flexible* otherwise.

The space $\mathcal{H}_{\text{fl}}(\mathcal{G})$ of all infinitesimal flexes of $\mathcal{G}$ is a vector subspace of $\mathcal{H}_v(\mathcal{G})$ that itself includes the subspace of all infinitesimal rigid motions $\mathcal{H}_{\text{rig}}(\mathcal{G})$.

The following proposition shows that infinitesimal flexes correspond to the velocity vectors that for small time t induce edge length changes of order $o(t)$.

Proposition 3.1.3 ([54])**.** *Let $\mathcal{G} = (G, p)$ be a finite framework in $\mathbb{R}^d$. The following are equivalent:*

(i) $u = (u_1, ..., u_n)$ is an infinitesimal flex of $\mathcal{G}$

(ii) $\|(p_i + tu_i) - (p_j + tu_j)\|_2^2 - \|p_i - p_j\|_2^2 = O(t^2)$ as $t \to 0$, for all $(i, j) \in E$.

Proof. $(i) \Rightarrow (ii)$: Let u be an infinitesimal flex of $\mathcal{G} = (G, p)$ then, $\langle p_i -$

$p_j, u_i - u_j\rangle = 0$ and

$$\begin{aligned}
&\|(p_i + tu_i) - (p_j + tu_j)\|_2^2 - \|p_i - p_j\|_2^2 \\
&= \langle (p_i - p_j) + t(u_i - u_j), (p_i - p_j) + t(u_i - u_j)\rangle - \|p_i - p_j\|_2^2 \\
&= \|p_i - p_j\|_2^2 + \langle t(u_i - u_j), (p_i - p_j)\rangle + \langle (p_i - p_j), t(u_i - u_j)\rangle \\
&+ \|t(u_i - u_j)\|_2^2 - \|p_i - p_j\|_2^2 \\
&= t^2\|(u_i - u_j)\|_2^2
\end{aligned}$$

Let $M = \max_{(i,j)\in E}\|(u_i - u_j)\|_2^2$, then

$$\|(p_i + tu_i) - (p_j + tu_j)\|_2^2 - \|p_i - p_j\|_2^2 \leq Mt^2$$

and $\|(p_i + tu_i) - (p_j + tu_j)\|_2^2 - \|p_i - p_j\|_2^2 = O(t^2)$ as required.

$(ii) \Rightarrow (i)$: Let $\|(p_i + tu_i) - (p_j + tu_j)\|_2^2 - \|p_i - p_j\|_2^2 = O(t^2)$ i.e. there exists $c > 0$ s.t.

$$|\ \|(p_i + tu_i) - (p_j + tu_j)\|_2^2 - \|p_i - p_j\|_2^2\ | \leq ct^2$$

for all $t \in [-\delta, \delta]$, which implies that

$$-ct^2 \leq \|(p_i + tu_i) - (p_j + tu_j)\|_2^2 - \|p_i - p_j\|_2^2 \leq ct^2$$

expanding the norms we have

$$-ct^2 \leq t\langle (u_i - u_j), (p_i - p_j)\rangle + t\langle (p_i - p_j), (u_i - u_j)\rangle + t^2\|(u_i - u_j)\|_2^2 \leq ct^2$$

and

$$-ct \leq \langle (u_i - u_j), (p_i - p_j)\rangle + \langle (p_i - p_j), (u_i - u_j)\rangle + t\|(u_i - u_j)\|_2^2 \leq ct.$$

Finally, taking limits as $t \to 0$ we conclude that $\langle (p_i - p_j), (u_i - u_j) \rangle = 0$ and u is an infinitesimal flex of $\mathcal{G} = (G, p)$. $\square$

Suppose that u is an infinitesimal flex of the framework $\mathcal{G} = (G, p)$ in $\mathbb{R}^d$, then u must satisfy

$$\langle p_i - p_j, u_i - u_j \rangle = 0,$$

for each edge $e = [p_i, p_j]$. Thus, corresponding to each edge in E, there is a linear equation that must be satisfied by u. It follows that the space of infinitesimal flexes of the framework is the solution space to this homogeneous system of $|E|$ equations in $d|V|$ variables.

The $|E| \times d|V|$ matrix of coefficients of this system is called the *rigidity matrix* of the framework and it is denoted by $R(G, p)$.

The rigidity matrix $R(G, p)$ is $\frac{1}{2}J(G, p)$ where $J(G, p)$ is the generalized Jacobian, evaluated at p.

Definition 3.1.4. The *rigidity matrix* of the finite framework $\mathcal{G} = (G, p)$ in $\mathbb{R}^2$ is the $|E| \times 2|V|$ matrix $R(G, p)$ with rows indexed by the framework edges and columns labelled by the vertices but with multiplicity two, namely the labels $v_1^x, v_1^y, v_2^x, v_2^y, \ldots$ and with entries $x_i - x_j, x_j - x_i, y_i - y_j, y_j - y_i$ occurring in the row with label $e = (v_i, v_j)$ with the respective column labels $v_i^x, v_i^y, v_j^x, v_j^y$ and with zero entries elsewhere. Rigidity matrices in higher dimension spaces are similarly defined.

It is straightforward to check that the infinitesimal flexes are the velocity vectors in the kernel of $R(G,p)$. Indeed we have

$$(R(G,p)u)_e = \langle p_i - p_j, u_i\rangle + \langle p_j - p_i, u_j\rangle = \langle p_i - p_j, u_i - u_j\rangle = 0.$$

The rigidity matrix defines a linear transformation

$$R(G,P) : \mathcal{H}_v(\mathcal{G}) = \textstyle\prod_{|V|} \mathbb{R}^d \to \mathcal{H}_e(\mathcal{G}) = \textstyle\prod_{|E|} \mathbb{R}.$$

Example 3.1.5. Let $\mathcal{G}$ be the two dimensional triangle framework in Figure 3.2. The rigidity matrix $R(G,P)$ can be formed as follows

$$R(G,p) = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} p_1^x & p_1^y & p_2^x & p_2^y & p_3^x & p_3^y \end{array} \\ \begin{pmatrix} x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 & 0 & 0 \\ 0 & 0 & x_2 - x_3 & y_2 - y_3 & x_3 - x_2 & y_3 - y_2 \\ x_1 - x_3 & y_1 - y_3 & 0 & 0 & x_3 - x_1 & y_3 - y_1 \end{pmatrix} \end{array}$$

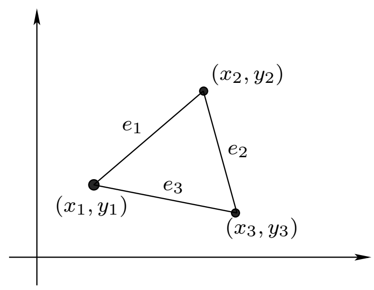

Figure 3.2: A finite framework in $\mathbb{R}^2$

Definition 3.1.6. A framework is *independent* if the row vectors of the rigidity matrix are linearly independent. A framework which is both independent and infinitesimally rigid is minimally infinitesimally rigid in the sense that the removal of any edge results in an infinitesimally flexible framework.

3.2 Generic Rigidity in the Plane

A fundamental result in planar rigidity theory is due to Laman (1970, [45]). This characterizes the rigidity of planar generic frameworks in purely combinatorial terms, that is, a property determined by the underlying graph regardless of the geometry implied by its placement. For more on Laman's Theorem and combinatorial rigidity see [45], [37], [66], [43] and [32].

A framework is said to be in *generic* position if the coordinates of its vertices are algebraically rationally independent. In more intuitive terms, a generic configuration has no degeneracy, i.e. no three points on the same line, no three lines going through the same point, etc. A *Laman graph* is a graph $G = (V, E)$ that satisfies $|E(V')| \leq 2|V'| - 3$ for all $V' \subseteq V$ with $|V'| \geq 4$. Such a graph is also called $(2, 3)$-sparse.

Theorem 3.2.1 ([37])**.** *Let (G, p) be a planar framework with $|V| \geq 2$ vertices. Then rank $R(G, p) \leq 2|V| - 3$. Furthermore, if equality holds, then (G, p) is rigid.*

Theorem 3.2.2 ([37])**.** *Let (G, p) be a planar framework. Then (G, p) is rigid if and only if (G, p) is infinitesimally rigid.*

Theorem 3.2.3. *Let (G, p) be a planar framework. Suppose that the rows of $R(G, p)$ are linearly independent. Then $|E(V')| \leq 2|V'| - 3$ for all $V' \subseteq V$ with $|V'| \geq 4$.*

Theorem 3.2.3 implies that all independent graphs are Laman. The proof for the reverse implication one needs some generic rigidity preserving graph operations (sometimes referred to as Henneberg rigidity preserving moves [36]). These moves can be used to prove the generic rigidity of infinite graphs if the graph can be obtained from a rigid framework by a series of rigidity preserving moves or what is defined in [54] as "sequential rigidity".

(i) ***Vertex addition,*** 0***-extension.*** Let $G = (V, E)$ be a graph such that $x, y \in V$, $v \notin V$. Then $H = (V \cup \{v\}, E \cup \{(v, x), (v, y)\})$ is a vertex addition of G.

(ii) ***Edge split,*** 1***-extension.*** Let $G = (V, E)$ be a graph such that $v \notin V$ and let $e = (x, y) \in E$. Then the graph $H' = (V', E')$ where $V' = V \cup \{v\}$ and $E' = (E - e) \cup \{(v, x), (v, y), (v, z)\}$ for some $z \in V$ is an edge split of G.

Theorem 3.2.4 (Laman [37])**.** *A graph $G = (V, E)$ is independent in $\mathbb{R}^2$ if and only if $|E(V')| \leq 2|V'| - 3$ for all $V' \subseteq V$ with $|V'| \geq 4$.*

Corollary 3.2.5 ([37])**.** *A planar graph $G = (V, E)$ is minimally rigid if and only if $|E| \leq 2|V| - 3$ and $|E(V')| \leq 2|V'| - 3$ for all $V' \subseteq V$ with $|V'| \geq 4$.*

Example 3.2.6. Note that it is necessary for each subgraph to satisfy the count in addition to the whole structure, this is to guarantee that no edges are wasted in over-bracing one of the subgraphs. Although the framework in Figure 3.3 satisfies the overall count, it is flexible.

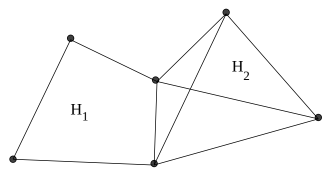

Figure 3.3: A flexible generic framework

From all the above, one can determine the rigidity of generic frameworks that share the same underlying graph. Asimov and Roth [2] complemented this result by proving that rigidity and infinitesimal rigidity are in fact equivalent for generic frameworks.

Generic Rigidity In 3D. Although Laman's theorem is considered one of the very early contributions towards rigidity theory, for frameworks in three dimensional space Laman's conditions are necessary but not sufficient and it still stands an open problem to find a combinatorial characterization for rigidity in three dimensional space.

Example 3.2.7 (The double banana)**.** A classic example is the well known *double banana* (Figure 3.4),[68]. The graph satisfies the count as for each subgraph V', $|E'| \leq 3|V'|-6$, and satisfies the overall count $|E(V)| = 3|V|-6$

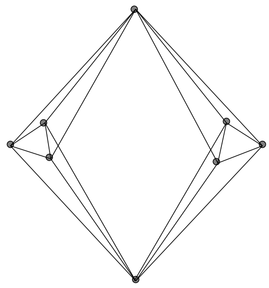

Figure 3.4: The double banana

but it is obviously flexible since each banana can rotate about the implied hinge (or the imaginary edge through the tips).

We see in the next section that this “Laman count” is in fact sufficient for the rigidity of certain three dimensional graphs.

3.3 The Rigidity of Frameworks Given by Convex Polyhedra

In 1776, Euler conjectured that “A closed spatial figure allows no changes, as long as it is not ripped apart” [31]. For convex polyhedra, this conjecture was later answered by Cauchy (1813) who proved that “If there is an isometry between the faces of two strictly convex polyhedra which is an isometry on each of the faces, then the two polyhedra are congruent”. A corollary of Cauchy’s Theorem is that all such convex polyhedra are in fact rigid.

Maxwell, 1864, proved Maxwell’s counting rule [51]: a sufficient condition for three dimensional rigidity is that the number of bars is greater than or equal to 3 times the number of points minus 6.

In 1958, Alexandrov [1] extended Cauchy's Theorem to include all frameworks given by convex polyhedra with faces triangulated by edges between additional vertices on the original edges. The resulting structure in this case, with triangles as faces and including the triangulating edges, is rigid. Gluck, 1975, proved that "almost all" polyhedra are rigid, and this settled Euler's conjecture for polyhedra in generic position.

Finally, in 1977 Euler's conjecture was proved to be false when Connelly [14], [15], [16] constructed a counterexample of a closed (but not convex) polyhedron with triangular faces that forms a flexible three dimensional framework. Further developments regarding convex polyhedra were carried out by Asimov and Roth [2], [3], [62].

These theorems determine the rigidity of a special class of mathematical frameworks, regardless of their geometric placements. For the sake of completion we will include proofs for the rigidity of convex polyhedra with triangular faces.

This rigidity result is also of significance to us as we can understand the rigidity of different three dimensional frameworks using the properties of their finite subframeworks. These examples include frameworks formed by corner connected convex polyhedra with triangular faces such as the "bipyramid" and the "octahedron net" frameworks. Other examples include frameworks formed by convex polyhedra with rigid faces such as the cube with a diagonal for each face. For such examples, it is the rigidity of individual polyhedral finite subframeworks that determines the analysis of the infinite structure. The theorems and proofs in this subsection can be found in [62] and [1].

Definition 3.3.1. A region of the plane or 3-space is *convex*, if for every pair of points of the region, the line segment joining them lies entirely in that region.

Definition 3.3.2. A *convex polyhedron* C in $\mathbb{R}^3$ is the convex hull of a finite set of non-coplanar points in $\mathbb{R}^3$.

Definition 3.3.3. A *supporting hyperplane* of a convex polyhedron C is a hyperplane containing C in one of its closed half spaces and containing a boundary point of C.

Definition 3.3.4. A *net* is an arbitrary finite collection of simple (non self-intersecting) open polygonal lines lying on a polyhedron and having no common points except possibly end points. A *region* is the collection of all points that can be joined one another by polygonal lines not intersecting the net. The simplest example of a net is the net of all edges of the polyhedron and the regions in this case are the faces of the polyhedron.

Theorem 3.3.5 (The Generalized Euler Theorem)**.** *Given a net on a close convex polyhedron, assume that v is the number of vertices, e the number of edges, c the number of connected components, and f the number of regions into which the net divides the polyhedron. Then*

$$v - e + f = c + 1.$$

In particular, if the net is connected, then $v - e + f = 2$.

Frameworks given by convex polyhedra Let C be a convex polyhedron in $\mathbb{R}^3$, a *vertex* of C is a point which is the intersection of C with a support plane, while an *edge* of C is a closed line segment which is the intersection of C with a support plane of C. If C has v vertices with coordinates $p_1, \ldots, p_v \in \mathbb{R}^3$, then $\mathcal{G} = (G, p)$ is the framework in $\mathbb{R}^3$ given by C where $G = (V, E)$ is the graph with

$$V = \{1, \ldots, v\},\ E = \{\{i, j\} : [p_i, p_j] \text{ is an edge of } C\}$$

and $p = (p_1, \ldots, p_v)$.

A *stress*, or a *self stress* to be precise, of a framework $\mathcal{G} = (G, p)$ is a collection of scalars $\omega_{\{i,j\}}$, one for each edge of $\mathcal{G} = (G, p)$, such that

$$\sum_{j \in a(i)} \omega_{\{i,j\}}(p_i - p_j) = 0, \text{ for } 1 \leq i \leq v$$

where $a(i) = \{j : [p_i, p_j] \text{ is an edge of } \mathcal{G}\}$. Stresses are vectors in the kernel of the transpose of the rigidity matrix. Thus "stress theory" is in many ways dual to infinitesimal flex theory.

Letting $\omega_{\{i,j\}} = 0$ for all edges gives the *trivial stress*, and a framework is *stress free* if it admits only a trivial stress. From the definitions of a stress and the rigidity matrix of a framework it follows that a framework in $\mathbb{R}^3$ is stress free if and only if the rank of $R(G, p)$ is equal to the number of edges of the framework. We say that the point $p \in \mathbb{R}^3$ is a *regular point* for the framework $\mathcal{G} = (G, p)$ if the rank of $R(G, p)$ is maximal.

The rigidity predictor. Let $\mathcal{G} = (G, p)$ be a framework in $\mathbb{R}^3$ where $(p_1, \ldots, p_v) \in \prod_{i=1}^{v} \mathbb{R}^3$ is a regular point of $\mathcal{G} = (G, p)$ and $p_1, \ldots, p_v$ do not lie on a hyperplane in $\mathbb{R}^3$. Then $\mathcal{G} = (G, p)$ is rigid in $\mathbb{R}^3$ if and only if $rank R(G, p) = 3v - 6$, where $R(G, p)$ is the rigidity matrix of the framework $\mathcal{G} = (G, p)$, and flexible in $\mathbb{R}^3$ of and only if $rank R(G, p) < 3v - 6$.

To determine which convex polyhedra in $\mathbb{R}^3$ give rigid frameworks in $\mathbb{R}^3$ and which give flexible frameworks we will start by using the signs of the coefficients $\omega_{\{i,j\}}$ of a stress to attach symbols $+$ and $-$ to some of the edges of C. The edge $\{i, j\}$ is marked $+$ if $\omega_{\{i,j\}} > 0$, $-$ if $\omega_{\{i,j\}} < 0$, and left unmarked if $\omega_{\{i,j\}} = 0$. Let G' be the graph on the surface ∂C of C such that the edges of G' are the marked edges of C and the vertices of G' are those of C incident with at least one marked edge.

Definition 3.3.6. The *index* of the vertex p_i, $I(p_i)$, is the number of sign changes encountered in the cycle of edges around p_i, say, when circled in the counter-clockwise direction. The *index* I of G, is the sum of indices of all the vertices of G, i.e.

$$I = \sum_{p_i \in G} I(p_i).$$

Lemma 3.3.7. *The index satisfies*

$$I \leq 4v' - 8$$

where v' is the number of vertices of G'.

Proof. Let e' be the number of edges of G' and f' the number of regions of $\partial C - G'$. Let f'_n be the number of regions with exactly n boundary edges. When we say a region has n edges, then each edge not separating that region from another one is counted twice. Clearly $f'_1 = 0$, and since there is no region bounded by two edges then $f'_2 = 0$. It follows that the total number of regions f' is equal to

$$f' = \sum_{n \geq 3} f'_n$$

and since each edge either belongs to two regions or is counted twice for a single region

$$2e' = \sum_{n \geq 3} n f'_n.$$

Now, we compute the index by counting regions rather than vertices. Since the number of sign changes in moving around a region cannot be greater than n, the number of its edges, and this number is even since when we complete this trip we return to the initial sign, it follows that the number of sign changes is an even number less than or equal to n, therefore

$$
\begin{aligned}
I &\leq 2f'_3 + 4f'_4 + 4f'_5 + 6f'_6 + 6f'_7 + \cdots \leq \sum_{n\geq 3}(2n-4)f'_n \\
&= 2\sum_{n\geq 3} nf'_n - 4\sum_{n\geq 3} f'_n \\
&= 2(2e') - 4f' \\
&= 4e' - 4f'.
\end{aligned}
$$

By Euler's formula 3.3.5, $v' - e' + f' \geq 2$. Therefore

$$4v' - 8 \geq 4e' - 4f'$$

substituting e' and f'

$$4v' - 8 \geq 4e' - 4f' \geq I$$

as required. $\square$

Lemma 3.3.8. *The index of every vertex of G' is greater than or equal to four.*

Proof. Recall that the index of a vertex is an even number, so to prove it is greater than or equal to four it suffices to show that it cannot be zero or two. Fix an arbitrary vertex p_i of $\mathcal{G} = (G, p)$, then

$$\sum_{j\in a(i)} \omega_{\{i,j\}}(p_i - p_j) = 0$$

and since if j is a vertex in G but not in G' then $\omega_{\{i,j\}} = 0$ it follows that

$$\sum_{j\in a'(i)} \omega_{\{i,j\}}(p_i - p_j) = 0$$

where $a'(i) = \{j : [p_i, p_j]$ is an edge of $G'\}$. We will first prove that the index of p_i cannot be zero. For if we let $I(p_i) = 0$, then the edges around p_i are either all marked $+$ or all marked $-$ for $j \in a'(i)$ since zero index means no change in the edge signs. By the convexity of C, there exists a hyperplane that passes through p_i and has no other points in common with C; say the equation of the plane is $n.(p_i - p_j) = 0$ where $n \in \mathbb{R}^3$ is its normal vector. Since all the vertices other than p_i lie in one side of the plane, $n.(p_i - p_j)$ is either positive for all $j \in a'(i)$ or negative for all $j \in a'(i)$. Therefore,

$$\sum_{j \in a'(i)} \omega_{\{i,j\}}[n.(p_i - p_j)] \neq 0$$

from which it follows that

$$\sum_{j \in a'(i)} \omega_{\{i,j\}}[n.(p_i - p_j)] = n.[\sum_{j \in a'(i)} \omega_{\{i,j\}}(p_i - p_j)]$$

this contradicts the earlier assertion that $\sum_{j \in a'(i)} \omega_{\{i,j\}}(p_i - p_j) = 0$ and the index of p_i cannot be zero.

On the other hand, the index of p_i cannot be two. For if $I(p_i) = 2$, then there is a set of edges marked $+$ followed by a set of edges marked $-$ in the cycle of edges around p_i. By the convexity of C, there exists a supporting hyperplane that passes through p_i with the edges marked $+$ on one side of the plane and those marked $-$ on the other side. If the equation of the plane is $n.(p_i - p_j) = 0$, then $n.(p_i - p_j)$ has one sign for all the $+$ edges and the opposite sign for the edges marked $-$. Therefore,

$$0 = n.[\sum_{j \in a'(i)} \omega_{\{i,j\}}(p_i - p_j)] = \sum_{j \in a'(i)} \omega_{\{i,j\}}[n.(p_i - p_j)] \neq 0$$

which is, again, a contradiction. $\square$

Theorem 3.3.9. *Let $\mathcal{G} = (G, p)$ be the framework in $\mathbb{R}^3$ given by a convex polyhedron C. Then*

$$rank R(G, p) = e$$

where $R(G, p)$ is the rigidity matrix of $\mathcal{G} = (G, p)$.

Proof. Assume that $\mathcal{G} = (G, p)$ admits a non trivial stress. We use the signs of this stress to mark the edges of C and let G' be the graph induced by the marked edges as before. By Lemma 3.3.7, $I \leq 4v' - 8$ and by Theorem 3.3.8, the index of each vertex is greater than or equal to 4, from which it follows that the index I is greater than 4 times the number of vertices. Therefore,

$$I \leq 4v' - 8 < 4v' \leq I.$$

This is a contradiction that shows that $\mathcal{G} = (G, p)$ is stress free and thus $rank R(G, p) = e$. $\square$

Lemma 3.3.10. *Let C be a convex polyhedron in $\mathbb{R}^3$ with v vertices, e edges, and f faces of which f_n have exactly n edges. Then $e \leq 3v - 6$ with equality if and only if every face of C is a triangle.*

Proof. By Euler's formula, $v = e - f + 2$ and therefore

$$3v - 6 = 3(v - 2) = 3(e - f) = e + (2e - 3f).$$

But

$$3f = 3\sum_{n\geq 3} f_n \leq \sum_{n\geq 3} nf_n = 2e$$

with equality if and only if $f = f_3$, i.e. every face is a triangle. Therefore $3v - 6 \leq e$ with equality if and only if every face is a triangle. $\square$

Corollary 3.3.11. *The framework $\mathcal{G} = (G, p)$ given by a convex polyhedron C is rigid in $\mathbb{R}^3$ if and only if every face of C is a triangle.*

Proof. By Theorem 3.3.9, $rankR(G, p) = e$, therefore $p = (p_1, \ldots, p_v)$ is a regular point and $p_1, \ldots, p_v$ are not coplanar. By the rigidity predictor, $\mathcal{G} = (G, p)$ is rigid if and only if $rankR(G, p) = 3v - 6$. This implies that $e = 3v - 6$ which happens if and only if every face of C is a triangle. $\square$

By Corollary 3.3.11, it is obvious that the bipyramid and the octahedron (sometimes known as a square bipyramid) are both convex polyhedrons with triangular faces and therefore rigid. These polyhedrons will later be used to construct infinite three dimensional crystals with special flex properties.

3.4 Calculating Flexibility Dimension

When a planar framework is in generic position, one can often deduce the dimension for the space of infinitesimal flexes using Laman's count. For example, a framework that satisfies the count would be rigid and would have a three dimensional infinitesimal flex space corresponding to rigid body motions. If the framework was one edge short of being Laman, then the framework would admit a non trivial flex and the space of infinitesimal flexes would be of dimension four. In the case that the framework was not in generic position, counting could give a clue but not an affirmative answer.

One way to thoroughly understand the infinitesimal flexibility of the framework is to identify a vector space basis for the space $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$ of all infinitesimal flexes. The following examples are a prelude to more subtle infinite framework considerations. The basic "linear algebra" technique used here for finite frameworks will be later extended to identify infinitesimal flex bases for various crystal frameworks.

The double square. Let $\mathcal{G} = (G, p)$ be the finite framework of two corner connected squares one inside the other (Figure 3.5). Here $p_4 = (0, 0)$ and $p_5 = (\frac{1}{4}, \frac{1}{4})$.

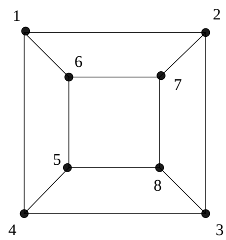

Figure 3.5: The double square

Proposition 3.4.1. *Let $\mathcal{G} = (G, p)$ be the double square framework in 2D. Then* $\dim \mathcal{H}_{\mathrm{fl}}(\mathcal{G}) = 5$.

Proof. We start by identifying velocity vectors associated with $\mathcal{G}$ as follows: Let

$$u^1 = ((1,0), \ldots, (1,0))$$

be the non-zero velocity vector of infinitesimal translation of the whole framework by 1 unit in the positive x direction. Also let

$$u^2 = ((0,1), \ldots, (0,1))$$

be the velocity vector of infinitesimal translation of $\mathcal{G}$ by 1 unit towards the positive y direction, let

$$u^3 = ((-1,0),(-1,1),(0,1),(0,0),(-\tfrac{1}{4},\tfrac{1}{4}),(-\tfrac{3}{4},\tfrac{1}{4}),(-\tfrac{3}{4},\tfrac{3}{4}),(-\tfrac{1}{4},\tfrac{3}{4}))$$

be the velocity vector of infinitesimal rotation about vertex p_4 of the framework. Now, Let u^4 be the non-trivial infinitesimal flex implied by assigning

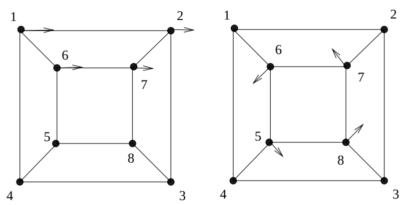

Figure 3.6: Infinitesimal flexes u^4 and u^5 of the double square

zero velocities to the vertices of the base edges for both squares and given by the vector

$$u^4 = ((1,0),(1,0),\underline{0},\underline{0},\underline{0},(1,0),(1,0),\underline{0})$$

as shown in Figure 3.6. Finally, let u^5 be the infinitesimal flex resulting by assigning zero velocities to vertices of the outer square and infinitesimally rotating the inner square (Figure 3.6)

$$u^5 = (\underline{0},\underline{0},\underline{0},\underline{0},(1,-1),(-1,-1),(-1,1),(1,1)).$$

We claim that

$$\mathcal{B} = \{u^1, u^2, u^3, u^4, u^5\}$$

is a spanning set for $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$.

To prove this we will consider an arbitrary flex u of $\mathcal{G}$,

$$\underline{u} = (u_1, u_2, \ldots, u_8), \;\; u_i = (u_i^x, u_i^y),$$

and we subtract appropriate multiples of elements of $\mathcal{B}$ so that we achieve a zero flex of the framework. In this way, we obtain a linear representation of u in terms of elements of $\mathcal{B}$.

First, start by subtracting $u_4^x u^1 + u_4^y u^2$. This results in a new flex, say v, such that $v_4 = (0, 0)$ and $v_3 = (0, v_3^y)$. With vertex p_4 admitting a zero velocity, the only option for p_3 would be to have a non-zero y velocity component and any flex of the structure will take the form

$$v = (*, *, (v_3^x, 0), \underline{0}, *, *, *, *).$$

Subtracting $v_3^y u^3$ results in a new flex, w, with $w_3 = w_4 = \underline{0}$ and $w_1 = w_2 = (w_1^x, 0)$. Indeed, both vertices p_3 and p_4 having zero velocities, results in the framework's edge $e = [p_4, p_3]$ being fixed which implies that vertices p_1 and p_2 can only admit non-zero x velocity components. A possible flex now would take the form

$$w = ((w_1^x, 0), (w_1^x, 0), \underline{0}, \underline{0}, *, *, *, *).$$

Proceeding in the same way, we subtract $w_1^x u^4$ which results in a new flex s such that all the vertices of the outer square are assigned zero velocities. The remaining possible flex now would look like

$$s = (\underline{0}, \underline{0}, \underline{0}, \underline{0}, (s_5^x, s_5^y), *, *, *).$$

The flex s has zero velocities on the outer framework vertices and it follows from the flex condition and the geometry that s is a multiple of u^5. Indeed subtracting $s_5^x u^5$, the new flex r takes the form

$$r = (\underline{0}, \underline{0}, \underline{0}, \underline{0}, (0, s_5^y + s_5^x), (s_6^x + s_5^x, s_6^y + s_5^x), (s_7^x + s_5^x, s_7^y - s_5^x), (s_8^x - s_5^x, s_8^y - s_5^x)).$$

From the fact that $r_1 = r_2 = r_3 = r_4 = \underline{0}$ and $r_5^x = 0$ we have $r_5^y = 0$. Thus it follows that $r_6 = r_7 = r_8 = \underline{0}$ and r is the zero flex of $\mathcal{G}$. From all the above, any flex u can be written as a linear combination of elements of $\mathcal{B}$. In fact the coefficients in the argument above are uniquely determined and so $\mathcal{B}$ is a basis and $\dim \mathcal{H}_{\mathrm{fl}}(\mathcal{G}) = 5$ as required. □

Continuous rigidity of the double square. In [62], Roth proved that for a framework in generic position, continuous rigidity and infinitesimal rigidity are equivalent. This fact, together with Laman's conditions implies that the double square is generically infinitesimally flexible, and equivalently, generically continuously flexible. In this case, $\dim \mathcal{H}_{\mathrm{fl}}(\mathcal{G}) = 4$.

From Proposition 3.4.1, the added symmetry adds to the infinitesimal flexibility of this framework. But unlike the generic case, none of the non-trivial infinitesimal flexes in the base $\mathcal{B}$ is derived from a continuous flex as the "symmetric" double square is in fact continuously rigid. This continuous rigidity problem was suggested by Professor Stephen C. Power and solved by Cruickshank, Kitson and Schulze (unpublished) as in the following proof.

Proposition 3.4.2. *The symmetric double square is continuously rigid.*

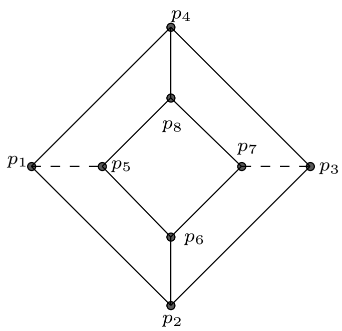

Figure 3.7: The double square is continuously rigid

Proof. Suppose that $p_1p_2p_3p_4$ is a square with diagonal of length 4 (so $|p_1p_2| = 2\sqrt{2}$), that $p_5p_6p_7p_8$ is a square with diagonal of length 2 (so $|p_5p_6| = \sqrt{2}$) and that $|p_1p_5| = |p_2p_6| = |p_3p_4| = |p_4p_8| = 1$ (Figure 3.7).

Using the fact that the diagonals of any equilateral quadrilateral are mutually perpendicular, by Pythagoras, the sum of the squares of the diagonals is four times the square of the side length. Now consider a flex of the framework consisting only of the solid bars. Suppose that in this flex $|p_1p_3| = 4 - x$ for some small $x > 0$. Then by Pythagoras we compute that

$$\begin{aligned}|p_2p_4|^2 &= 4|p_1p_2|^2 - |p_1p_3|^2\\ &= 4(2\sqrt{2})^2 - (4-x)^2\\ &= 16 + 8x - x^2\end{aligned}$$

and therefore,

$$|p_2p_4| = 2\sqrt{4 + 2x - \tfrac{x^2}{4}}.$$

Since, $|p_2p_6| = |p_4p_8| = 1$, the triangle inequality implies that

$$|p_6p_8| \geq 2\sqrt{4 + 2x - \tfrac{x^2}{4}} - 2.$$

Now, by applying Pythagoras to the quadrilateral $p_5p_6p_7p_8$ we have

$$\begin{aligned}|p_5p_7|^2 &= 4|p_5p_6|^2 - |p_6p_8|^2\\ &\leq 4(\sqrt{2})^2 - (2\sqrt{4 + 2x - \frac{x^2}{4}} - 2)^2\\ &\leq -12 - 8x + x^2 + 8\sqrt{4 + 2x - \frac{x^2}{4}}\end{aligned}$$

Note that for $0 < x < 2$,

$$-12 - 8x + x^2 + 8\sqrt{4 + 2x - \tfrac{x^2}{4}} < (2-x)^2$$

Therefore, $|p_5p_7|^2 < (2-x)^2$ and we conclude that

$$|p_5p_7| < 2 - x.$$

But recalling that $|p_1p_3| < 4 - x$, we conclude, by applying the triangle inequality, that $|p_1p_5| + |p_3p_7| > 2$ in this flex of the framework consisting only of the solid bars. Therefore this flex is not the restriction of any flex of the entire framework including the dotted bars, since in that framework $|p_1p_5| = |p_3p_7| = 1$ and the required continuous rigidity follows. $\square$

The double cube. The cube is an example of a flexible convex three dimensional framework. Using the same method used for the double square, one can directly show that $\dim \mathcal{H}_{\mathrm{fl}}(\mathcal{G}_{\mathrm{cube}}) = 12$ for the cubic framework. A generalization of the double square example in three dimensions is the double cube $\mathcal{G}_{\mathrm{2cube}}$, i.e., two corner connected cubes one inside the other (Figure 3.8). In generic position, the space of all infinitesimal flexes dimension is $3|V| - |E| = 48 - 32 = 16$ (There is no "Laman theorem" that predicts this but it can be verified by computer calculation). In the following proposition we calculate the dimension for the space $\mathcal{H}_{\mathrm{fl}}(\mathcal{G}_{\mathrm{2cube}})$ using a similar argument to that for the double square. The following table gives a choice of placement for the double cube:

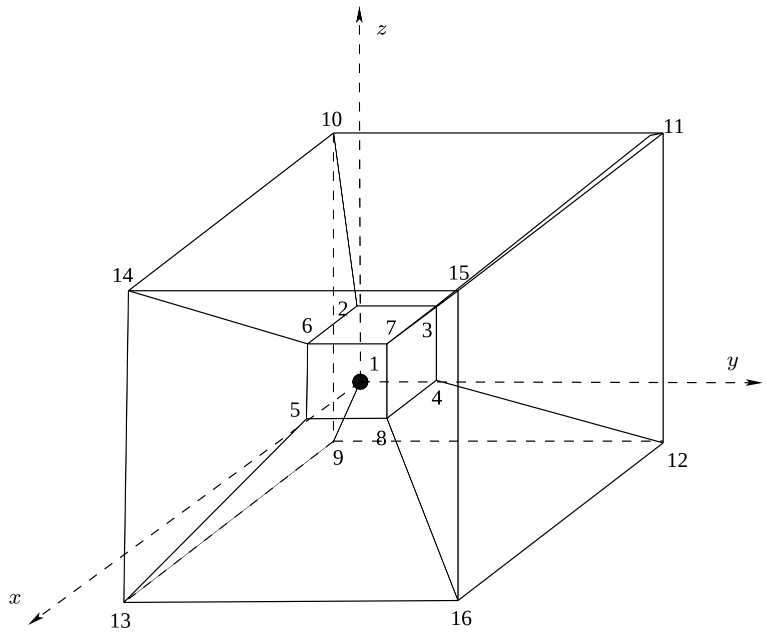

Figure 3.8: The double cube (with adjusted scaling)

inner cube (rear face)	inner cube (front face)	outer cube (rear face)	outer cube (front face)
$p_1 = (0, 0, 0)$	$p_5 = (2, 0, 0)$	$p_9 = (-1, -1, -1)$	$p_{13} = (3, -1, -1)$
$p_2 = (0, 0, 2)$	$p_6 = (2, 0, 2)$	$p_{10} = (-1, -1, 3)$	$p_{14} = (3, -1, 3)$
$p_3 = (0, 2, 2)$	$p_7 = (2, 2, 2)$	$p_{11} = (-1, 3, 3)$	$p_{15} = (3, 3, 3)$
$p_4 = (0, 2, 0)$	$p_8 = (2, 2, 0)$	$p_{12} = (-1, 3, -1)$	$p_{16} = (3, 3, -1)$

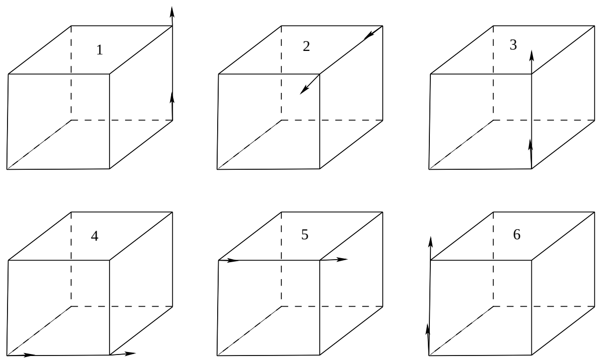

Figure 3.9: Base elements u^k for $\mathcal{G}_{\text{2cube}} = (G, p)$ (restricted to the interior cube)

Proposition 3.4.3. *Let* $\mathcal{G}_{\text{2cube}} = (G, p)$ *be the double cube framework in 3D. Then* $\dim \mathcal{H}_{\text{fl}}(\mathcal{G}_{\text{2cube}}) = 17$.

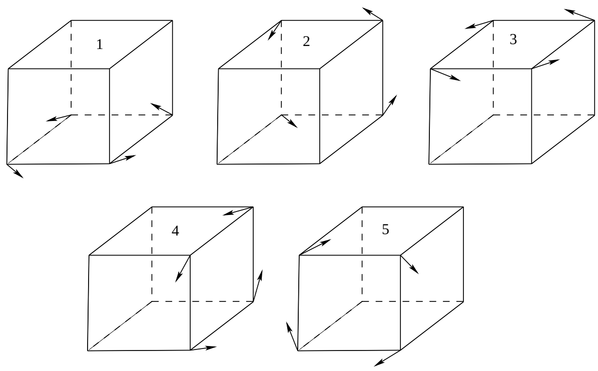

Figure 3.10: Base elements s^k for $\mathcal{G}_{\text{2cube}} = (G, p)$ (restricted to the exterior cube)

Proof. We proceed by calculating $\dim \mathcal{H}_{\text{fl}}(\mathcal{G}_{\text{cube}})$ and then we use this to calculate $\dim \mathcal{H}_{\text{fl}}(\mathcal{G}_{\text{2cube}})$. Let z be a general flex of $\mathcal{G}_{\text{cube}}$ and let $w^k = (w^k_x, w^k_y, w^k_z)$ be the velocity vector at vertex p_k. Subtracting appropriate multiples of the three rigid body translations, $\vec{x}$, $\vec{y}$ and $\vec{z}$, we can arrange for the new flex to have a zero velocity at vertex p_1. This implies that p_2 admits a zero z velocity component. Subtracting an appropriate linear combination of r^x and r^y (the rigid body rotations about the axes x and y respectively), p_2 is now assigned a zero velocity vector. The zero velocity at p_1 implies that p_4 is only allowed non-zero x and z velocity components. Subtracting a multiple of r^z, the rigid body rotation about the z axis, p_4's velocity vector only has a non-zero z component. To achieve a zero velocity at p_4, we subtract a multiple of the infinitesimal flex u^1 (Figure 3.9) for which the restriction of u^1 to the vertices of the inner cube satisfies $u^1|_{p_4} = u^1|_{p_3} = (0, 0, 1)$, with the remaining vertices of the inner cube having zero velocities. This results in vertices p_1, p_2 and p_4 having zero velocities (so far, we have used 7 base elements). The zero velocities at p_2 and p_4 imply that p_3 can only have a non-zero x velocity component. Subtracting an appropriate multiple of the flex u^2 (the flexes u^k are defined in a similar manner as u^1 and are illustrated in Figure 3.9), we can arrange for the velocity at p_3 to be zero. The zero velocity at p_4 implies that p_8 is only allowed non-zero y and z velocity components and they can be similarly illuminated by subtracting a linear combination of u^3 and u^4. At this point, all the vertices p_1, p_2, p_3, p_4 and p_8 admit zero velocities leaving one "flex" option for p_7 which is to have a non-zero y velocity component. Subtracting a multiple of u^5 we can arrange for the velocity at p_7 to be zero. The zero velocities at all the vertices p_1, p_2, p_3, p_4, p_7 and p_8 leave the only option for the flex at p_5 and p_6 to be a constant multiple of u^6, by subtracting this we arrive at a zero flex of the inner cube. It follows from the argument above, that any flex of the cube can be written as a linear combination of elements of the set $\mathcal{B} = \{\vec{x}, \vec{y}, \vec{z}, r^x, r^y, r^z, u^1, \ldots, u^6\}$. Since the required coefficients are uniquely determined, we can deduce that the set $\mathcal{B}$ is a base and $\dim \mathcal{H}_{\text{fl}}(\mathcal{G}_{\text{cube}}) = 12$.

Now suppose that z' is a flex of the framework $\mathcal{G}_{2\text{cube}}$ such that z' assigns zero velocities to the vertices of the inner cube. Thus,

$$z' = (\underline{0}, \ldots, \underline{0}, w^9, w^{10}, \ldots, w^{16}).$$

With our specific choice of geometry we can use the infinitesimal flex condition to deduce the properties of the velocity vectors implied by z'. For example, applying the flex condition to p_1 and p_9, we have

$$\begin{aligned}\langle p_1 - p_9, w^1\rangle + \langle p_9 - p_1, w^9\rangle &= \langle (1,1,1), (0,0,0)\rangle + \langle (-1,-1,-1), (w_x^9, w_y^9, w_z^9)\rangle \\ &= -w_x^9 - w_y^9 - w_z^9.\end{aligned}$$

Therefore, a velocity vector w^9 at p_9 must satisfy $w_x^9 = -w_y^9 - w_z^9$. In the same way we can deduce the equations for the remaining velocity vectors:

w^9:	$w_x^9 = -w_y^9 - w_z^9$	w^{10}:	$w_z^{10} = w_x^{10} + w_y^{10}$
w^{11}:	$w_x^{11} = w_y^{11} + w_z^{11}$	w^{12}:	$w_y^{12} = w_x^{12} + w_z^{10}$
w^{13}:	$w_x^{13} = w_y^{13} + w_z^{13}$	w^{14}:	$w_y^{14} = w_x^{14} + w_z^{14}$
w^{15}:	$w_x^{15} = -w_y^{15} - w_z^{15}$	w^{16}:	$w_z^{16} = w_x^{16} + w_y^{16}$

Let s^k, $k = 1, \ldots, 5$, be the flexes of infinitesimal rotation of a single face for the outer cube and such that all the other vertices are assigned zero velocities (these flexes are illustrated in Figure 3.10). For each vertex on the outer cube, s^k assigns one zero velocity component and the other two are equal in magnitude with signs determined by the equations above. Starting at p_9, we may subtract an appropriate multiple of s^1 so that p_9 has a zero velocity x component. By subtracting a multiple of s^2, p_9 now has zero x and y velocity components and this implies that the z velocity component is zero from the equation above. With a zero velocity vector at p_9, p_{10} can only admit non-zero x and y velocity components and this flex is in fact a multiple of s^3 which we may subtract to achieve a zero velocity at p_{10}. Moving on to p_{11}, we note that the only possibility is for it to have non-zero x and z velocity components. Subtracting a multiple of s^4 the velocity at p_{11} is zero. Since we have zero velocities for the vertices p_4, p_{10} and p_{11}, the equation for p_{12} implies that the velocity vector for p_{12} has to be zero. It follows from all the above that the remaining flex of $\mathcal{G}_{2\text{cube}}$ has to be a constant multiple of s^5 and by subtracting that we arrive at a zero flex of $\mathcal{G}_{\text{2cube}}$. Finally, the coefficients in the above argument are uniquely determined, from which we can deduce that the set $\mathcal{B}' = \mathcal{B} \cup \{s^1, \ldots, s^5\}$ is a base and $\dim \mathcal{H}_{\text{fl}}(\mathcal{G}_{\text{2cube}}) = 17$. $\square$

Chapter 4

Infinite Bar-joint Frameworks

In this chapter, a framework is viewed as an infinite structure in $\mathbb{R}^d$. Formally, a countable *infinite bar-joint framework* in $\mathbb{R}^d$ is the pair $\mathcal{G} = (G, p)$ where $G = (V, E)$ has countable vertex set V and edge set E, and where $p = (p_1, p_2, ...)$ with $p_i \in \mathbb{R}^d$ for all i, is the *framework vector* of $\mathcal{G}$ associated with an enumeration $V = \{v_1, v_2, ...\}$. After the definitions we give a number of original illustrating examples and we analyse a class of infinite "strip" frameworks with special infinitesimal flex properties. Furthermore, we obtain two sufficient conditions for the boundedness of the rigidity matrix $R(\mathcal{G})$ as a Hilbert space operator. The first of these, Theorem 4.4.2, applies to all crystal frameworks. The second condition covers certain infinite tree frameworks where the degrees of the vertices are unbounded.

4.1 Forms of Rigidity

In this section we give a range of definitions related to infinite bar-joint frameworks and different classes of infinitesimal flexes.

Definition 4.1.1. A framework $\mathcal{G} = (G, p)$ in $\mathbb{R}^d$ is *locally finite* if the degree of each vertex of $\mathcal{G}$ is finite.

Definition 4.1.2. A countably infinite framework $\mathcal{G} = (G, p)$ is *edge vanishing* if the sequence $(d_{e_i})_{i=1}^{\infty}$ formed by all bar lengths has no lower bound $m > 0$. $\mathcal{G} = (G, p)$ is *edge unbounded* if $(d_{e_i})_{i=1}^{\infty}$ has no upper bound $M > 0$. $\mathcal{G} = (G, p)$ is *distance-regular* if $(d_{e_i})_{i=1}^{\infty}$ has a lower bound $m > 0$ and an upper bound $M > 0$. That is, if there exist $0 < m < M$ such that for all edges (i, j),

$$m < |p_i - p_j| \leq M.$$

Definition 4.1.3. A countably infinite framework $\mathcal{G} = (G, p)$ is *bounded* or *unbounded* if the sequence p has this property.

Example 4.1.4. It is possible for a framework to be edge vanishing without being bounded. For example, the framework in Figure 4.1, is an infinite linear framework in $\mathbb{R}^2$ with vertices

$$p_1 = (0, 0), p_2 = (1, 0), p_3 = (1 + \tfrac{1}{2}, 0), \ldots, p_n = (\textstyle\sum_{k=1}^{n-1} \tfrac{1}{k}, 0), \ldots$$

But

$$(d_{e_i})_{i=1}^{\infty} = (\tfrac{1}{n})_{n=1}^{\infty}$$

Figure 4.1: An edge vanishing framework that is not bounded

and $\mathcal{G}$ is edge vanishing. Moreover, $p_n \to \infty$ as $n \to \infty$ and $\mathcal{G}$ is unbounded.

Definition 4.1.5. Let $\mathcal{G} = (G, p)$ be an infinite framework in $\mathbb{R}^2$ with connected abstract graph $G = (V, E)$, $V = \{v_1, v_2, ...\}$ and $p = (p_1, p_2, ...)$. A base-fixed continuous flex, or, simply, a flex of $\mathcal{G} = (G, p)$, is a function $p(t) = (p_1(t), p_2(t), ...)$ from $[0, 1]$ to $\prod_V \mathbb{R}^2$ with the following properties:

(i) $p(0) = 0$.

(ii) Each coordinate function $p_i : [0, 1] \to \mathbb{R}^2$ is continuous.

(iii) For some *base edge* (v_a, v_b) with $|p_a - p_b| \neq 0$, $p_a(t) = p_a(0)$ and $p_b(t) = p_b(0)$ for all t.

(iv) Each edge distance is conserved: $|p_i(t) - p_j(t)| = |p_i(0) - p_j(0)|$ for all edges (v_i, v_j), and al t.

(v) $p(t) \neq p$ for some $t \in (0, 1]$.

The framework $\mathcal{G}$ is *flexible*, or more precisely, *continuously flexible*, if it possesses a base-fixed continuous flex. The framework $\mathcal{G}$ is *rigid*, or *continuously rigid*, if it is not flexible.

In a similar way one can define a continuous flex of a framework in $\mathbb{R}^3$, although for it to be a "base-fixed" or "non-trivial" continuous flex, requires a base of three framework vertices as a three dimensional framework could infinitesimally rotate about a fixed edge as a rigid body.

Definition 4.1.6. Let $\mathcal{G} = (G, p)$ be a finite or infinite framework in $\mathbb{R}^2$. The *edge function* of $\mathcal{G}$ is defined to be

$$f_G : \prod_V \mathbb{R}^2 \to \prod_E \mathbb{R}, \ \ f_G(q) = (|q_i - q_j|^2)_{e=(v_i,v_j)}$$

and depends only on the abstract graph G.

Definition 4.1.7. The *solution set* of a framework $\mathcal{G} = (G, p)$, denoted $V(G, p)$, is the set $f_G^{-1}(f_G(p))$. This is the set of all framework vectors q that satisfy the distance constraint equations

$$|q_i - q_j|^2 = |p_i - p_j|^2, \text{ for all edges } e = (v_i, v_j).$$

In Definition 4.1.12 we define some forms of infinitesimal flexes. For completeness we also record some related forms of continuous flexes in the sense of the following definition.

Definition 4.1.8. A continuous base-fixed two-sided flex $p(t) : t \in [-1, 1]$ of a framework $\mathcal{G} = (G, p)$ in $\mathbb{R}^d$ is a *smooth flex* if each coordinate function $p_i(t)$ is infinitely differentiable.

Definition 4.1.9. A continuous flex $p(t) = (p_k(t))_{k=1}^{\infty}$, $t \in [0, 1]$ of an infinite framework $\mathcal{G} = (G, p)$ in $\mathbb{R}^d$ is said to be:

(i) a *bounded* flex if for some $M > 0$ and every k and t,

$$|p_k(t) - p_k(0)| \leq M,$$

(ii) a *colossal flex* if it is not bounded,

(iii) a *vanishing flex* if $p(t)$ is a bounded flex and if the maximal displacement

$$\|p_k - p_k(0)\|_\infty = \sup_{t\in[0,1]}|p_k(t) - p_k(0)|$$

tends to zero as $k \to \infty$,

(iv) a *square-summable flex* if

$$\sum_{k=1}^{\infty}\|p_k - p_k(0)\|_\infty^2 < \infty,$$

(v) a *summable flex* if

$$\sum_{k=1}^{\infty}\|p_k - p_k(0)\|_\infty < \infty,$$

(vi) an *internal flex* if for all but finitely many k the function $p_k(t)$ is constant.

Definition 4.1.10. Let $\mathcal{G} = (G, p)$ be an infinite framework in $\mathbb{R}^d$. An *infinitesimal flex* of $\mathcal{G}$ is a vector in $\mathcal{H}_v(\mathcal{G}) = \prod_V \mathbb{R}^d$ for which, as in the finite case, $\langle p_i - p_j, u_i - u_j\rangle = 0$ holds for each edge $e = [p_i, p_j]$.

Using the same notation as in the finite case, let $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$ denote the linear space of all infinitesimal flexes. This contains the space of rigid body motions $\mathcal{H}_{\mathrm{rig}}(\mathcal{G})$ which in two dimensions is spanned by two translations and a rotation.

Example 4.1.11. Figure 4.2 is an example of a 5-regular infinite framework that is both continuously and infinitesimally flexible. Later on we find that it defines a "crystal framework" and we identify a "base" for the space of all infinitesimal flexes $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$.

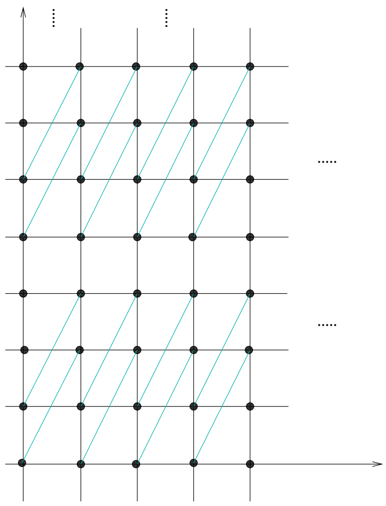

Figure 4.2: A 5-regular infinitesimally flexible infinite framework

Now that we are familiar with the notion of an infinitesimal flex of a framework, we can identify more specific classes of infinitesimal flexes (in the sense of the following definition). This enables us to develop a good understanding of various frameworks in terms of their flexibility as we later see how the existence of certain types of flexes determine the identification of a base for the space $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$.

Definition 4.1.12. An infinitesimal flex $u = (u_i) \in \prod \mathbb{R}^d$ of an infinite framework $\mathcal{G} = (G, p)$ is said to be:

(i) a *bounded* infinitesimal flex if for some $M > 0$ and every i,

$$\|u_i\| \leq M,$$

(ii) a *vanishing infinitesimal flex* if

$$\|u_i\| \text{ tends to zero as } i \to \infty,$$

(iii) a *square-summable infinitesimal flex* if

$$\textstyle\sum_{i=1}^{\infty} \|u_i\|^2 < \infty,$$

(iv) a *summable flex* if

$$\textstyle\sum_{i=1}^{\infty} \|u_i\| < \infty.$$

4.2 The Generic Case

To complement the earlier brief discussion on generic rigidity for finite frameworks, we mention here a recent contribution by Kitson and Power [41] towards the rigidity of general countable simple graphs with respect to both

Euclidean and non Euclidean ℓ_p norms [42]. We restrict our attention here to their characterization of rigidity of countable graphs with generic placements in the Euclidean plane as they obtained a Laman-type Theorem for such graphs. Defining a graph to be *generic* whenever every finite subframework is generic, they proved that any countably infinite graph in fact admits a generic placement in $\mathbb{R}^2$.

Here we give two original examples of graphs that are generically infinitesimally rigid, but as we move on to the analysis of "crystal frameworks", the added symmetry results in both examples being infinitesimally flexible.

Definition 4.2.1 ([52])**.** If P is a property for a class of finite, simple, connected graphs then a graph G is *sequentially* P if G is the union of graphs in some increasing sequence of vertex induced finite subgraphs $G_1 \subseteq G_2 \subseteq \dots$, and each graph G_k has property P.

Theorem 4.2.2 ([41])**.** *Let G be a countable simple graph. The following statements are equivalent:*

(i) G is generically infinitesimally rigid in $\mathbb{R}^2$.

(ii) G is sequentially generically infinitesimally rigid in $\mathbb{R}^2$.

Example 4.2.3. Let G be the countably infinite graph of a 5-regular grid. This graph is sequentially generically infinitesimally rigid in $\mathbb{R}^2$ (Figure 4.3) and therefore is generically infinitesimally rigid. In Chapter 5 we consider a non-generic placement of this graph and we identify a non-trivial base for the space of all infinitesimal flexes $\mathcal{H}_{\text{fl}}$ (Proposition 8.3.2).

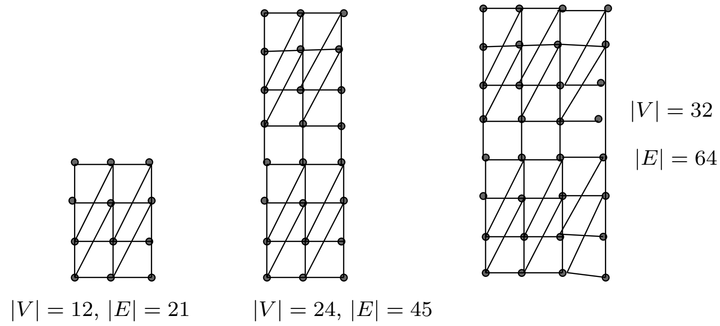

Figure 4.3: The 5-regular grid infinite graph is sequentially infinitesimally rigid

A *tower of graphs* is a sequence of finite graphs $\{G_k : k \in \mathbb{N}\}$ such that G_k is a subgraph of G_{k+1} for all $k \in \mathbb{N}$. A countable graph G contains a *vertex-complete tower* $\{G_k : k \in \mathbb{N}\}$ if each G_k is a subgraph of G and $V(G) = \cup_{k \in \mathbb{N}} V(G_k)$. Moreover, if each G_k is a $(2,3)$-tight subgraph of G, then G contains a $(2,3)$-tight vertex-complete tower.

The following theorem can be viewed as a generalization of Laman's Theorem for finite frameworks:

Theorem 4.2.4 ([41]). *Let G be a countable simple graph. The following statements are equivalent:*

(i) G is generically infinitesimally rigid in $\mathbb{R}^2$.

(ii) G contains a $(2,3)$-tight vertex-complete tower.

Example 4.2.5. Let G be the countably infinite graph of double corner connected triangles (Figure 4.4). This graph can be viewed as a $(2,3)$-tight vertex-complete tower and therefore is generically infinitesimally rigid. In Chapter 5 we consider a non-generic placement of this graph resulting in a framework that admits infinitesimal flexes with finite support.

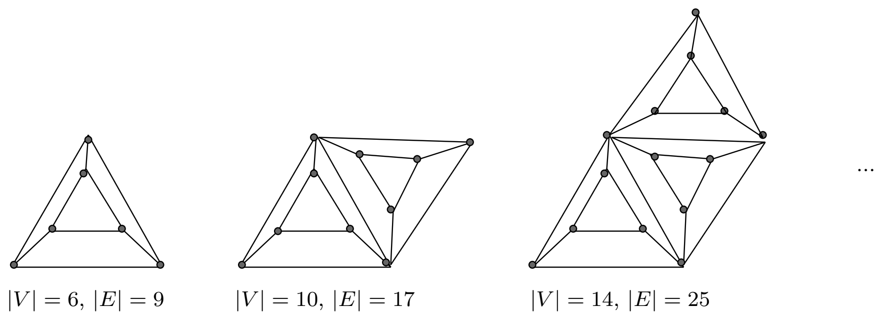

Figure 4.4: The double triangle infinite graph has a $(2,3)$-tight vertex-complete tower

Although there is a generalization of Laman's Theorem for countably infinite generic graphs, the infinitesimal rigidity and continuous rigidity of such graphs are not equivalent as in the finite case.

4.3 Infinite Strip Frameworks

In this section we introduce an infinite "strip" framework that admits a special infinitesimal flex with an "input-output" behaviour that could serve as a building block for examples tailored to a specific choice of infinitesimal flexes. In Chapter 5 we introduce a crystal framework formed by joining copies of the infinite strip to the basic square grid. Further infinitesimal flex analysis for the "joined" crystal framework is obtained in Chapter 8.

Example 4.3.1. Let $\mathcal{G}$ be the infinite, "lever" bar-joint framework consisting of connected upright rigid triangles, as in Figure 4.5. With the base fixed, this framework admits a one dimensional space of non-trivial infinitesimal flexes which is uniquely determined by the velocity vector assigned to vertex p_1.

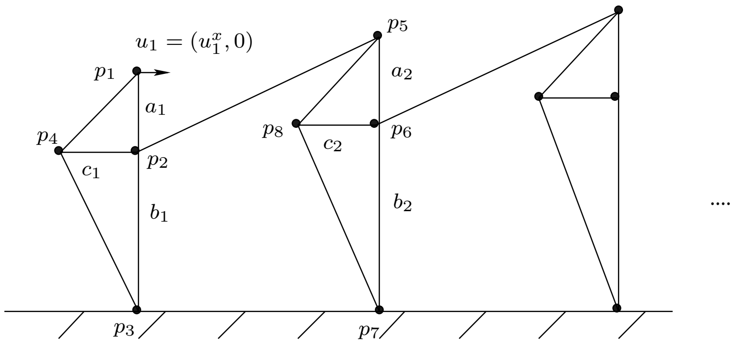

Figure 4.5: An infinite lever framework

Let u_i be the flex acting on the point p_i, then using the infinitesimal flex condition together with the fact that $u_1 = (u_1^x, 0), u_3 = u_7 = (0, 0)$ we have:

$$u_5 = (\frac{1}{1 + \frac{a_1}{b_1}} u_1^x, 0).$$

Example 4.3.2. Let $\mathcal{G}$ be the infinite, base fixed, pin-bar framework suggested by Figure 4.6. In this case, the vertices p_2, p_5, etc. are *pin joints* and the edges are rigid bars. This allows us to draw simple diagrams with equivalent rigidity properties. From Example 4.3.1 we can choose the lengths of the bars to achieve specific types of infinitesimal flexes. For example

$$(u_{3n})_{n=1}^{\infty} = \begin{cases} (1,0) & \text{if} \quad n = 1, \\ \\ (\prod_{k=1}^{n-1}(\dfrac{1}{1+\frac{a_k}{b_k}}), 0) & \text{if} \quad n \geq 2 \end{cases}.$$

is an infinitesimal flex of $\mathcal{G}$. With the choice

$$a_k = \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}} \text{ and } b_k = 1 - a_k,$$

we have

$$\begin{aligned} u_{3n}^{x} &= \prod_{k=1}^{n-1}(\frac{1}{1+\frac{a_k}{b_k}}) \\ &= \prod_{k=1}^{n-1}(\frac{1}{1+\frac{a_k}{1-a_k}}) \\ &= \prod_{k=1}^{n-1}(1-a_k) \\ &= \prod_{k=1}^{n-1}(\frac{\sqrt{k}}{\sqrt{k+1}}) \\ &= \frac{1}{\sqrt{2}}.\frac{\sqrt{2}}{\sqrt{3}}.\dots\frac{\sqrt{n-2}}{\sqrt{n-1}}.\frac{\sqrt{n-1}}{\sqrt{n}} \\ &= \frac{1}{\sqrt{n}} \end{aligned}$$

and therefore $(u_{3n})_{n=1}^{\infty} = ((\frac{1}{\sqrt{n}}, 0))_{n=1}^{\infty}$. In this way we obtain an infinitesimal flex of $\mathcal{G}$ that is vanishing but not square summable.

In the same way, if we choose

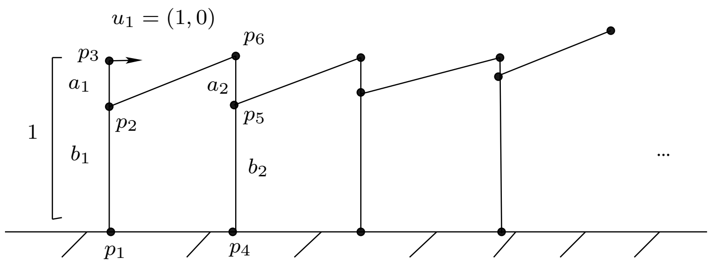

Figure 4.6: An infinite pin-bar framework

$$a_k = \frac{\sqrt{(k+1)^{\frac{5}{6}}} - \sqrt{k^{\frac{5}{6}}}}{\sqrt{(k+1)^{\frac{5}{6}}}} \text{ and } b_k = 1 - a_k,$$

then:

$$(u_{3n})_{n=1}^{\infty} = ((\frac{1}{\sqrt{n^{\frac{5}{6}}}}, 0))_{n=1}^{\infty}.$$

It follows that we obtain in this way an infinitesimal flex of $\mathcal{C}$ that is in ℓ^3 but not square summable.

4.4 Infinite Frameworks and $R(G, p)$

As with finite frameworks one can define the *rigidity matrix* of the infinite framework $\mathcal{G} = (G, P)$ in $\mathbb{R}^2$. This is the matrix $R(G, p)$ with rows indexed by the framework edges and columns labelled by the vertices but with multiplicity two, namely the labels $v_1^x, v_1^y, v_2^x, v_2^y, \ldots$ and with entries $x_i - x_j, x_j - x_i, y_i - y_j, y_j - y_i$ occurring in the row with label $e = (v_i, v_j)$ with the respective column labels $v_i^x, v_i^y, v_j^x, v_j^y$ and with zero entries elsewhere.

Note that $R(G,p)$ defines a linear transformation from $\mathcal{H}_v(\mathcal{G}) = \prod_V \mathbb{R}^2$ to $\mathcal{H}_e(\mathcal{G}) = \prod_E \mathbb{R}$ and it follows that a vector u in $\mathcal{H}_v(\mathcal{G})$ is an infinitesimal flex if and only if $R(G,p)u = 0$. The rigidity matrix for a framework in $\mathbb{R}^d$ is similarly defined.

In the following definition we note the form of a *self stress* for an infinite bar-joint framework. However, we do not need to develop this "dual perspective" in this thesis.

Definition 4.4.1. A *self-stress* of a finite or infinite framework $(\mathcal{G}) = (G, P)$ is a vector

$$w = (w_e) \in \mathcal{H}_e(\mathcal{G}) = \prod_E \mathbb{R}$$

such that w lies in the nullspace of the transpose matrix $R(G,p)^T$.

Theorem 4.4.2 ([54]). *Let $\mathcal{G} = (G,p)$ be a distance regular framework in $\mathbb{R}^2$ such that the degrees of the vertices are uniformly bounded, Then its rigidity matrix determines a bounded Hilbert space transformation R.*

While the proof below is lengthy it is essentially elementary, being repeated use of the Cauchy-Schwarz inequality.

Proof. Let $f \in \prod \mathbb{R}^2$ then,

$$\begin{aligned}
\|R(G,p)f\|_2^2 &= \| \sum_{(i,j)\in E} (x_i - x_j) f_i^x e_i^x + (y_i - y_j) f_i^y e_i^y + (x_j - x_i) f_j^x e_j^x + (y_j - y_i) f_j^y e_j^y \|^2 \\
&= \sum_{(i,j)\in E} |(x_i - x_j) f_i^x + (y_i - y_j) f_i^y + (x_j - x_i) f_j^x + (y_j - y_i) f_j^y|^2 \\
&\leq \sum_{(i,j)\in E} (\, |x_i - x_j||f_i^x| + |y_i - y_j||f_i^y| + |x_j - x_i||f_j^x| + |y_j - y_i||f_j^y| \,)^2.
\end{aligned}$$

Since the framework is distance regular:

$$\begin{aligned}
&\leq a^2 \sum_{(i,j)\in E} (\, |f_i^x| + |f_i^y| + |f_j^x| + |f_j^y| \,)^2 \\
&= a^2 \sum_{(i,j)\in E} (\, |f_i^x|^2 + |f_i^y|^2 + |f_j^x|^2 + |f_j^y|^2 \\
&+ 2|f_i^x||f_i^y| + 2|f_j^x||f_j^y| + 2|f_i^x||f_j^x| \\
&+ 2|f_i^y||f_j^x| + 2|f_i^x||f_j^y| + 2|f_i^y||f_j^y| \,)
\end{aligned}$$

and since the degrees of the framework vertices are uniformly bounded:

$$\begin{aligned}
&\leq a^2 [\, 4b\|f\|_2^2 + 2 \sum_{(i,j)\in E} |f_i^x||f_i^y| + 2 \sum_{(i,j)\in E} |f_j^x||f_j^y| \\
&+ 2 \sum_{(i,j)\in E} |f_i^x||f_j^x| + 2 \sum_{(i,j)\in E} |f_i^y||f_j^x| \\
&+ 2 \sum_{(i,j)\in E} |f_i^x||f_j^y| + 2 \sum_{(i,j)\in E} |f_i^y||f_j^y| \,]
\end{aligned}$$

$$
\begin{aligned}
&\leq 2a^2[\,2b\|f\|_2^2 + \sum_{(i,j)\in E} |f_i^x||f_i^y| + \sum_{(i,j)\in E} |f_j^x||f_j^y| \\
&+ \sum_{i\in V}(\sum_{j:(i,j)\in E} |f_i^x||f_j^x|\,) + \sum_{i\in V}(\sum_{j:(i,j)\in E} |f_i^y||f_j^x|\,) \\
&+ \sum_{i\in V}(\sum_{j:(i,j)\in E} |f_i^x||f_j^y|\,) + \sum_{i\in V}(\sum_{j:(i,j)\in E} |f_i^y||f_j^y|\,)\,]
\end{aligned}
$$

$$
\begin{aligned}
&= 2a^2[\,2b\|f\|_2^2 + \sum_{(i,j)\in E} |f_i^x||f_i^y| + \sum_{(i,j)\in E} |f_j^x||f_j^y| \\
&+ \sum_{i\in V} |f_i^x|\,(\sum_{j:(i,j)\in E} |f_j^x|\,) + \sum_{i\in V} |f_i^y|\,(\sum_{j:(i,j)\in E} |f_j^x|\,) \\
&+ \sum_{i\in V} |f_i^x|\,(\sum_{j:(i,j)\in E} |f_j^y|\,) + \sum_{i\in V} |f_i^y|\,(\sum_{j:(i,j)\in E} |f_j^y|\,)\,]
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2a^2[\,2b\|f\|_2^2 + \sum_{(i,j)\in E} |f_i^x||f_i^y| + \sum_{(i,j)\in E} |f_j^x||f_j^y| \\
&+ \sum_{i\in V} |f_i^x||(f_i^x)^*| + \sum_{i\in V} |f_i^y||(f_i^x)^*| \\
&+ \sum_{i\in V} |f_i^x||(f_i^y)^*| + \sum_{i\in V} |f_i^y||(f_i^y)^*|\,].
\end{aligned}
$$

Using the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&\leq 2a^2\{\ 2b\|f\|_2^2 \\
&+ [\sum_{(i,j)\in E} |f_i^x|^2\ ]^{\frac{1}{2}} [\sum_{(i,j)\in E} |f_i^y|^2\ ]^{\frac{1}{2}} \\
&+ [\sum_{(i,j)\in E} |f_j^x|^2\ ]^{\frac{1}{2}} [\sum_{(i,j)\in E} |f_j^y|^2\ ]^{\frac{1}{2}} \\
&+ [\sum_{i\in V} |f_i^x|^2\ ]^{\frac{1}{2}} [\sum_{i\in V} |(f_i^x)^*|^2\ ]^{\frac{1}{2}} \\
&+ [\sum_{i\in V} |f_i^y|^2\ ]^{\frac{1}{2}} [\sum_{i\in V} |(f_i^x)^*|^2\ ]^{\frac{1}{2}} \\
&+ [\sum_{i\in V} |f_i^x|^2\ ]^{\frac{1}{2}} [\sum_{i\in V} |(f_i^y)^*|^2\ ]^{\frac{1}{2}} \\
&+ [\sum_{i\in V} |f_i^y|^2\ ]^{\frac{1}{2}} [\sum_{i\in V} |(f_i^y)^*|^2\ ]^{\frac{1}{2}}\ \}
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2a^2(2b\|f\|_2^2 + 6b\|f\|_2^2) \\
&= 16a^2b\|f\|_2^2.
\end{aligned}
$$

Where $a = \max\{|e| : e \in E\}$ is the maximum of edge lengths, $b = \max\{\deg v : v \in V\}$ is the maximum degree of the vertices and $f_i^* = \max_{(i,j)\in E}\{f_i\}$. $\square$

The following examples illustrate how an infinite framework that does not satisfy one of the conditions above results in an unbounded rigidity matrix.

Example 4.4.3. Let $\mathcal{G} = (G, p)$ be the infinite framework (Figure 4.7) with edges:

$$e_1 = [p_1, p_2] = [(0,0),(0,1)], e_2 = [p_1, p_3] = [(0,0),(1,0)],$$

$$e_3 = [p_3, p_4] = [(1,0),(1,2)], e_4 = [p_4, p_2] = [(1,2),(0,1)],$$

$$e_5 = [p_3, p_5] = [(1,0),(2,0)], e_6 = [p_5, p_6] = [(2,0),(2,3)], \ldots$$

Here the degrees of the vertices are uniformly bounded but the framework is not distance regular.

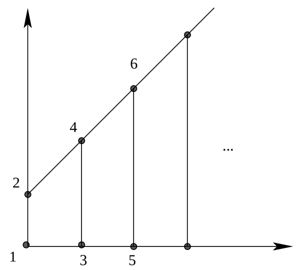

Figure 4.7: A framework that is not distance regular

The rigidity matrix for this framework is

$$R(G,p) = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & \ldots \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & \\ \vdots & \vdots & & & \ddots & \ddots & & & \end{bmatrix}$$

Let f be the vector $f = (0,0,0,1,0,0,0,\frac{1}{2},0,0,0,\frac{1}{3},...)$ in $\prod \mathbb{R}^2$. Then

$$\|R(G,p)f\|_2^2 = \|(1,0,1,-\frac{1}{2},0,1,-\frac{1}{6},...)\|_2^2 > \sum_1^\infty 1$$

which diverges.

Example 4.4.4. Let $\mathcal{G} = (G, p)$ be the infinite framework (Figure 4.8) such that

$$p_1 = (0,0) \text{ and } p_n = (\cos \tfrac{\pi}{2^{n-1}}, \sin \tfrac{\pi}{2^{n-1}}), n \geq 2,$$

together with framework edges $e_n = [p_1, p_{n+1}]$, $n \in \mathbb{N}$. This is a distance regular framework but the degrees of the vertices are not uniformly bounded.

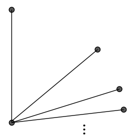

Figure 4.8: A distance regular framework where the degrees of the vertices are not uniformly bounded

$$R(G,p) = \begin{bmatrix} -\cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} & \sin\frac{\pi}{2} & 0 & 0 & 0 & 0 & \dots \\ -\cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 & 0 & \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 & 0 & \dots \\ -\cos\frac{\pi}{8} & -\sin\frac{\pi}{8} & 0 & 0 & 0 & 0 & \cos\frac{\pi}{8} & \sin\frac{\pi}{8} & \dots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & & & \end{bmatrix}$$

Note that the first column in $R(G, p)$ is not square summable and that is a necessary condition for it to be bounded.

In the following theorem, we give a sufficient condition for the boundedness of the rigidity matrix of a countably infinite tree framework in $\mathbb{R}^2$. Note that the degrees of the framework vertices need not be uniformly bounded.

Theorem 4.4.5. *Let $\mathfrak{T} = (V, E)$ be a tree graph in $\mathbb{R}^2$ with vertex set*

$$V = \{v_\phi, v_{\underline{n}}, \underline{n} \in S\}$$

where

$$S = \{\underline{n} : \underline{n} = (n_1, n_2, ..., n_d),\ n_i \in [1, N_i],\ N_i \in \mathbb{N} \cup \{\infty\},\ d = 1, 2, ...\}$$

and with edge set

$$E = \{e_{\underline{n}, n_{d+1}},\ \underline{n} \in S\}\ ,\ e_{\underline{n}, n_{d+1}} = [p_{\underline{n}}, p_{\underline{n}, n_{d+1}}] = [p_{\underline{n}}, p_{\underline{N}}].$$

Then $R(\mathfrak{T}, p)$ defines a bounded operator if there exists $M > 0$ such that

$$(\textstyle\sum_{\underline{n} \in S} |e_{\underline{n}, n_{d+1}}|^2)^{\frac{1}{2}} \leq M \textit{ for all } p_{\underline{n}}.$$

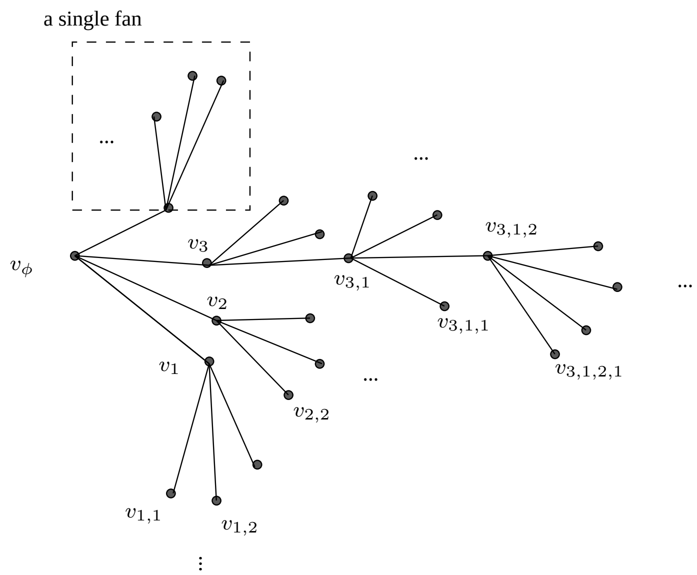

Figure 4.9: An infinite tree

Proof. Each row in $R(\mathfrak{T}, p)$ has four entries: the first two correspond to a single "fan" 's base point and the other two correspond to one of the fan's ends (Figure 4.9). We have

$$\begin{aligned} R(\mathfrak{T},p)_e = [\, 0 \;\; \ldots \;\; 0 \;\; & x_{\underline{n}} - x_{\underline{n},n_{d+1}} \quad y_{\underline{n}} - y_{\underline{n},n_{d+1}} \\ & 0 \;\; \ldots \;\; 0 \quad x_{\underline{n},n_{d+1}} - x_{\underline{n}} \quad y_{\underline{n},n_{d+1}} - y_{\underline{n}} \;\; 0 \;\; \ldots \;\; 0 \,] \end{aligned}$$

and therefore $R(\mathfrak{T}, p)$ can be written as the sum of two block diagonal infinite matrices:

$$R(\mathfrak{T}, p) = A + B.$$

The matrix A is

$$A = \begin{bmatrix} A_1 = \begin{bmatrix} * & * \\ * & * \\ \vdots & \vdots \end{bmatrix} & & \\ & A_2 = \begin{bmatrix} * & * \\ * & * \\ \vdots & \vdots \end{bmatrix} & \\ & & \ddots \\ & & \end{bmatrix}$$

and A is the direct sum of the sub-matrices A_i where each A_i is a two column matrix such that each row corresponds to the first two entries of $R(\mathfrak{T}, p)_e$.

We have $A = \bigoplus_{i\in\mathbb{N}} A_i$ and so the operator norm of A satisfies

$$
\begin{aligned}
\|A\|_{\mathrm{op}} &= \sup_{i\in\mathbb{N}}\{\ \|A_i\|_{\mathrm{op}}\ \} \\
&\leq \sup_{i\in\mathbb{N}}\{\ \|A_i\|_{\mathrm{HS}}\ \} \\
&\leq \sup_{i\in\mathbb{N}}\{(\sum_{\underline{n}\in S} |e^x_{\underline{n},n_{d+1}}|^2\,)^{\frac{1}{2}} + (\sum_{\underline{n}\in S} |e^y_{\underline{n},n_{d+1}}|^2\,)^{\frac{1}{2}}\} \\
&\leq 2M.
\end{aligned}
$$

On the other hand,

$$
B = \begin{bmatrix}
B_1 = \begin{bmatrix} 0 & 0 & * & * & & & \\ 0 & 0 & & & * & * & \\ \vdots & \vdots & & & & & \ddots \end{bmatrix} & & \\
& B_2 = \begin{bmatrix} * & * & & & \\ & & * & * & \\ & & & & \ddots \end{bmatrix} & \\
& & \ddots
\end{bmatrix}
$$

can be written as the direct sum of the sub-matrices B_i where each B_i is itself a block diagonal matrix such that the blocks $B_{i,j}, j \in \mathbb{N}$ are 1×2 matrices and their entries are the last two non-zero entries from $R(\mathfrak{T}, p)_e$. Therefore,

$$B = \bigoplus_{i \in \mathbb{N}} B_i \text{ and } B_i = \bigoplus_{j \in \mathbb{N}} B_{i,j}.$$

Also,

$$\begin{aligned}
\|B\|_{\text{op}} &= \sup_{i \in \mathbb{N}} \{ \|B_i\|_{\text{op}} \} \\
&\leq \sup_{i \in \mathbb{N}} \{ \|B_{i,j}\|_{\text{op}}, j \geq 1 \} \\
&\leq \sup_{i \in \mathbb{N}} \{ \|B_{i,j}\|_{\text{HS}} j \geq 1 \} \\
&\leq \sup_{i \in \mathbb{N}} \{ (\sum_{\underline{n} \in S} |-e^x_{\underline{n}, n_{d+1}}|^2)^{\frac{1}{2}} + (\sum_{\underline{n} \in S} |-e^y_{\underline{n}, n_{d+1}}|^2)^{\frac{1}{2}} \} \\
&\leq 2M.
\end{aligned}$$

From the triangle inequality:

$$\|R(\mathfrak{T}, p)\|_{\text{op}} = \|A + B\|_{\text{op}} \leq \|A\|_{\text{op}} + \|B\|_{\text{op}} \leq 4M.$$

Thus $R(\mathfrak{T}, p)$ is bounded as required. □

4.5 Factorization of $R(G, p)$

Another area where rigidity theory is becoming more involved is in the study of robots and formation control. The robots referred to here are "agents" and a formation is a group of these agents moving in dimensions two or three. For more on this subject one can refer to [27]. In their papers [71],[73] Zelazo et al obtained a factorization of the rigidity matrix for a finite framework. Here we give a precise proof of this factorization of $R(G,p)$ following an example by Owen and Power [54] which also applies for infinite frameworks. We use this factorization to give another proof for the rigidity matrix boundedness (Theorem 4.4.2).

To develop the complete setting, let $G=(V,E)$ be a simple, oriented, finite or countably infinite, locally finite graph. For each vertex v_j, define a *local incidence matrix* $E(v_j) \in \mathbb{R}^{|E|\times|V|}$ such that the row entry corresponding to edge e_k and vertex column v_i is equal to 1 if $i=j$ and v_j is the source vertex of e_k and it is equal to -1 if $i=j$ and v_j is the range vertex of e_k and zeros otherwise. Note that zero rows correspond to edges not adjacent to v_j. Let p be any position vector of G in $\mathbb{R}^2$ viewed in the finite case as the $|V|\times 2$ matrix

$$p = \begin{bmatrix} p_1^x & p_1^y \\ p_2^x & p_2^y \\ \vdots & \vdots \\ p_{|V|}^x & p_{|V|}^y \end{bmatrix}$$

or as a 2-column infinite matrix when G is infinite.

Proposition 4.5.1. *Let p be any position vector of G in $\mathbb{R}^2$. Then*

$$R(G,p) = E(v)D(p)$$

where $E(v) \in \mathbb{R}^{|E|\times|V|^2}$, $E(v) = \begin{bmatrix} E(v_1) & E(v_2) & \dots \end{bmatrix}$ *and* $D(p) \in \mathbb{R}^{|V|^2\times 2|V|}$ *is the block diagonal matrix* $diag(p, p, \dots)$.

It is straightforward to check that the above product is indeed equal to the rigidity matrix by understanding how the matrices $E(v)$ and $D(p)$ are indexed. For example, in the finite case, the row of $E(v)$ corresponding to edge $e_k = [v_i, v_j]$ would look like

$$\begin{array}{lccccc} & & v_i & & v_j & \\ & \dots & \dots v_i \dots v_j \dots & \dots & \dots v_i \dots v_j \dots & \dots \\ [E(v)]_{e_k} = & (\quad \dots & 1 \quad -1 & \dots & -1 \quad 1 & \dots \quad) \end{array}$$

Note that the entries corresponding to v_i and v_j have opposite signs every time. Therefore, $[E(v)D(p)]_{e_k}$ is

$$\begin{array}{ccccc} & v_i & & v_j & \\ \dots & \dots v_i \dots v_j \dots & \dots & \dots v_i \dots v_j \dots & \dots \\ (\quad \dots & p_i^x - p_j^x \quad p_i^y - p_j^y & \dots & p_j^x - p_i^x \quad p_j^y - p_i^y & \dots \quad) \end{array}$$

which is equal to the e_k-th row in $R(G, p)$.

Example 4.5.2. Let G be a simple, oriented graph of a triangle (Figure 4.10) and let

$$p = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \in \mathbb{R}^{|V| \times 2}$$

be a position vector of G.

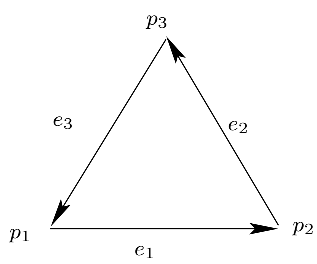

Figure 4.10: A simple oriented graph of a triangle

The local incidence matrices

$$E(v_1) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \; E(v_2) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } E(v_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

From the above discussion $R(G, p)$ is equal to the product

$$\begin{bmatrix} 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \end{bmatrix}.$$

And

$$R(G,p) = \begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

Another simple factorization of $R(G,p)$ was observed by Owen and Power [54] in the following example.

Example 4.5.3. Let $(\mathbb{N}, p)$ denote a semi-infinite framework in $\mathbb{R}^2$ whose abstract graph is a tree with a single branch and $p = (p_i)$, $p_i = (x_i, y_i)$,

$i = 1, 2, \dots$. Writing $x_i - x_j = x_{ij}$ and $y_i - y_j = y_{ij}$ the rigidity matrix with respect to the natural ordered basis takes the form

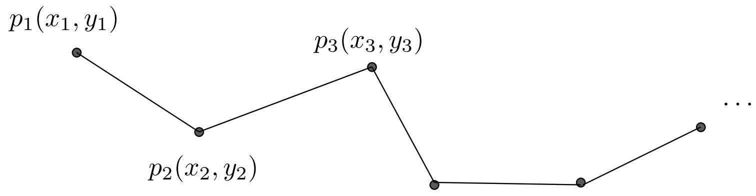

Figure 4.11: $(\mathbb{N}, p)$

$$R(\mathbb{N}, p) = \begin{bmatrix} x_{12} & y_{12} & x_{21} & y_{21} & 0 & \dots & & \\ 0 & 0 & x_{23} & y_{23} & x_{32} & y_{32} & 0 & \dots \\ 0 & 0 & 0 & 0 & x_{34} & \dots & & \\ \vdots & & & & \ddots & & & \end{bmatrix}.$$

With respect to the coordinate decomposition $\mathcal{H}_v = \mathcal{H}_x \oplus \mathcal{H}_y$ the rigidity matrix can be written as

$$R(G, p) = \begin{bmatrix} R_x & R_y \end{bmatrix} = \begin{bmatrix} D_x & D_y \end{bmatrix} \begin{bmatrix} T & \\ & T \end{bmatrix}$$

such that $R_x = D_x T$, $R_y = D_y T$ and D_x, D_y are the diagonal matrices

$$D_x = \begin{bmatrix} x_{12} & 0 & 0 & \dots & \\ 0 & x_{23} & 0 & \dots & \\ 0 & 0 & x_{34} & 0 & \dots \\ \vdots & & & \ddots & \end{bmatrix}, \; D_y = \begin{bmatrix} y_{12} & 0 & 0 & \dots & \\ 0 & y_{23} & 0 & \dots & \\ 0 & 0 & y_{34} & 0 & \dots \\ \vdots & & & \ddots & \end{bmatrix}$$

and

$$T = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots \\ 0 & 1 & -1 & \dots & \\ 0 & 0 & 1 & -1 & \dots \\ \vdots & & & \ddots & \end{bmatrix}.$$

In the light of the example above, one can deduce a similar form for any rigidity matrix $R(G, p)$, for a finite or countably infinite graph, with respect to $\mathcal{H}_v = \mathcal{H}_x \oplus \mathcal{H}_y$.

Proposition 4.5.4. *Let (G, p) be a finite or countably infinite framework in $\mathbb{R}^2$. Then with respect to the coordinate decomposition $\mathcal{H}_v = \mathcal{H}_x \oplus \mathcal{H}_y$,*

$$R(G, p) = \begin{bmatrix} R_x & R_y \end{bmatrix} = \begin{bmatrix} D_x & D_y \end{bmatrix} \begin{bmatrix} E(G) & \\ & E(G) \end{bmatrix}.$$

Where $E(G)$ is the $|E| \times |V|$ incidence matrix determined by an orientation of G, and D_x, D_y are the $|E| \times |E|$ matrices determined by the same orientation. Note that, as before, the e, v-th entry in $E(G)$ is 1 if v is a source of e and -1 if it is the range of e.

Checking this fact, amounts to giving a short proof of Proposition 2 in Zelazo et al [71]. Indeed with this factorization it follows that $R(G,p)^* R(G,p)$ has the form

$$\begin{bmatrix} E(G)^T & \\ & E(G)^T \end{bmatrix} \begin{bmatrix} D_x^2 & D_y D_x \\ D_x D_y & D_y^2 \end{bmatrix} \begin{bmatrix} E(G) & \\ & E(G) \end{bmatrix}.$$

Remark 4.5.5. One way of using $R(G,p)$ in factorized form, is to give a short proof of Theorem 4.4.2.

Proof of Theorem 4.4.2. We will prove the boundedness of $R(G,p)$ by proving that both factors in the formula of Proposition 4.5.4 are bounded as follows:

Both of D_x and D_y are $|E| \times |E|$ diagonal matrices, and

$$\begin{aligned} \|D_x\| &= \sup_{e=[v_i,v_j]} |x_e| \\ &= \sup_{e=[v_i,v_j]} |p^x_{v_i} - p^x_{v_j}| \\ &\leq \sup_{e=[v_i,v_j]} \|p_{v_i} - p_{v_j}\|_2 \\ &\leq a. \end{aligned}$$

Similarly, $\|D_y\| \leq a$.

To prove that $\begin{bmatrix} E(G) & \\ & E(G) \end{bmatrix}$ is bounded, note that

$$\|\begin{bmatrix} E(G) & \\ & E(G) \end{bmatrix}\|_2^2 \leq \|E(G)\|_2^2$$

and

$$\begin{aligned} \|E(G)\begin{bmatrix} f_1 \\ f_2 \\ \vdots \end{bmatrix}\|_2^2 &= \|\sum_{i,j:(v_i,v_j)\in E}(f_i - f_j)\|^2 \\ &\leq \sum_{i,j:(v_i,v_j)\in E}|f_i - f_j|^2 \\ &\leq \sum_{i,j:(v_i,v_j)\in E}|f_i|^2 + \sum_{i,j:(v_i,v_j)\in E}|f_j|^2 + 2\sum_{i,j:(v_i,v_j)\in E}|f_i||f_j| \\ &\leq 2b\|f\|_2^2 + 2\sum_{i,j:(v_i,v_j)\in V\times V}|f_i||f_j| \\ &\leq 4b\|f\|_2^2. \end{aligned}$$

Where a is the maximum of edge lengths and $b = \max\{\deg v_i : v_i \in V\}$ is the maximum degree of the vertices.

From all the above, it follows that $\begin{bmatrix} E(G) & \\ & E(G) \end{bmatrix}$ is bounded, and therefore $R(G, p)$ is bounded. $\square$

Chapter 5

Crystal Frameworks

In this chapter we begin an analysis of infinite bar-joint frameworks with high symmetry, particularly those with translational periodic symmetry known as "crystal frameworks". In the terminology of chemists, a crystal is built up by arranging atoms and groups of atoms in regular patterns and the basic arrangement of atoms that describes the crystal is identified as the unit cell. Formally, a periodic framework can be identified in terms of "quotient graphs" [65], or "gain graphs" [60], for example. Here we follow the mathematical identification introduced by Owen and Power [54] and Power [58] where a crystal framework is determined by a finite "motif" and discrete "translation group".

After formal definitions we build up the understanding of the infinitesimal flexibility of crystal frameworks through a range of contrasting examples. In particular, we see that the high symmetry of crystal frameworks gives rise to special classes of infinitesimal flexes. We start by identifying some of the basic infinitesimal flexes of various examples such as "local, band limited" before moving on to classes of infinitesimal flexes that are exclusive to crystal frameworks, mainly, strictly periodic and super cell periodic flexes.

The determination of general periodic flexes depends on the analysis of a matrix function for the framework. This can be used for the identification and analysis of the "RUM spectrum". Where material scientists mostly rely on laboratory experiments ([25]) or computer analysis ([33]) to determine the RUM spectrum, Power [58] introduced the matrix function associated with a crystal framework and developed a purely mathematical method to determine the RUM spectrum using techniques from functional analysis and operator theory. Here we build up on the theory of periodic flexes introduced in [58] and we prove some relations between the spaces of "supercell n-fold periodic" and "phase periodic" flexes. For further considerations of periodic rigidity and related results we refer the reader to Owen and Power [54],[52],[53], Power [56],[58],[57], Borcea and Streinu [11],[10], Ross [59], [60], Ross, Schulze and Whiteley [61] and Malestein and Theran [49].

5.1 Definitions and Examples

This section can be considered as a mathematical background for crystal frameworks. We state some of the definitions introduced in [54] and [58] before giving a variety of examples. As we move on towards more specific classes we see how these frameworks admit different flexing properties.

Definition 5.1.1. An *isometry* of $\mathbb{R}^3$ is a distance preserving map:

$$T : \mathbb{R}^3 \to \mathbb{R}^3 \text{ such that } \|Tx - Ty\| = \|x - y\|, \ \forall x, y \in \mathbb{R}^3.$$

Definition 5.1.2. A *full rank translation group* is a set of translation isometries $\{T_k : k \in \mathbb{Z}^3\}$ with $T_{k+l} = T_k + T_l$ for all k, l, $T_k \neq I$ if $k \neq 0$, and such that the three *period vectors*

$$a = T_{\gamma_1}0, \ \ b = T_{\gamma_2}0, \ \ c = T_{\gamma_3}0$$

associated with the generators $\gamma_1 = (1, 0, 0), \gamma_2 = (0, 1, 0), \gamma_3 = (0, 0, 1)$ of $\mathbb{Z}^3$ are not coplanar. Full rank translation groups in $\mathbb{R}^d$ are similarly defined.

Definition 5.1.3. A *crystal framework* $\mathcal{C} = (F_v, F_e, \mathcal{T})$ in $\mathbb{R}^d$, with *full rank translation group* $\mathcal{T} = \{T_k : k \in \mathbb{Z}^d\}$ and *motif* (F_v, F_e), is a countable bar-joint framework with framework points $p_{\kappa,k}$, for $1 \leq \kappa \leq t, k \in \mathbb{Z}^d$, such that

(i) F_v is a finite set of framework vertices, $\{p_{\kappa,0} : 1 \leq \kappa \leq t\}$ in $\mathbb{R}^d$, and F_e is a finite set of framework edges,

(ii) for each κ and k the point $p_{\kappa,k}$ is the translate $T_k p_{\kappa,0}$,

(iii) the set $\mathcal{C}_v$ of framework points is the disjoint union of the sets $T_k(F_v)$ for $k \in \mathbb{Z}^d$,

(iv) the set $\mathcal{C}_e$ of framework edges is the disjoint union of the sets $T_k(F_e)$ for $k \in \mathbb{Z}^d$.

A *unit cell* for $\mathcal{C}$ is defined by the period vectors. Multiple cells of a given unit cell associated with a translation group $\mathcal{T}$ give super cells associated with a subgroup $\mathcal{T}' \subseteq \mathcal{T}$.

Definition 5.1.4. A crystal framework $\mathcal{C}$ in $\mathbb{R}^d$ is said to be *in Maxwell counting equilibrium* if $d|F_v| = |F_e|$. If $d|F_v| < |F_e|$ then $\mathcal{C}$ is said to be *edge rich* while if $d|F_v| > |F_e|$ then $\mathcal{C}$ is said to be *edge sparse*.

If a choice of motif for a crystal framework is in Maxwell counting equilibrium, then every other motif choice within the same translation group will be in Maxwell counting equilibrium. It follows that a crystal framework in Maxwell counting equilibrium admits a square "matrix function" and we may compute the determinant that is used to form the "crystal polynomial" as we shall see in the following sections.

In the case of a crystal framework in $\mathbb{R}^d$ a velocity vector is a doubled-indexed sequence v of vectors $v_{\kappa,k}$ in $\mathbb{R}^d$ regarded as instantaneous velocities applied to the frameworks vertices $p_{\kappa,k}$. Let $\mathcal{H}_v(\mathcal{C})$ be the vector space of all velocity vectors:

$$\mathcal{H}_v(\mathcal{C}) = \prod_{\kappa,k} \mathbb{R}^d.$$

Definition 5.1.5. Let $\mathcal{C}$ be a crystal framework with framework vertices $p_{\kappa,k}$ as defined before. A real *infinitesimal flex* of $\mathcal{C}$ is a set of velocity vectors $u_{\kappa,k} \in \mathcal{H}_v(\mathcal{C})$, for each vertex, such that for each edge $e = [p_{\kappa,k}, p_{\tau,l}]$,

$$\langle p_{\kappa,k} - p_{\tau,l}, u_{\kappa,k} - u_{\tau,l} \rangle = 0.$$

The set of all infinitesimal flexes form a vector subspace $\mathcal{H}_{\mathrm{fl}}(\mathcal{C})$ of $\mathcal{H}_v(\mathcal{C})$. Note that each isometry of $\mathbb{R}^d$ gives rise to a one-dimensional vector subspace of $\mathcal{H}_{\mathrm{fl}}(\mathcal{C})$.

The rigidity matrix $R(\mathcal{C})$ of the crystal framework $\mathcal{C}$ is a real infinite matrix defined as in the finite framework case. It has rows labelled by the framework edges $e = [p_{\kappa,k}, p_{\tau,l}]$ and columns labelled by the framework point coordinate indices $(\kappa, x, k), (\kappa, y, k)$. The row for edge e takes the form

$$\begin{array}{c}
\begin{array}{cccccccccc}
 & & \kappa, x & \kappa, y & & & & \tau, x & \tau, y & \\
\end{array} \\
\left(\begin{array}{ccccccccccc}
\dots & 0 & p^x_{\kappa,k} - p^x_{\tau,l} & p^y_{\kappa,k} - p^y_{\tau,l} & 0 & \dots & 0 & p^x_{\tau,l} - p^x_{\kappa,k} & p^y_{\tau,l} - p^y_{\kappa,k} & 0 & \dots
\end{array}\right)
\end{array}$$

The definition of $R(\mathcal{C})$ for $d = 3, 4, \dots$ and also for general countably infinite bar-joint frameworks is essentially the same.

Let

$$\mathcal{H}_e(\mathcal{C}) = \prod_{e \in \mathcal{C}_e} \mathbb{R} = \prod_{e \in F_e, k \in \mathbb{Z}^d} \mathbb{R}$$

be the space of real sequences $w = (w_{e,k})_{e\in F_e, k\in\mathbb{Z}^d}$ labelled by the framework edges. Then $R(\mathcal{C})$ defines a linear transformation

$$R : \mathcal{H}_v(\mathcal{C}) \to \mathcal{H}_e(\mathcal{C}).$$

Each row of R has at most $2d$ non-zero entries and the image $R(u)$ is given by the well-defined matrix multiplication $R(\mathcal{C})(u)$.

The proof of the following Proposition is the same as in the case of finite frameworks.

Proposition 5.1.6. *The infinitesimal flexes of the crystal framework $\mathcal{C}$ are the velocity vectors in $\mathcal{H}_v(\mathcal{C})$ that lie in the null space of the linear transformation $R(\mathcal{C})$.*

The determination of the following flexes of a crystal framework introduced in [58] will be used later on for the identification of a bases for the space $\mathcal{H}_{\mathrm{fl}}(\mathcal{C})$ of all infinitesimal flexes.

Definition 5.1.7. Let u be a flex of the crystal framework $\mathcal{C}$, then u is said to be

i. *band limited* if u is supported by a set of framework vertices within a finite distance from a hyperplane,

ii. a *local infinitesimal flex* if $u_{\kappa,k} = 0$ for all but finitely many values of κ, k.

From the definition above, a local flex sequence has finite support. A band limited flex, for a planar framework, is supported by the framework vertices between two parallel lines.

In the following examples, we suggest a choice of motif and translation group for a variety of crystal frameworks. Although some frameworks can be derived from familiar ones by the addition of edges or vertices, we find that they have different infinitesimal flexing properties even with a small change in some cases.

The triangulated grid $\mathcal{C}_{\text{tri}}$. The framework $\mathcal{C}_{\text{tri}}$ presents the regular tiling of the plane by triangles. It is sequentially infinitesimally rigid in the sense that it can be viewed as an increasing sequence of infinitesimally rigid finite subframeworks. This is an example of an infinitesimally rigid, and hence rigid, crystal framework.

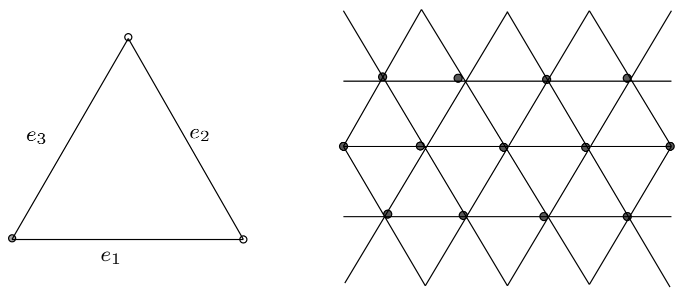

Figure 5.1: The triangulated grid $\mathcal{C}_{\text{tri}}$

The translation group for $\mathcal{C}_{\text{tri}}$:

$\mathcal{T} = \{T_k : k \in \mathbb{Z}^2\}$, $T_k(x,y) = (x,y) + k_1 a + k_2 b$, $a = (1,0)$, $b = (\frac{1}{2}, \frac{\sqrt{3}}{2})$.

A motif choice for the triangulated grid $\mathcal{C}_{\text{tri}}$ (Figure 5.1)		
Motif vertices F_v	Motif edges F_e	period vectors
$p_1 = (0,0)$	$e_1 = [p_1, p_{1,(1,0)} = (1,0)]$ $e_2 = [p_{1,(1,0)}, p_{1,(0,1)} = (\frac{1}{2}, \frac{\sqrt{3}}{2})]$ $e_3 = [p_{1,(0,1)}, p_{1,(1,0)}]$	$\underline{a} = (1,0)$ $\underline{b} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$
Infinitesimal flexibility: infinitesimally rigid		

We note that $2|F_v| = 2 < |F_e|$ and the framework is edge rich.

The alternating double triangles framework $\mathcal{C}_{2\text{tri}}$. This framework can be viewed as $\mathcal{C}_{\text{tri}}$ but with alternating corner connected double triangles. Here, the presence of a local flex creates a framework with a rigid crystal subframework which is infinitesimally flexible.

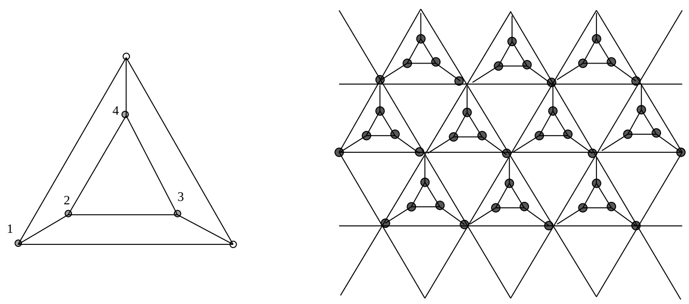

Figure 5.2: The alternating double triangles framework $\mathcal{C}_{2\text{tri}}$

A motif choice for the framework $\mathcal{C}_{2\mathrm{tri}}$ (Figure 5.2)		
Motif vertices F_v	Motif edges F_e	period vectors
$p_1 = (0,0)$ $p_2 = (\frac{1}{3}, \frac{\sqrt{3}}{6})$ $p_3 = (\frac{2}{3}, \frac{\sqrt{3}}{6})$ $p_4 = (\frac{1}{2}, \frac{\sqrt{3}}{3})$	$e_1 = [p_1, p_{1,(1,0)} = (1,0)]$ $e_2 = [p_{1,(1,0)}, p_{1,(0,1)} = (\frac{1}{2}, \frac{\sqrt{3}}{2})]$ $e_3 = [p_{1,(0,1)}, p_1]$ $e_4 = [p_2, p_3]$ $e_5 = [p_3, p_4]$ $e_6 = [p_4, p_2]$ $e_7 = [p_1, p_2]$ $e_8 = [p_{1,(1,0)}, p_3]$ $e_9 = [p_{1,(0,1)}, p_4]$	$\underline{a} = (1,0)$ $\underline{b} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$
Basic infinitesimal flexes: local		

We note that $2|F_v| = 8 < |F_e|$ and the framework is edge rich.

The basic grid $\mathcal{C}_{\mathbb{Z}^2}$. This framework presents the regular square tiling of the plane. There are no local infinitesimal flexes of the basic grid but this framework admits horizontal and vertical band limited infinitesimal flexes supported by the lines of vertices parallel to both axis.

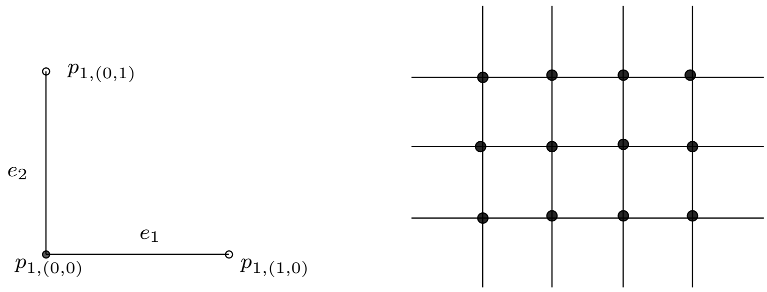

Figure 5.3: The basic grid $\mathcal{C}_{\mathbb{Z}^2}$

A motif choice for the basic grid $\mathcal{C}_{\mathbb{Z}^2}$ (Figure 5.3)		
Motif vertices F_v	Motif edges F_e	period vectors
$p_1 = (0,0)$	$e_1 = [p_{1,(0,0)}, p_{1,(1,0)} = (1,0)]$ $e_2 = [p_{1,(0,0)}, p_{1,(0,1)} = (0,1)]$	$\underline{a} = (1,0)$ $\underline{b} = (0,1)$
Infinitesimal flexibility: band limited		

We note that $2|F_v| = 2 = |F_e|$ and the framework is in Maxwell counting Equilibrium.

The squares framework $\mathcal{C}_{\mathbf{sq}}$. This framework can be obtained from the basic grid $\mathcal{C}_{\mathbb{Z}^2}$ by adding diagonals to alternative squares. It admits a one dimensional space of non-trivial infinitesimal flexes where the rigid squares alternately rotate with equal magnitude and opposite direction.

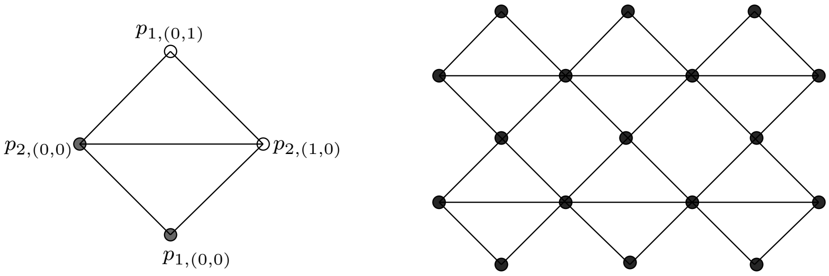

Figure 5.4: The squares framework $\mathcal{C}_{\text{sq}}$

A motif choice for the squares framework $\mathcal{C}_{\text{sq}}$ (Figure 5.4)		
Motif vertices F_v	Motif edges F_e	period vectors
$p_1 = (\frac{1}{2}, 0)$ $p_2 = (0, \frac{1}{2})$	$e_1 = [p_{1,(0,0)}, p_{2,(0,0)}]$ $e_2 = [p_{1,(0,0)}, p_{2,(1,0)}]$ $e_3 = [p_{2,(0,0)}, p_{1,(0,1)}]$ $e_4 = [p_{1,(0,1)}, p_{2,(1,0)}]$ $e_5 = [p_{2,(0,0)}, p_{2,(1,0)}]$	$\underline{a} = (1, 0)$ $\underline{b} = (0, 1)$
Basic infinitesimal flexes: alternating rotation infinitesimal flex (full support)		

Let $F_v = \{p_1, p_2\}$ with translation group $\mathcal{T} = \{T_k : k \in \mathbb{Z}^2\}$

$$\text{such that } p_{\kappa,k} = T_k p_{\kappa,0}, \text{ for } \kappa \in \{1, 2\}, k \in \mathbb{Z}^2$$

$$T_{k=(n,m)}(x, y) = (x, y) + n\underline{a} + m\underline{b},\ \underline{a} = (1, 0),\ \underline{b} = (0, 1).$$

for example when $k = (1, 0)$:

$$p_{1,(1,0)} = T_{(1,0)}p_1 = (\tfrac{3}{2}, 0)$$
$$p_{2,(1,0)} = T_{(1,0)}p_2 = (1, \tfrac{1}{2})$$

and for $k = (0,1)$:

$$p_{1,(0,1)} = T_{(0,1)}p_1 = (\tfrac{1}{2}, 1)$$
$$p_{2,(0,1)} = T_{(0,1)}p_2 = (0, \tfrac{3}{2})$$

this defines the natural periodic labelling of framework edges:

$$e_{j,k} = T_k e_j, \quad j \in \{1,2,3,4,5\}, k \in \mathbb{Z}^2.$$

We note that $2|F_v| = 4 < 5 = |F_e|$ and the framework is edge rich.

The double-squares framework $\mathcal{C}_{\mathbf{2sq}}$. This framework can be derived from the basic grid by augmenting inner squares and connecting their corners with those of the basic grid. $\mathcal{C}_{2\mathrm{sq}}$ admits local flexes where the inner squares can infinitesimally rotate about their centres.

We remark that the earlier observation that the finite double square framework is continuously rigid implies that $\mathcal{C}_{2\mathrm{sq}}$ is sequentially continuously rigid and therefore continuously rigid. On the other hand, the frameworks $\mathcal{C}_{\mathbb{Z}^2}$ and $\mathcal{C}_{\mathrm{sq}}$ are not continuously rigid.

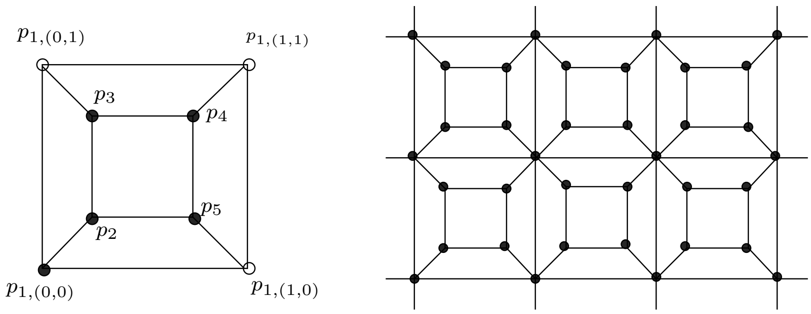

Figure 5.5: The double-squares framework $\mathcal{C}_{2sq}$

A motif choice for the double squares framework $\mathcal{C}_{2sq}$ (Figure 5.5)		
Motif vertices F_v	Motif edges F_e	period vectors
$p_1 = (0,0)$ $p_2 = (\frac{1}{4}, \frac{1}{4})$ $p_3 = (\frac{1}{4}, \frac{3}{4})$ $p_4 = (\frac{3}{4}, \frac{3}{4})$ $p_5 = (\frac{3}{4}, \frac{1}{4})$	$e_1 = [p_{1,(0,0)}, p_{1,(0,1)}]$ $e_2 = [p_{1,(0,0)}, p_{1,(1,0)}]$ $e_3 = [p_{2,(0,0)}, p_{3,(0,0)}]$ $e_4 = [p_{3,(0,0)}, p_{4,(0,0)}]$ $e_5 = [p_{4,(0,0)}, p_{5,(0,0)}]$ $e_6 = [p_{2,(0,0)}, p_{5,(0,0)}]$ $e_7 = [p_{1,(0,0)}, p_{2,(0,0)}]$ $e_8 = [p_{3,(0,0)}, p_{1,(0,1)}]$ $e_9 = [p_{4,(0,0)}, p_{1,(1,1)}]$ $e_{10} = [p_{5,(0,0)}, p_{1,(1,0)}]$	$\underline{a} = (1,0)$ $\underline{b} = (0,1)$
Infinitesimal flexibility: local, band limited		

We note that $2|F_v| = 10 = |F_e|$ and the framework is in Maxwell counting Equilibrium.

The 5-regular grid framework $\mathcal{C}_{5\mathbf{grid}}$. In this framework, the basic grid $\mathcal{C}_{\mathbb{Z}^2}$ is augmented with diagonal lines creating "strip" subframeworks.

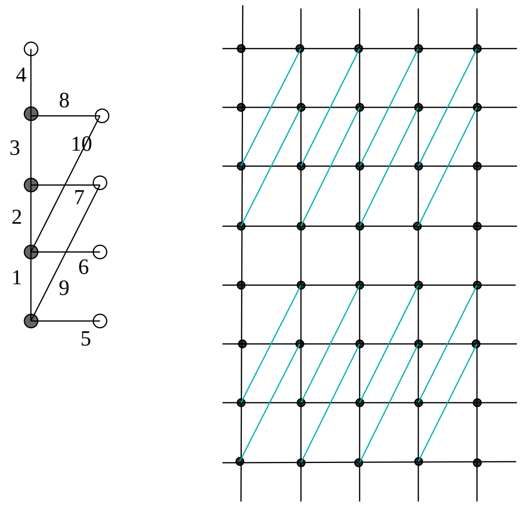

Figure 5.6: The 5-regular grid framework $\mathcal{C}_{5\mathrm{grid}}$

A motif choice for the 5-regular grid framework $\mathcal{C}_{5\text{grid}}$ (Figure 5.6)		
Motif vertices F_v	Motif edges F_e	period vectors
$p_1 = (0,0)$ $p_2 = (0,1)$ $p_3 = (0,2)$ $p_4 = (0,3)$	$e_1 = [p_{1,(0,0)}, p_{2,(0,0)}]$ $e_2 = [p_{2,(0,0)}, p_{3,(0,0)}]$ $e_3 = [p_{3,(0,0)}, p_{4,(0,0)}]$ $e_4 = [p_{4,(0,0)}, p_{1,(0,1)}]$ $e_5 = [p_{1,(0,0)}, p_{1,(1,0)}]$ $e_6 = [p_{2,(0,0)}, p_{2,(1,0)}]$ $e_7 = [p_{3,(0,0)}, p_{3,(1,0)}]$ $e_8 = [p_{4,(0,0)}, p_{4,(1,0)}]$ $e_9 = [p_{1,(0,0)}, p_{3,(1,0)}]$ $e_{10} = [p_{2,(0,0)}, p_{4,(1,0)}]$	$\underline{a} = (1,0)$ $\underline{b} = (0,4)$
Infinitesimal flexibility: band limited		

We note that $2|F_v| = 8$ and $|F_e| = 10$ and the framework is edge rich.

The augmented grid+strip framework $\mathcal{C}^+_{\mathbb{Z}^2}$. In this example the basic grid is augmented with countably many copies of the strip framework. This creates an example with "geometric growth flexes" that we will explore in more detail when we identify a "free basis" for the space of all infinitesimal flexes.

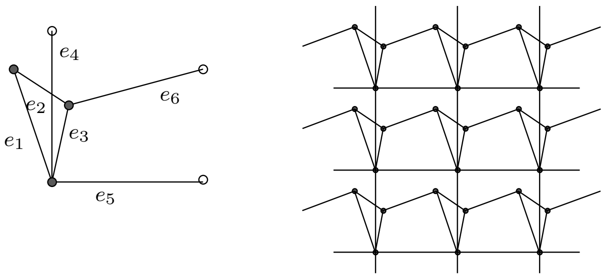

Figure 5.7: The augmented grid+strip framework $\mathcal{C}^+_{\mathbb{Z}^2}$

A motif choice for the augmented grid+strip framework $\mathcal{C}^+_{\mathbb{Z}^2}$(Figure 5.7)		
Motif vertices F_v	Motif edges F_e	period vectors
$p_1 = (0, 0)$ $p_2 = (\frac{-1}{4}, \frac{3}{4})$ $p_3 = (\frac{1}{8}, \frac{1}{2})$	$e_1 = [p_{1,(0,0)}, p_{2,(0,0)}]$ $e_2 = [p_{2,(0,0)}, p_{3,(0,0)}]$ $e_3 = [p_{1,(0,0)}, p_{3,(0,0)}]$ $e_4 = [p_{1,(0,0)}, p_{1,(0,1)}]$ $e_5 = [p_{1,(0,0)}, p_{1,(1,0)}]$ $e_6 = [p_{3,(0,0)}, p_{2,(1,0)}]$	$\underline{a} = (1, 0)$ $\underline{b} = (0, 1)$
Infinitesimal flexibility: band limited		

The kagome framework $\mathcal{C}_{\mathbf{kag}}$. This framework presents the tiling of the plane by regular triangles and hexagons. It can be formed by corner connected equilateral triangles joined in a hexagonal manner.

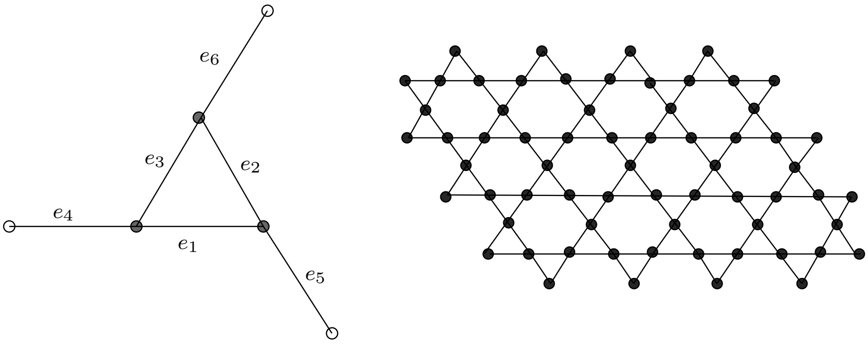

Figure 5.8: The kagome framework $\mathcal{C}_{\text{kag}}$

A motif choice for the kagome framework $\mathcal{C}_{\text{kag}}$ (Figure 5.8)		
Motif vertices F_v	Motif edges F_e	period vectors
$p_1 = (0,0)$ $p_2 = (\frac{1}{2}, 0)$ $p_3 = (\frac{1}{4}, \frac{\sqrt{3}}{4})$	$e_1 = [p_{1,(0,0)}, p_{2,(0,0)}]$ $e_2 = [p_{2,(0,0)}, p_{3,(0,0)}]$ $e_3 = [p_{3,(0,0)}, p_{1,(0,0)}]$ $e_4 = [p_{1,(0,0)}, p_{2,(-1,0)}]$ $e_5 = [p_{2,(0,0)}, p_{3,(1,-1)}]$ $e_6 = [p_{3,(0,0)}, p_{1,(0,1)}]$	$\underline{a} = (1,0)$ $\underline{b} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$
Infinitesimal flexibility: band limited (we consider these flexes in more detail in Theorem 8.3.3)		

We note that $2|F_v| = 6 = |F_e|$ and the framework is in Maxwell counting Equilibrium.

The basic 3 dimensional grid $\mathcal{C}_{\mathbb{Z}^3}$. This framework is the generalization of the basic planar grid $\mathcal{C}_{\mathbb{Z}^2}$.

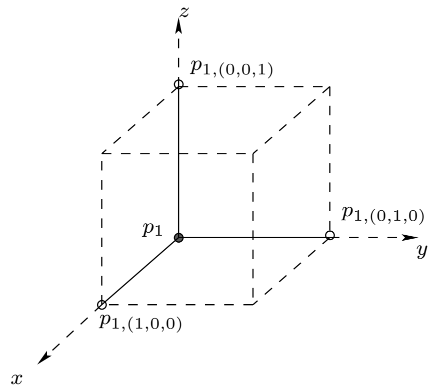

Figure 5.9: The basic 3 dimensional grid $\mathcal{C}_{\mathbb{Z}^3}$

A motif choice for the basic 3 dimensional grid $\mathcal{C}_{\mathbb{Z}^3}$ (Figure 5.9)		
Motif vertices F_v	number of motif edges	period vectors
$p_1 = (0,0,0)$	$\lvert F_e \rvert = 3$	$\underline{a} = (1,0,0)$ $\underline{b} = (0,1,0)$ $\underline{c} = (0,0,1)$
Infinitesimal flexibility: band limited		

Let $F_v = \{p_1\}$ with translation group $\mathcal{T} = \{T_k : k \in \mathbb{Z}^3\}$

$$\text{such that } p_{\kappa,k} = T_k p_{\kappa,0}, \text{ for } \kappa = 1, k \in \mathbb{Z}^3$$

$$T_{k=(n,m,l)}(x,y,z) = (x,y,z) + n\underline{a} + m\underline{b} + l\underline{c},\ \underline{a} = (1,0,0),\ \underline{b} = (0,1,0),$$
$$\underline{c} = (0,0,1)$$

and with motif edge set $F_e = \{e_1, e_2, e_3\}$. We note that $3|F_v| = 3 = |F_e|$ and the framework is in Maxwell counting Equilibrium.

The regular octahedron net framework $\mathcal{C}_{\mathrm{Oct}}$. This *square bypiramed framework* is formed by layers of corner connected regular octahedra. Because these octahedra are rigid, the squares framework $\mathcal{C}_{\mathrm{sq}}$ can be viewed as a subframework of $\mathcal{C}_{\mathrm{Oct}}$. This idea leads to the identification of a "crystal basis" later on.

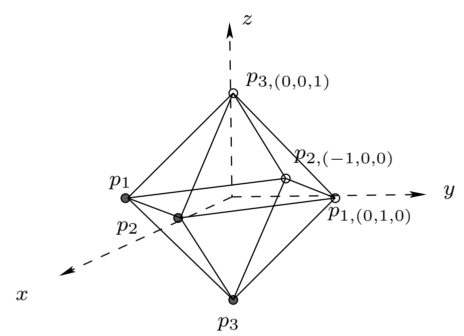

Figure 5.10: The regular octahedron net framework $\mathcal{C}_{\mathrm{Oct}}$

The regular octahedron net framework $\mathcal{C}_{\mathrm{Oct}}$ (Figure 5.10)		
Motif vertices F_v	number of motif edges	period vectors
$p_1 = (0, -1, 0)$ $p_2 = (1, 0, 0)$ $p_3 = (0, 0, -1)$	$\|F_e\| = 12$	$\underline{a} = (2, 0, 0)$ $\underline{b} = (0, 2, 0)$ $\underline{c} = (0, 0, 2)$
Infinitesimal flexibility: band limited		

We note that $3|F_v| = 9 < |F_e|$ and the framework is edge rich.

The bipyramid framework $\mathcal{C}_{\mathrm{Bipyr}}$. This framework is formed by layers of corner connected regular bipyramids. Because these bipyramids are rigid, the triangulated grid framework $\mathcal{C}_{\mathrm{tri}}$ can be viewed as a subframework of $\mathcal{C}_{\mathrm{Bipyr}}$.

The bipyramid framework $\mathcal{C}_{\mathrm{Bipyr}}$ (Figure 5.11)		
Motif vertices F_v	number of motif edges	period vectors
$p_1 = (0, 0, 0)$ $p_2 = (\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{-\sqrt{3}}{2})$	$\|F_e\| = 9$	$\underline{a} = (1, 0, 0)$ $\underline{b} = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ $\underline{c} = (0, 0, \sqrt{3})$

We note that $3|F_v| = 6 < |F_e|$ and the framework is edge rich.

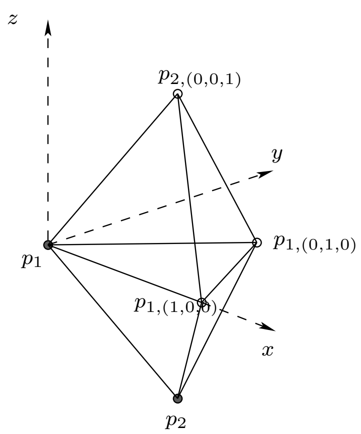

Figure 5.11: The bipyramid framework $\mathcal{C}_{\text{Bipyr}}$

The framework $\mathcal{C}_{\text{Bipyr}}$ is infinitesimally flexible, although it contains infinitely many copies of the infinitesimally rigid planar framework $\mathcal{C}_{\text{tri}}$. A detailed account of the flexibility of $\mathcal{C}_{\text{Bipyr}}$ is obtained in [4]. In fact this framework admits unbounded flexes with "geometric growth".

Remark 5.1.8. Although the examples above are viewed as mathematical crystal frameworks, some do appear in regular and semi-regular tilings for example. For chemistry related definitions and examples, one can refer to [20], [21] and [22].

5.2 The Matrix Function $\Phi_{\mathcal{C}}(z)$ and Crystal Polynomials

Given a crystal framework $\mathcal{C}$ we now associate the matrix valued function given in Owen and Power [54] and Power [58]. This is associated with a choice of periodicity group $\mathcal{T}$ and a motif (F_v, F_e) and is also called the *symbol function* of $\mathcal{C}$ (and $\mathcal{T}$). We shall see that this object plays a key role in the determination of strictly periodic and phase periodic flexes.

Denote the general points of the d-torus by $z = (z_1, \dots, z_d)$ where $z_i \in \mathbb{C}$ and $|z_i| = 1$. Write z^k for the monomial function $z \to z^k$ from $\mathbb{T}^d$ to $\mathbb{C}$. We may think of general monomials z^δ as products of z_i or $\overline{z_i}$ with just non-negative powers since $z_i^{-k} = \overline{z_i}^k$ for points on the circle $\mathbb{T}$.

For the directed edge $e = [p_{\kappa,k}, p_{\tau,l}]$ we define *the edge vector* v_e by $v_e = p_{\kappa,k} - p_{\tau,l}$ and write $v_{e,\sigma}$ for the σ-coordinate of v_e, $1 \leq \sigma \leq d$.

Definition 5.2.1. Let $\mathcal{C}$ be a crystal framework in $\mathbb{R}^d$ with motif sets

$$F_v = \{p_{\kappa,\underline{0}} : 1 \leq \kappa \leq |F_v|\}, \;\; F_e = \{e_i : 1 \leq i \leq |F_e|\}.$$

Then $\Phi_{\mathcal{C}}(z)$ is the *matrix valued function* on $\mathbb{T}^d$ with rows labelled by the edges $e = [p_{\kappa,k}, p_{\tau,l}] \in F_e$ and with columns labelled by pairs κ, σ. As a matrix of scalar functions the entries are given by

$$(\Phi_{\mathcal{C}}(z))_{e,(\kappa,\sigma)} = v_{e,\sigma}\overline{z}^k,$$
$$(\Phi_{\mathcal{C}}(z))_{e,(\tau,\sigma)} = -v_{e,\sigma}\overline{z}^l$$

if $\kappa \neq \tau$, while for a reflexive edge, with $\kappa = \tau$,

$$(\Phi_{\mathcal{C}}(z))_{e,(\kappa,\sigma)} = v_{e,\sigma}(\overline{z}^k - \overline{z}^l),$$

with the remaining entries in each row equal to the zero function.

In the following examples we calculate the matrix functions for the motif choices in the previous section. Here we simply write (z, w) for a general point in $\mathbb{T}^2$.

The matrix function for $\mathcal{C}_{\mathrm{tri}}$ is

$$\Phi_{\mathcal{C}_{\mathrm{tri}}}(z, w) = \begin{bmatrix} \overline{z} - 1 & 0 \\ \frac{1}{2}(\overline{z} - \overline{w}) & \frac{\sqrt{3}}{2}(\overline{w} - \overline{z}) \\ \frac{1}{2}(\overline{w} - 1) & \frac{\sqrt{3}}{2}(\overline{w} - 1) \end{bmatrix}.$$

The matrix function for $\mathcal{C}_{\mathbb{Z}^2}$ is

$$\Phi_{\mathcal{C}_{\mathbb{Z}^2}}(z, w) = \begin{bmatrix} 1 - \overline{z} & 0 \\ 0 & 1 - \overline{w} \end{bmatrix}.$$

Since this framework is in Maxwell counting equilibrium the matrix function is square and we may compute

$$\det \Phi_{\mathcal{C}_{\mathbb{Z}^2}}(z, w) = (\overline{z} - 1)(\overline{w} - 1).$$

For the squares framework $\mathcal{C}_{\mathrm{sq}}$ the matrix function is

$$\Phi_{\mathcal{C}_{\mathrm{sq}}}(z,w) = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & -1 & \overline{z} & \overline{z} \\ -\overline{w} & -\overline{w} & 1 & 1 \\ -\overline{w} & \overline{w} & \overline{z} & -\overline{z} \\ 0 & 0 & -1+\overline{z} & 0 \end{bmatrix}.$$

The matrix function $\Phi_{\mathcal{C}_{5\mathrm{grid}}}(z,w)$ for the 5-regular grid framework takes the form

$$\begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & \overline{w} & 0 & 0 & 0 & 0 & 0 & -1 \\ -1(1-\overline{z}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1(1-\overline{z}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1(1-\overline{z}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1(1-\overline{z}) & 0 \\ -1 & -2 & 0 & 0 & \overline{z} & 2\overline{z} & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 & \overline{z} & 2\overline{z} \end{bmatrix}.$$

The matrix function $\Phi_{\mathcal{C}_{2\mathrm{sq}}}(z,w)$ for the double square framework takes the form

$$\frac{1}{2}\begin{bmatrix} 0 & -2+2\overline{w} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2+2\overline{z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\overline{w} & 0 & 0 & 0 & \frac{-1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2}\overline{zw} & \frac{1}{2}\overline{zw} & 0 & 0 & 0 & 0 & \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{-1}{2}\overline{z} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$$

and in this case the determinant is zero.

The matrix function for the kagome framework $\mathcal{C}_{\text{kag}}$ takes the form

$$\Phi_{\mathcal{C}_{\mathrm{kag}}}(z,w) = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\sqrt{3} & -1 & \sqrt{3} \\ -1 & -\sqrt{3} & 0 & 0 & 1 & \sqrt{3} \\ 2 & 0 & -2z & 0 & 0 & 0 \\ 0 & 0 & -1 & \sqrt{3} & \overline{z}w & -\sqrt{3}\overline{z}w \\ \overline{w} & \sqrt{3}\overline{w} & 0 & 0 & -1 & -\sqrt{3} \end{bmatrix}$$

and in this case the determinant is a constant multiple of

$$\overline{zw}(z-1)(w-1)(z-w).$$

Polynomials for crystal frameworks. Let $\mathcal{C}$ be a crystal framework in $\mathbb{R}^d$ with given isometry group $\mathcal{T}$. If $\mathcal{C}$ is in Maxwell counting equilibrium then we may form the polynomial $\det(\Phi_{\mathcal{C}}(z))$ of the matrix function associated with a particular motif. This is a polynomial in the coordinate functions z_i and their complex conjugates $\overline{z_i}$, and therefore is a multi-variable trigonometric polynomial on $\mathbb{T}^d$ and is possibly identically zero. In the nonzero case we remove dependence on the motif and formally define the *crystal polynomial* $p_{\mathcal{C}}(z)$.

Definition 5.2.2. The *crystal polynomial* $p_{\mathcal{C}}(z)$ associated with the pair $\mathcal{C}, \mathcal{T}$ and a lexicographic ordering is the product $\alpha z^\gamma \det(\Phi_{\mathcal{C}}(z))$ where the multi-power γ and the scalar α are chosen so that

(i) $p_{\mathcal{C}}(z)$ is a linear combination of non-negative power monomials,

$$\alpha z^{\gamma} \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha z^\alpha,$$

(ii) $p_{\mathcal{C}}(z)$ has minimum total degree,

(iii) $p_{\mathcal{C}}(z)$ has leading monomial with coefficient 1.

Here the monomials are ordered lexicographically, so that, for example, the monomial function $z_1^2 z_2$ has higher multi-degree than $z_1 z_2^3$. Generally, the monomial $z^{i_1} \dots z^{i_n}$ has higher multi-degree than $z^{j_1} \dots z^{j_n}$ if either

i. $i_1 > j_1$ or

ii. there exists $1 < k < n$ such that $i_k > j_k$ and $i_l = j_l$ for all $1 < l < k$.

In this way one defines the leading term of a multi-variable polynomial.

Remark 5.2.3. Different motifs for $(\mathcal{C}, \mathcal{T})$ give matrix functions that are equivalent in a natural way. A different motif for the same translation group can be obtained by translation or by choosing different motif edges. Either way, this results in the multiplication of the appropriate rows (respectively columns) by a monomial. The crystal polynomial is multiplied by a monomial and this leaves its zero set unchanged.

From all the above, it follows that the crystal polynomial for the grid framework $\mathcal{C}_{\mathbb{Z}^2}$ is

$$p_{\mathbb{Z}^2}(z, w) = (z - 1)(w - 1).$$

For the kagome framework and the translation group, as above, the crystal polynomial is

$$p_{\text{kag}}(z,w) = (z-1)(w-1)(z-w).$$

Finally, for the double square framework the determinant is zero and the crystal polynomial is the zero polynomial.

5.3 $\Phi_{\mathcal{C}}(z)$ and the RUM Spectrum

Let $\mathcal{K}_v(\mathcal{C})$, $\mathcal{K}_e(\mathcal{C})$ (or simply $\mathcal{K}_v$ and $\mathcal{K}_e$ when the framework in question is understood) be the complex scalar versions of $\mathcal{H}_v(\mathcal{C})$, $\mathcal{H}_e(\mathcal{C})$. Also, let $\mathcal{K}_v^\omega(\mathcal{C})$ be the complex vector subspace of complex velocity vectors $\tilde{v} = (\tilde{v}_{\kappa,k})$ such that $\tilde{v}_{\kappa,k} = \omega^k v_{\kappa,0}$, $\omega^k = (\omega_1^{k_1}, \ldots, \omega_d^{k_d})$ for $\kappa \in F_v$, $k \in \mathbb{Z}^d$ where $v_{\kappa,0} \in \mathbb{R}^d$ and $v = (v_{\kappa,0}) \in \mathbb{R}^{d|F_v|}$. Note that $\mathcal{K}_v^\omega(\mathcal{C})$ is a finite dimensional subspace of $\mathcal{K}_v(\mathcal{C})$ with $\dim \mathcal{K}_v^\omega(\mathcal{C}) = d|F_v|$.

Similarly let $\mathcal{K}_e^\omega(\mathcal{C}) \subset \mathcal{K}_e(\mathcal{C})$ be the subspace of complex sequences $w = (w_e)_{e\in F_e}$ labelled by the framework edges which are phase periodic in this way for the phase ω. Note that the rigidity matrix $R(\mathcal{C})$ provides a linear transformation $R(\mathcal{C}) : \mathcal{K}_v^\omega(\mathcal{C}) \to \mathcal{K}_e^\omega(\mathcal{C})$ by restriction (see also Theorem 5.3.1).

Let $\{\xi_{\kappa,\sigma} : \kappa \in F_v, \sigma \in \{x,y,z\}\}$ be the standard basis for the vector space $\mathbb{C}^{3|F_v|}$. Write $\xi_{\kappa,\sigma}^\omega$ for the velocity vectors in $\mathcal{K}_v^\omega(\mathcal{C})$ which extend the basis elements $\xi_{\kappa,\sigma}$. Formally

$$(\xi^{\omega}_{\kappa,\sigma})_{\kappa',k} = \delta_{\kappa,\kappa'}\omega^k \xi_{\kappa,\sigma}$$

where

$$\delta_{\kappa,\kappa'} = \begin{cases} 0 & \text{if} \quad \kappa \neq \kappa', \\ \\ 1 & \text{if} \quad \kappa = \kappa' \end{cases}.$$

Similarly let $\eta_e, e \in F_e$ be the standard basis for $\mathbb{C}^{|F_e|}$ and write $\eta^{\omega}_e, e \in F_e$ for the natural associated basis for $\mathcal{K}^{\omega}_e(\mathcal{C})$, with

$$(\eta^{\omega}_e)_{e',k} = \omega^k \delta_{e,e'}.$$

The next theorem gives the connection between $\Phi_{\mathcal{C}}(z)$ and the infinitesimal flex properties of $\mathcal{C}$. Here the rigidity matrix $R(\mathcal{C})$ is viewed as the linear transformation

$$R(\mathcal{C}) : \mathcal{K}_v(\mathcal{C}) \to \mathcal{K}_e(\mathcal{C}).$$

Theorem 5.3.1 ([56]). *The restriction of the rigidity matrix $R(\mathcal{C})$ to the finite-dimensional vector space $\mathcal{K}^{\omega}_v(\mathcal{C})$ has representing matrix $\Phi_{\mathcal{C}}(\overline{\omega})$ with respect to natural vector space basis.*

Proof. Let $\tilde{u}$ be a velocity vector in $\mathcal{K}^{\omega}_v(\mathcal{C})$ determined by $u \in \mathbb{C}^{d|F_v|}$ as defined before. Thus we can write $\tilde{u}_{\kappa,k} = \omega^k u_{\kappa,0}$. Let $e \in F_e$ be an edge of the form $[p_{\kappa,k}, p_{\tau,l}]$ and let $\langle .,. \rangle$ denote the bilinear form on $\mathbb{C}^d$. Note that the $(e,0)^{th}$ entry of $R(\mathcal{C})(\tilde{u})$ can be written as

$$(R(\mathcal{C})(\tilde{u}))_{(e,0)} = \langle v_e, \tilde{u}_{\kappa,k}\rangle + \langle -v_e, \tilde{u}_{\tau,l}\rangle.$$

Therefore, the $(e, k')^{th}$ entry of $R(\mathcal{C})(\tilde{u})$ is

$$\begin{aligned}
(R(\mathcal{C})(\tilde{u}))_{(e,k')} &= \langle v_e, \tilde{u}_{\kappa,k'+k}\rangle + \langle -v_e, \tilde{u}_{\tau,k'+l}\rangle \\
&= \langle v_e, \omega^{k'+k} u_\kappa\rangle + \langle -v_e, \omega^{k'+l} u_\tau\rangle \\
&= \omega^{k'}(\langle \omega^k v_e, u_\kappa\rangle + \langle -\omega^l v_e, u_\tau\rangle) \\
&= \omega^{k'}(\Phi_{\mathcal{C}}(\overline{\omega})u)_e
\end{aligned}$$

if $\kappa \neq \tau$. While for a reflexive edge ($\kappa = \tau$)

$$\begin{aligned}
(R(\mathcal{C})(\tilde{u}))_{(e,k')} &= \omega^{k'}(\langle \omega^k v_e, u_\kappa\rangle + \langle -\omega^l v_e, u_\tau\rangle) \\
&= \omega^{k'}\langle (\omega^k - \omega^l) v_e, u_\kappa\rangle \\
&= \omega^{k'}(\Phi_{\mathcal{C}}(\overline{\omega})u)_e.
\end{aligned}$$

$\square$

Definition 5.3.2. The rigid unit mode spectrum (RUM spectrum) of the crystal framework $\mathcal{C}$ in $\mathbb{R}^d$, with translation group $\mathcal{T}$, is the set $\Omega(\mathcal{C})$ of points $\omega = (\omega_1, \ldots, \omega_d)$ in $\mathbb{T}^d$ for which there is a non-zero vector u in $\mathcal{K}_v^\omega(\mathcal{C})$ which is an infinitesimal flex for $\mathcal{C}$. The rigid unit modes are the nonzero infinitesimal flexes that give rise to points in the spectrum. The mode multiplicity function is an integer valued function on $\Omega(\mathcal{C})$ defined by

$$\mu(\omega) = \sum \dim \ker \Phi(\overline{\omega}).$$

From the theorem we have

$$\Omega(\mathcal{C}) = \{\omega \in \mathbb{T}^d : \ker \Phi(\overline{\omega}) \neq 0\}.$$

The rigid unit modes themselves are the non-zero infinitesimal flexes giving rise to points in the RUM spectrum and for a framework in Maxwell counting equilibrium $\Omega(\mathcal{C})$ is the zero set of $p_{\mathcal{C}}(z)$.

Remark 5.3.3. Topologically, the d-torus $\mathbb{T}^d$ is homeomorphic to the d-hypercube, this allows us to view the RUM spectrum as subset of $[0,1)^d$. These *wave vectors* in $[0,1)^d$ can be obtained by simply taking logarithms coordinatewise.

For the basic grid framework $\mathcal{C}_{\mathbb{Z}^2}$ the polynomial is $(z-1)(w-1)$ and for $\Omega(\mathcal{C}_{\mathbb{Z}^2})$ we obtain the set which is the union of the two curves in $\mathbb{T}^2$ defined by $z = 1$ and $w = 1$. In terms of wave vectors this translates to the subset of $[0,1)^2$ shown in Figure 5.12.

For the kagome framework the polynomial is $(z-1)(w-1)(z-w)$ and we obtain the set which is the union of the three curves in $\mathbb{T}^2$ defined by $z = 1$, $w = 1$ and $z = w$ or the subset of the unit square shown in Figure 5.12.

The crystal polynomial for the double squares framework is the zero polynomial and in this case the RUM spectrum is the entire Torus $\mathbb{T}^2$.

For the edge rich triangulated grid $\mathcal{C}_{\text{tri}}$, the squares framework $\mathcal{C}_{\text{sq}}$ and the 5-regular grid framework $\mathcal{C}_{\text{5grid}}$, the matrix function $\Phi_{\mathcal{C}}(z,w)$ is not square. For non-square $\Phi_{\mathcal{C}}(z,w)$ one can instead form the finite set of polynomials for the $d|F_v| \times |F_v|$ square submatrices and in this case the RUM spectrum will be a subset of intersections of the zero sets of these polynomials on the torus $\mathbb{T}^d$.

The Rum spectrum for $\mathcal{C}_{\text{tri}}$ is trivial, $\Omega(\mathcal{C}_{\text{tri}}) = \{(1,1)\}$. For $\mathcal{C}_{\text{sq}}$ the RUM spectrum is $\Omega(\mathcal{C}_{\text{sq}}) = \{(1,1),(-1,-1)\}$. Finally, For $\mathcal{C}_{\text{5grid}}$ the RUM spectrum is the subset of $\mathbb{T}^2$ defined by $z = 1$.

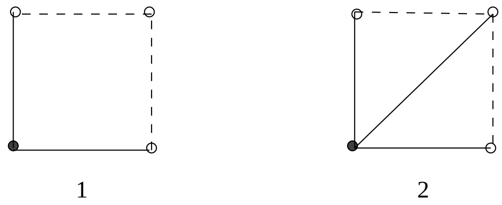

Figure 5.12: The RUM spectrum for the basic grid (1), and for the kagome framework (2).

For the augmented grid+strip framework, the matrix $\Phi_{\mathcal{C}(z,w)}$ is square and calculation shows that the RUM spectrum is the set which is the union of the two curves in $\mathbb{T}^2$ defined by $z = 1$ and $w = 1$.

The framework obtained from the augmented grid+strip by connecting the horizontal strips as in Figure 5.13 has the same matrix for the augmented grid+strip in addition to one row corresponding to the additional edge. In this case the intersection of the zero sets of the polynomials for the square submatrices is the singleton set $\{(1,1)\}$ and the RUM spectrum is trivial.

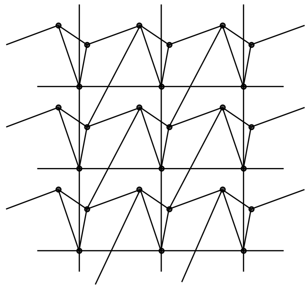

Figure 5.13: The rigid grid+strip framework

5.4 Periodic Rigidity

This section is dedicated to a special class of infinitesimal flexes associated with crystal structures, namely, periodic infinitesimal flexes. Such flexes can be strictly periodic, supercell n-fold periodic or phase periodic. Depending on the choice of translation group, we will see how these flexes relate to each other.

Definition 5.4.1. Let $\mathcal{C}$ be a crystal framework with translation group $\mathcal{T}$. A complex valued infinitesimal flex $u \in \mathcal{K}_{\mathrm{fl}}(\mathcal{C})$ of $\mathcal{C}$ in $\mathbb{R}^d$ is said to be

(i) *strictly periodic* (or simply *periodic*) if it satisfies the periodicity condition

$$u_{\kappa,k} = u_{\kappa,0} \text{ for all } k \in \mathbb{Z}^d,$$

(ii) *supercell n-fold periodic*, for $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$, if it satisfies the periodicity condition

$$u_{\kappa,k} = u_{\kappa,k+n} \text{ for all } k \in \mathbb{Z}^d,$$

(iii) *ω-phase periodic*, $\omega \in \mathbb{T}^d$ if it satisfies

$$u_{\kappa,k} = \omega^k u_{\kappa,0} \text{ for all } \kappa \in F_v, k \in \mathbb{Z}^d.$$

If a crystal framework admits any of the above flexes in a non-trivial way, then it is said to be *flexible* in that sense, otherwise it is considered *rigid*. For example, we say "*strictly periodically infinitesimally flexible*" and so on.

Remark 5.4.2. Let $\mathcal{C}$ be a crystal framework in $\mathbb{R}^d$. If $\mathcal{C}$ has a nonzero local infinitesimal flex and ω was a multi-phase in $\mathbb{T}^d$ then the new flex

$$\tilde{u} = \sum_{k \in \mathbb{Z}^d} \omega^k T_k u$$

is well defined. Also, it is a phase periodic infinitesimal flex that is nonzero for almost every ω. This implies that the RUM spectrum for $\mathcal{C}$ is $\mathbb{T}^d$.

The periodic rigidity matrix. It follows from Theorem 5.3.1 that we can identify real or complex strictly periodic flexes using the kernel of the *periodic rigidity matrix*, $\Phi(1, \dots, 1)$. This matrix is defined by

$$R^1 = \Phi_{\mathcal{C}}(1, \dots, 1).$$

We can define the spaces of strictly periodic flexes and ω-phase periodic flexes in terms of the matrix function as follows

$$\mathcal{K}^{\omega}_{\mathrm{fl}}(\mathcal{C}) = \{((\omega^k a)_k)_{k\in\mathbb{Z}^d},\ a \in \ker \Phi(\overline{\omega})\}$$

with $\omega = (1,\ldots,1)$ in case of strictly periodic flexes.

We will denote the subspace of $\mathcal{K}_v(\mathcal{C})$ ($\mathcal{K}_{\mathrm{fl}}(\mathcal{C})$) of all complex valued supercell n-fold periodic velocities (flexes) by $\mathcal{K}^n_v(\mathcal{C})$ ($\mathcal{K}^n_{\mathrm{fl}}(\mathcal{C})$), or simply $\mathcal{K}^n_{\mathrm{fl}}$ when the relevant crystal framework is understood, and the subspace of ω-phase periodic velocities (flexes) by $\mathcal{K}^{\omega}_v(\mathcal{C})$ ($\mathcal{K}^{\omega}_{\mathrm{fl}}(\mathcal{C})$).

In the next theorem we prove that the space of supercell n-fold periodic velocities can be written as the direct sum of spaces of various ω-phase periodic velocities. This observation has also been indicated recently in Power [58]. In the proof we make use of similar formulae to that of the discrete Fourier transform ([50], [40]).

Theorem 5.4.3. *For a crystal framework $\mathcal{C}$ in $\mathbb{R}^d$,*

$$\mathcal{K}_v^{(n_1,\ldots,n_d)} = \bigoplus_{0\le m_i\le n_i-1} \mathcal{K}_v^{(\omega_{n_1}^{m_1},\ldots,\omega_{n_d}^{m_d})}$$

where $\omega_{n_i} = e^{2\pi i/n_i}$ and $\mathcal{K}_v^{(\omega_{n_1}^{m_1},\ldots,\omega_{n_d}^{m_d})}$ is the subspace of $\mathcal{K}_v$ of all $(\omega_{n_1}^{m_1},\ldots,\omega_{n_d}^{m_d})$-phase periodic velocity sequences, i.e. the velocities satisfying

$$v_{\kappa,k} = \omega_{n_1}^{m_1k_1}\ldots\omega_{n_d}^{m_dk_d} v_{\kappa,0}.$$

Proof. For simplicity, we will prove the theorem for $d = 2$, i.e. we will prove that

$$\mathcal{K}_v^{(n_1,n_2)} = \bigoplus_{0\le m_i\le n_i-1} \mathcal{K}_v^{(\omega_{n_1}^{m_1},\omega_{n_2}^{m_2})}.$$

For higher dimensions the proof is an immediate generalization. On one hand, it is clear that $(\omega_{n_1}^{m_1}, \omega_{n_2}^{m_2})$-phase periodic velocity sequences and their combinations are supercell (n_1, n_2)-fold periodic and so

$$\bigoplus_{0 \le m_i \le n_i - 1} \mathcal{K}_v^{(\omega_{n_1}^{m_1}, \omega_{n_2}^{m_2})} \subseteq \mathcal{K}_v^{(n_1,n_2)}.$$

On the other hand, let $v = (v_{\kappa,(k_1,k_2)})$ be a supercell (n_1, n_2)-fold periodic sequence in $\mathcal{K}_v^{(n_1,n_2)}$. Let $v^{[r_1,r_2]} = (v_{\kappa,(k_1,k_2)}^{[r_1,r_2]})$ be the translation of $v = (v_{\kappa,(k_1,k_2)})$ by r_1 steps to the right and r_2 steps upwards, i.e. $v_{\kappa,(k_1,k_2)}^{[r_1,r_2]} = v_{\kappa,(k_1+r_1,k_2+r_2)}$. Then the sequence $v = (v_{\kappa,(k_1,k_2)})$ can be written as a combination of $(\omega_{n_1}^{m_1}, \omega_{n_2}^{m_2})$-phase periodic velocity sequences as follows:

$$v_{\kappa,(k_1,k_2)} = \sum_{0 \le m_i \le n_i - 1} (\omega_{n_1}^{m_1})^{k_1} (\omega_{n_2}^{m_2})^{k_2} z_{\kappa,(k_1,k_2)}^{[m_1,m_2]} \text{ for all } (k_1, k_2) \in \mathbb{Z}^2$$

where $\omega_{n_i} = e^{2\pi i/n_i}$ and $z^{[m_1,m_2]} = (z_{\kappa,(k_1,k_2)}^{[m_1,m_2]}) \in \mathcal{K}_v^{(\omega_{n_1}^{m_1}, \omega_{n_2}^{m_2})}$ is defined by

$$z_{\kappa,(k_1,k_2)}^{[m_1,m_2]} = \frac{1}{n_1 n_2} \sum_{0 \le r_i \le n_i - 1} (\omega_{n_1}^{m_1})^{-r_1} (\omega_{n_2}^{m_2})^{-r_2} v_{\kappa,(k_1,k_2)}^{[r_1,r_2]}.$$

First, we will prove that the sequences $z^{[m_1,m_2]} = (z_{\kappa,(k_1,k_2)}^{[m_1,m_2]})$ are in fact

$(\omega_{n_1}^{m_1}, \omega_{n_2}^{m_2})$-phase periodic.

$$\begin{aligned}
z_{\kappa,(k_1,k_2)}^{[m_1,m_2]} &= \frac{1}{n_1 n_2} \sum_{0 \le r_i \le n_i - 1} (\omega_{n_1}^{m_1})^{-r_1} (\omega_{n_2}^{m_2})^{-r_2} v_{\kappa,(k_1,k_2)}^{[r_1,r_2]} \\
&= \frac{1}{n_1 n_2} \sum_{0 \le r_i \le n_i - 1} (\omega_{n_1}^{m_1})^{-r_1} (\omega_{n_2}^{m_2})^{-r_2} v_{\kappa,(k_1+r_1,k_2+r_2)} \\
&= \frac{1}{n_1 n_2} \sum_{0 \le r_i \le n_i - 1} (\omega_{n_1}^{m_1})^{-r_1} (\omega_{n_2}^{m_2})^{-r_2} v_{\kappa,(0,0)}^{k_1+r_1,k_2+r_2} \\
&= \frac{1}{n_1 n_2} (\omega_{n_1}^{m_1})^{k_1} (\omega_{n_2}^{m_2})^{k_2} \sum_{0 \le r_i \le n_i - 1} (\omega_{n_1}^{m_1})^{-(k_1+r_1)} (\omega_{n_2}^{m_2})^{-(k_2+r_2)} v_{\kappa,(0,0)}^{[k_1+r_1,k_2+r_2]} \\
&= (\omega_{n_1}^{m_1})^{k_1} (\omega_{n_2}^{m_2})^{k_2} z_{\kappa,(0,0)}^{[m_1,m_2]}
\end{aligned}$$

and therefore we have $z^{[m_1,m_2]} = (z_{\kappa,(k_1,k_2)}^{[m_1,m_2]}) \in \mathcal{K}_v^{(\omega_{n_1}^{m_1}, \omega_{n_2}^{m_2})}$. Using the definition of $z^{[m_1,m_2]} = (z_{\kappa,(k_1,k_2)}^{[m_1,m_2]})$ we have for all (k_1, k_2),

$$\begin{aligned}
&\sum_{0 \le m_i \le n_i - 1} (\omega_{n_1}^{m_1})^{k_1} (\omega_{n_2}^{m_2})^{k_2} z_{\kappa,(k_1,k_2)}^{[m_1,m_2]} \\
&= \frac{1}{n_1 n_2} \sum_{0 \le m_i \le n_i - 1} (\omega_{n_1}^{m_1})^{k_1} (\omega_{n_2}^{m_2})^{k_2} \Big(\sum_{0 \le r_i \le n_i - 1} (\omega_{n_1}^{m_1})^{-r_1} (\omega_{n_2}^{m_2})^{-r_2} v_{\kappa,(k_1,k_2)}^{[r_1,r_2]} \Big) \\
&= \frac{1}{n_1 n_2} \sum_{0 \le r_i \le n_i - 1} v_{\kappa,(k_1,k_2)}^{[r_1,r_2]} \Big(\sum_{0 \le m_i \le n_i - 1} (\omega_{n_1}^{m_1})^{k_1 - r_1} (\omega_{n_2}^{m_2})^{k_2 - r_2} \Big).
\end{aligned}$$

Since the sum of the nth roots of unity is zero, we have

$$\sum_{0 \le m_i \le n_i - 1} (\omega_{n_1}^{m_1})^{k_1 - r_1} (\omega_{n_2}^{m_2})^{k_2 - r_2} = \begin{cases} 0 & \text{if} \quad k_1 \ne r_1 \text{ or } k_2 \ne r_2, \\ n_1 n_2 & \text{if} \quad k_1 = r_1 \text{ and } k_2 = r_2 \end{cases} .$$

Therefore,

$$\frac{1}{n_1 n_2} \sum_{0 \le r_i \le n_i - 1} v_{\kappa,(k_1,k_2)}^{[r_1,r_2]} \Big(\sum_{0 \le m_i \le n_i - 1} (\omega_{n_1}^{m_1})^{k_1 - r_1} (\omega_{n_2}^{m_2})^{k_2 - r_2} \Big)$$
$$= v_{\kappa,(k_1,k_2)}^{[0,0]}$$
$$= v_{\kappa,(k_1,k_2)}.$$

From all the above,

$$v_{\kappa,(k_1,k_2)} = \sum_{0 \le m_i \le n_i - 1} (\omega_{n_1}^{m_1})^{k_1} (\omega_{n_2}^{m_2})^{k_2} z_{\kappa,(k_1,k_2)}^{[m_1,m_2]}.$$

Finally, note that any $(\omega_{n_1}^{m_1}, \omega_{n_2}^{m_2})$-phase periodic velocity that is $(\omega_{n_1}^{m'_1}, \omega_{n_2}^{m'_2})$-phase periodic has to be zero, and we have a direct sum. $\square$

It follows from Theorem 5.4.3 above that a similar direct sum relation holds between space of supercell n-fold periodic flexes and spaces of various ω-phase periodic flexes. This can be immediately deduced from the proof of Theorem 5.4.3 by observing that translates and sums of flexes give flexes. The same concept will be later used for proofs regarding almost periodic rigidity theory.

Corollary 5.4.4. *For a crystal framework $\mathcal{C}$ in $\mathbb{R}^d$,*

$$\mathcal{K}_{\mathrm{fl}}^{(n_1,\ldots,n_d)} = \bigoplus_{0 \le m_i \le n_i - 1} \mathcal{K}_{\mathrm{fl}}^{(\omega_{n_1}^{m_1},\ldots,\omega_{n_d}^{m_d})}$$

where $\omega_{n_i} = e^{2\pi i/n_i}$ and $\mathcal{K}_{\mathrm{fl}}^{(\omega_{n_1}^{m_1},\ldots,\omega_{n_d}^{m_d})}$ is the subspace of $\mathcal{K}_v^{(\omega_{n_1}^{m_1},\ldots,\omega_{n_d}^{m_d})}$ of all $(\omega_{n_1}^{m_1},\ldots,\omega_{n_d}^{m_d})$-phase periodic flexes.

Definition 5.4.5. Let $M_1 = (F_v^{(1)}, F_e^{(1)})$ be a motif for $\mathcal{C}$ in $\mathbb{R}^2$ with translation group $\mathcal{T}^{(1)}$, We say that the motif $M_2 = (F_v^{(2)}, F_e^{(2)})$ with translation group $\mathcal{T}^{(2)}$ is a *supercell (n_1, n_2)-fold inflation* of M_1 if

$$F_v^{(2)} = \{p_{\kappa,0}^{[r_1,r_2]} : p_{\kappa,0}^{[r_1,r_2]} = p_{\kappa,(r_1,r_2)}, p_{\kappa,0} \in F_v^{(1)}, 0 \leq r_i \leq n_i - 1\}$$

and

$$\begin{aligned} F_e^{(2)} = \{e_{j,0}^{[r_1,r_2]} : e_{j,0}^{[r_1,r_2]} &= [p_{\kappa,0}^{[r_1,r_2]}, p_{\tau,l}^{[r_1,r_2]}], \\ e_{j,0} = [p_{\kappa,0}, p_{\tau,l}] &\in F_e^{(1)}, 0 \leq r_i \leq n_i - 1\}. \end{aligned}$$

It is clear from the definition above that $M_1 \subseteq M_2$ and $\mathcal{T}^{(1)} \supseteq \mathcal{T}^{(2)}$.

Example 5.4.6. Consider the basic grid, $\mathcal{C}_{\mathbb{Z}^2}$. Let M_1 be the minimal motif consisting of a single point, M_2 be a supercell $(1, 2)$-fold inflation of M_1 and M_3 a supercell $(2, 2)$-fold inflation of M_1 (Figure 5.14). Note that M_3 can be considered as a supercell $(2, 1)$-fold inflation of M_2 too.

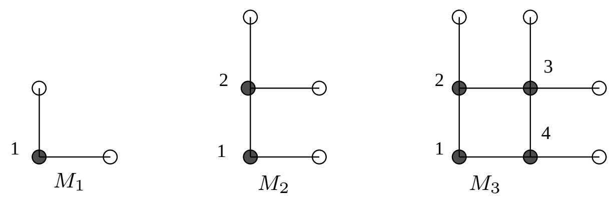

Figure 5.14: Different motifs for the basic grid $\mathcal{C}_{\mathbb{Z}^2}$

$\mathcal{C}_{\mathbb{Z}^2}$ with M_1 is infinitesimally strictly periodically rigid, as the only strictly periodic flexes are translations. This is because $\ker \Phi^{(1)}(1, 1) = \mathbb{R}^2$, but M_1 has a single point, and therefore any motion applied to that point applies to the rest of the vertices. If we consider the motif choice M_2, then

$$\ker \Phi^{(2)}(1,1) = \text{span}\{((0,1),(0,1)),((1,0),(0,0)),((0,0),(1,0))\}.$$

With motif choice M_3 we find that

$$\ker \Phi^{(3)}(1,1) = \text{span}\{((1,0),(1,0),(0,0),(0,0)),((0,1),(0,0),(0,1)(0,0)),$$
$$((0,0),(0,1),(0,0),(0,1)),((0,0),(0,0),(1,0),(1,0))\}.$$

Therefore, with both motif choices M_2 and M_3, $\mathcal{C}_{\mathbb{Z}^2}$ is infinitesimally strictly periodically flexible.

Example 5.4.7. With any choice of motif, the brick framework $\mathcal{C}_{\text{brick}}$ (Figure 5.15) is infinitesimally strictly periodically flexible.

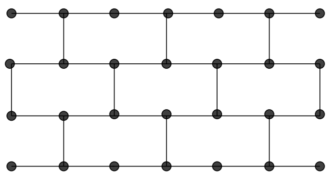

Figure 5.15: The brick framework $\mathcal{C}_{\text{brick}}$

The direct sum relation in Corollary 5.4.4 implies that the corresponding kernels of the matrix functions can also be expressed in a similar way and we have the following result.

Corollary 5.4.8. *Let M_1 and M_2 be two motifs of $\mathcal{C}$ in $\mathbb{R}^2$ as above (or M_2 be a supercell (n_1, n_2)-fold inflation of M_1). If $\Phi^{(1)}$, $\Phi^{(2)}$ are the symbol functions for $\mathcal{C} = (F_v^{(1)}, F_e^{(1)}, \mathcal{T}^{(1)})$, $\mathcal{C} = (F_v^{(2)}, F_e^{(2)}, \mathcal{T}^{(2)})$ respectively, then*

$$\ker \Phi^{(2)}(1,1) = \bigoplus_{0 \le m_i \le n_i - 1} \ker \Phi^{(1)}(\overline{\omega}_{n_1}^{m_1}, \overline{\omega}_{n_2}^{m_2})$$

where $\omega_{n_i} = e^{2\pi i/n_i}$.

Example 5.4.9. It follows from the previous corollary, that one can deduce the strictly periodic flexes of the basic grid $\mathcal{C}_{\mathbb{Z}^2}$ with respect to different motif choices using the matrix function associated with the minimal motif.

Corollary 5.4.8 implies that the dimension of the space of supercell n-fold periodic flexes is in fact the sum of dimensions of the relevant ω-phase periodic flexes in the sense of the following corollary.

Corollary 5.4.10. *Let* M_1 *and* M_2 *be two motifs of* $\mathcal{C}$ *in* $\mathbb{R}^2$ *as above (or* M_2 *be a supercell* (n_1, n_2)*-fold inflation of* M_1 *). If* $\Phi^{(1)}$, $\Phi^{(2)}$ *are the symbol functions for* $\mathcal{C} = (F_v^{(1)}, F_e^{(1)}, \mathcal{T}^{(1)})$, $\mathcal{C} = (F_v^{(2)}, F_e^{(2)}, \mathcal{T}^{(2)})$ *respectively, then*

$$\dim\ker\Phi^{(2)}(1,1) = \sum_{0\leq m_i\leq n_i-1} \dim\ker\Phi^{(1)}(\overline{\omega}_{n_1}^{m_1}, \overline{\omega}_{n_2}^{m_2})$$

where $\omega_{n_i} = e^{2\pi i/n_i}$.

Remark 5.4.11. For the basic grid, $\mathcal{C}_{\mathbb{Z}^2}$, with M_1, the framework is strictly periodically rigid and the subspace of such flexes is of two dimensions corresponding to translations in each coordinate direction. With the motif M_2, the basic grid admits a strictly periodic infinitesimal flex and therefore the subspace of these flexes is three dimensional. A higher dimension for the strictly periodic flex subspace results when M_3 is the chosen motif. In Figure 5.16, each number corresponds to the dimension of $\dim\ker\Phi(\overline{\omega})$ with respect to the phase ω and the dimension of the subspace of strictly periodic flexes is equal to the sum of the dimensions of the relevant subspaces of phase periodic flexes.

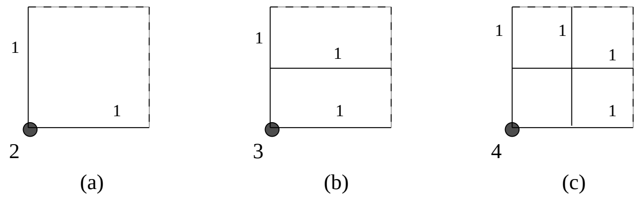

Figure 5.16: (a) The RUM spectrum for $\mathcal{C}_{\mathbb{Z}^2}$ with motif choice M_1, (b) with motif M_2 and (c) with M_3

Example 5.4.12. Consider the double triangles framework $\mathcal{C}_{2\text{tri}}$. This framework is rich in local infinitesimal flexes and therefore the RUM spectrum in this case is $\mathbb{T}^2$. $\mathcal{C}_{2\text{tri}}$ admits a 1-dimensional subspace of non-trivial strictly periodic infinitesimal flexes. Precisely we have

$$\ker \Phi(1,1) = \text{span}\{((1,0),(1,0),(1,0),(1,0)),((0,1),(0,1),(0,1),(0,1)),$$
$$((0,0),(-\frac{\sqrt{3}}{2},1),(-\frac{\sqrt{3}}{2},-1),(\frac{2-\sqrt{3}}{2},0))\}$$

And $\dim \ker \Phi(1,1) = 3$. Now, we fill up the whole framework with triangles as in Figure 5.17.

In this case, the framework admits a 4-dimensional subspace of strictly periodic flexes, 2 corresponding to translations and the other 2 come from the infinitesimal rotation of the inner triangles either both in one direction or in different directions. These double triangles frameworks, being "rigid but infinitesimally flexible", result in the multiplicity being dependent on

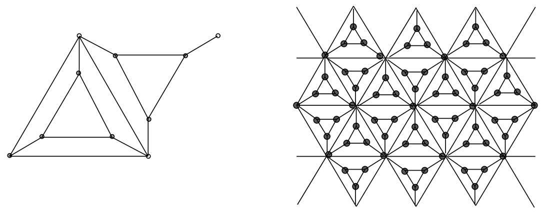

Figure 5.17: A 2-double triangle framework

the number of triangles in the motif that we fill in. We will have 2 translations and then for n triangles, the multiplicity becomes $2+n$. That is because we will get n linearly independent flexes each with one non zero flex corresponding to one triangle and zero otherwise.

Chapter 6

Almost Periodic Functions

The notion of an almost periodic function was introduced by H. Bohr in 1924 [8] as a generalization of the notion of a purely periodic function. Recall that a function $f : \mathbb{R} \to \mathbb{C}$ is τ-*periodic* if $f(x) = f(x + \tau)$ for all $x \in \mathbb{R}$. Bohr's work was preceded by the investigations of P. Bohl and E. Esclangon. Subsequently, during the 1920s and 1930s, Bohr's theory was substantially developed by S. Bochner, H. Weyl, A. Besicovitch and others [6]. In particular, the compactness property of an almost periodic function was discovered by Bochner in 1927.

In this chapter we aim to develop an understanding of multi-variable almost periodic functions and the approximation by trigonometric polynomials theorem. With this understanding we will be able to adapt this theory in the next chapter to obtain the counterpart theory of almost periodic sequences

$(a_n)_{n\in\mathbb{Z}}$, and for almost periodic multivariate sequences $(a_k)_{k\in\mathbb{Z}^d}$. In particular, we will obtain there an explicit approximation by trigonometric sequences theorem, with explicit kernels providing the approximating sequences. We remark that we found the explicit formalism of Partington [55] to be more useful for our purposes than the general theory of almost periodic functions on locally compact abelian groups ([7], [67]).

6.1 Single Variable Almost Periodic Functions

In this section, we state the definitions and basic properties of almost periodic functions of one variable to familiarize the reader with the theorems we aim to understand in higher dimensions. For more details see [19], [55], [6], [18], [46], [48], [64], [30] and [40]. In fact, as we note after Theorem 6.2.20, this one variable theory is contained within the multi-variable theory of the following section.

An example of an almost periodic function is $f(x) = 2-(\cos x+\cos(\sqrt{2}x))$, noticing that $f(x)$ has the value 0 only at $x = 0$ we see that f is not periodic.

Definition 6.1.1. The class $AP(\mathbb{R})$ of *uniformly almost periodic functions* is the closed linear span in $L^\infty(\mathbb{R})$ of the set of functions $(e_\lambda)_{\lambda\in\mathbb{R}}$ where $e_\lambda(x) = e^{i\lambda x}$.

An equivalent definition is the following:

Definition 6.1.2 (Approximation by trigonometric polynomials)**.** Let f : $\mathbb{R} \to \mathbb{C}$ we say that f is *uniformly almost periodic* if for all $\epsilon > 0$ there exists a trigonometric polynomial $P : \mathbb{R} \to \mathbb{C}$ such that:

$$|f(x) - P(x)| < \epsilon \text{ for all } x \in \mathbb{R}.$$

Since the uniform limit of continuous functions is continuous, then it follows that $AP(\mathbb{R}) \subset C_b(\mathbb{R})$, the class of all continuous bounded functions on $\mathbb{R}$.

Definition 6.1.3. Let $f : \mathbb{R} \to \mathbb{C}$ be any function on $\mathbb{R}$, the *right shift* R_λ of f is defined by

$$(R_\lambda f)(x) = f(x - \lambda).$$

Definition 6.1.4. Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function. Then a number $\lambda \in \mathbb{R}$ is called an *ϵ-translation number* of f if

$$\|R_\lambda f - f\|_\infty \leq \epsilon.$$

Definition 6.1.5. A set $S \subseteq \mathbb{R}$ is said to be *relatively dense* if there exists an $L > 0$ such that every interval of length L contains an element of S.

Example 6.1.6. The set of numbers $S = \{\pm n, n \in \mathbb{Z}_+\}$ is relatively dense in $\mathbb{R}$. On the other hand, the set of numbers $S = \{\pm n^2, n \in \mathbb{Z}_+\}$ is not relatively dense since $(n + 1)^2 - n^2 \to \infty$ as $n \to \infty$.

Definition 6.1.7. Let f be a function $f : \mathbb{R} \to \mathbb{C}$. Then

i. f is said to be a *Bohr function* if f is continuous and for every $\epsilon > 0$ the set of ϵ-translation numbers of f is relatively dense.

ii. f satisfies *Bochner's condition* if it is continuous and bounded and the set of translates $\{R_\lambda f,\ \lambda \in \mathbb{R}\}$ is relatively compact in $C_b(\mathbb{R})$. That is, f satisfies Bochner's condition if every sequence in $\{R_\lambda f,\ \lambda \in \mathbb{R}\}$ has a uniformly convergent subsequence.

Theorem 6.1.8. *The Bohr functions form a closed linear subspace of $C_b(\mathbb{R})$ and hence every function in $AP(\mathbb{R})$ is a Bohr function.*

Corollary 6.1.9. *The Bohr functions form a closed subalgebra of $C_b(\mathbb{R})$.*

It follows from the corollary above that the sum of two Bohr functions is a Bohr function. For example, the function

$$f(x) = a + be^{ix} + ce^{\sqrt{2}ix}$$

is almost periodic [47]. To prove this directly we will make use of the definitions and the following theorems

- Kronecker's Density Theorem [12]: If the real number θ is distinct from each rational multiple of π, then the set $\{e^{in\theta} : n \in \mathbb{Z}\}$ is dense in the unit circle.

- The Sharpened Dirichlet-Kronecker Theorem [30]: If $t, a_1, \ldots, a_k$ are nonzero real numbers and if ϵ is a positive number, then there exists a relatively dense set N of integers such that $n \in N$ implies the existence of integers $m_1, \ldots, m_k$ for which

$$|nt - m_i a_i| < \epsilon \text{ for } i = 1, \ldots, k.$$

Let $\theta = \dfrac{2\pi}{\sqrt{2}}$, then by Kronecker's Density Theorem $e^{im\theta}$ is dense in $\mathbb{T}$, so we need to find m such that $|e^{im\theta} - 1| < \epsilon$.

By the periodicity of e^{ix},

$$\begin{aligned}
|f(x-\tau) - f(x)| &= |be^{i(x-\tau+2n\pi)} + ce^{i\sqrt{2}(x-\tau+\frac{2m\pi}{\sqrt{2}})} - be^{ix} - ce^{i\sqrt{2}x}| \\
&\leq |b||e^{ix}||e^{-\tau+2n\pi} - 1| + |c||e^{i\sqrt{x}}||e^{\sqrt{2}i(-\tau+\frac{2m\pi}{\sqrt{2}})} - 1|.
\end{aligned}$$

Now for the non-zero numbers θ and 2π, and for $\epsilon > 0$ the Sharpened Dirichlet-Kronecker Theorem implies that there exists integer numbers m_0, n_0 such that

$$|m_0 \frac{2\pi}{\sqrt{2}} - n_0 2\pi| < \epsilon.$$

Thus, for $m = m_0$, $n = n_0$ we see that $\tau = n_0 2\pi$ is a ϵ-period of f. It follows that

$$\begin{aligned}
|f(x-\tau) - f(x)| &\leq |b||e^{ix}||e^{-\tau+2n\pi} - 1| + |c||e^{i\sqrt{x}}||e^{\sqrt{2}i(-\tau+\frac{2m\pi}{\sqrt{2}})} - 1| \\
&\leq |b|\epsilon_1 + |c|\epsilon_2.
\end{aligned}$$

Choosing $\epsilon > \max\{2|b|\epsilon_1, 2|c|\epsilon_2\}$, the difference $f(x-\tau) - f(x)$ is smaller than ϵ for all $x \in \mathbb{R}$ as required.

Proposition 6.1.10. *Let $f \in C_b(\mathbb{R})$ be a Bohr function. Then*

$$[f,1] = \lim_{T\to\infty} [f,1]_T = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\, dx$$

exists. Hence

$$[f,g] = \lim_{T\to\infty} [f,g]_T = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{g(x)}\, dx$$

is well defined for all Bohr functions f and g.

The *Bohr transform* of f is the function

$$f^B(\lambda) = [f, e_\lambda].$$

The *mean value* of f (sometimes referred to as $M(f)$) is given by

$$M(f) = f^B(0) = [f,1].$$

For example, if $f(x) = a + be^{ix} + ce^{\sqrt{2}ix}$, then

$$\begin{aligned}
M(f) = [f,1] &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (a + be^{ix} + ce^{\sqrt{2}ix})\, dx \\
&= \lim_{T\to\infty} \frac{1}{2T} [ax]_{-T}^{T} + \frac{b}{i}[e^{ix}]_{-T}^{T} + \frac{c}{i\sqrt{2}}[e^{i\sqrt{2}x}]_{-T}^{T} \\
&= \lim_{T\to\infty} (a + \frac{b}{T}\frac{e^{iT} - e^{-iT}}{2i} + \frac{c}{\sqrt{2}T}\frac{e^{i\sqrt{2}T} - e^{-i\sqrt{2}T}}{2i}) \\
&= \lim_{T\to\infty} (a + b\frac{\sin T}{T} + \frac{c}{\sqrt{2}}\frac{\sin\sqrt{2}T}{T}) \\
&= a.
\end{aligned}$$

Definition 6.1.11. Let $f \in C_b(\mathbb{R})$ be a Bohr function. The *Bohr spectrum* of f is the set

$$\Lambda_f = \{\lambda \in \mathbb{R} : [f, e_\lambda] \neq 0\}.$$

Theorem 6.1.12. *Let f be a Bohr function. Then Bessel's inequality holds, in the following form:*

$$[f, f] \geq \sum_{k=1}^{n} |[f, e_{\lambda_k}]|^2$$

for all distinct $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Hence $[f, e_\lambda] \neq 0$ for at most a countable set of $\lambda \in \mathbb{R}$.

Theorem 6.1.13 (Uniqueness theorem)**.** *Let f be a Bohr function such that $[f, e_\lambda] = 0$ for all $\lambda \in \mathbb{R}$. Then f is identically zero.*

Theorem 6.1.14 (Parseval's identity)**.** *Let f be a Bohr function. Then*

$$[f, f] = \sum_{\lambda \in \mathbb{R}} |f, e_\lambda|^2.$$

The main theorem in the theory of almost periodic functions states that every almost periodic function in the sense of Bohr is the uniform limit of trigonometric polynomials. The approach to finding the approximating polynomials, is analogous to Fejér's theorem for periodic functions.

Theorem 6.1.15. *Let f be a Bohr function. Then f can be approximated uniformly by trigonometric polynomials.*

6.2 Multi-Variable Almost Periodic Functions

In this section we develop the theory of almost periodic functions of two variables in complete detail, following the approach of Partington [55] and using similar terminology. Once the two variables theory is complete, the extension to higher dimensions is a routine generalization.

Definition 6.2.1. The class $AP(\mathbb{R}^2)$ of *uniformly almost periodic functions* is the closed linear span in $L^\infty(\mathbb{R}^2)$ of the set of functions $(e_{(\lambda,\gamma)})_{(\lambda,\gamma)\in\mathbb{R}^2}$, $e_{(\lambda,\gamma)}(x,y) = e^{i(\lambda x+\gamma y)}$.

An evidently equivalent definition is the following:

Definition 6.2.2 (Approximation by trigonometric polynomials)**.** Let $f : \mathbb{R}^2 \to \mathbb{C}$ we say that f is *uniformly almost periodic* if for all $\epsilon > 0$ there exists a trigonometric polynomial $P : \mathbb{R}^2 \to \mathbb{C}$ such that:

$$|f(x,y) - P(x,y)| < \epsilon \text{ for all } x, y \in \mathbb{R}.$$

Since the uniform limit of a continuous function is continuous, it follows that the set of almost periodic functions $AP(\mathbb{R}^2)$ is a subset of the class $C_b(\mathbb{R}^2)$ of all continuous bounded functions on $\mathbb{R}^2$.

Definition 6.2.3. Let $f : \mathbb{R}^2 \to \mathbb{C}$ be any function on $\mathbb{R}^2$ and λ, $\gamma \in \mathbb{R}$. The *right shift* $R_{(\lambda,\gamma)}$ of f is defined by

$$(R_{(\lambda,\gamma)}f)(x,y) = f(x-\lambda, y-\gamma).$$

Definition 6.2.4. Let $f : \mathbb{R}^2 \to \mathbb{C}$ be a continuous function. A vector $(\lambda,\gamma) \in \mathbb{R}^2$ is called an *ϵ-translation vector* of f if

$$\|R_{(\lambda,\gamma)}f - f\|_\infty \leq \epsilon.$$

Definition 6.2.5. A set $S \subseteq \mathbb{R}^2$ is said to be *relatively dense* if there exists an $L > 0$ such that every square of side length L contains an element of S.

Definition 6.2.6. Let f be a function $f : \mathbb{R}^2 \to \mathbb{C}$. Then

i. f is a *Bohr function* if f is continuous and for every $\epsilon > 0$ the set of ϵ-translation vectors of f is relatively dense.

ii. f satisfies *Bochner's condition* if it is continuous and bounded and the set of translates $\{R_{(\lambda,\gamma)}f,\ (\lambda,\gamma) \in \mathbb{R}^2\}$ is relatively compact in $C_b(\mathbb{R}^2)$.

In the development below translation vectors in $\mathbb{R}^2$ replace translation lengths in the one-variable theory, and squares in $\mathbb{R}^2$ replace intervals. At certain points in the development we have to do extra work to obtain the desired estimates, such as in the integral estimates for Proposition 6.2.10. However, we are able to generalize the entire single variable theory indicated above with natural adaptations of the one variable theory.

Theorem 6.2.7. *The class of Bohr functions coincides with the class of functions satisfying the Bochner condition.*

Proof. 1. To prove that Bohr functions satisfy Bochner's condition we need to prove:

a. Every Bohr function is bounded;

b. Every Bohr function is uniformly continuous.

Proof of a: Let $f : \mathbb{R}^2 \to \mathbb{C}$ be a Bohr function, and let L be such that every square of side length L contains a 1-translation vector of f. If $(x, y) \in \mathbb{R}^2$ is arbitrary, then there exists a 1-translation vector (λ, γ) of f for which (λ, γ) lies in the square $[x - L, x] \times [y - L, y]$. Since f is continuous, there exists $M > 0$ such that $|f(x, y)| \leq M$ for all $x, y \in [0, L]$. Now we have

$$|f(x,y)| \leq |f(x-\lambda, y-\gamma)| + |f(x-\lambda, y-\gamma) - f(x,y)| \leq M + 1,$$

and f is bounded.

Proof of b: Let f be a Bohr function and $\epsilon > 0$, and let L be such that every square of edge length L contains an $\epsilon/3$-translation vector of f. Since f is uniformly continuous on the closed square $[0, L+1]^2$, there exists $\delta = \delta(\epsilon/3)$, $0 < \delta < 1$ such that:

$$|f(x_1, y_1) - f(x_2, y_2)| < \epsilon$$

whenever

$$(x_1, y_1), (x_2, y_2) \in [0, L+1]^2 \text{ and } \|(x_1, y_1) - (x_2, y_2)\| < \delta.$$

Given any $(x_1', y_1'), (x_2', y_2') \in \mathbb{R}^2$ with $\|(x_1', y_1') - (x_2', y_2')\|_2 < \delta$,
we can find an $\epsilon/3$-translation vector $(\lambda, \gamma) \in \mathbb{R}^2$ such that the points $(x_1, y_1) = (x_1', y_1') - (\lambda, \gamma), (x_2, y_2) = (x_2', y_2') - (\lambda, \gamma)$ lie in the square $[0, L+1]^2$. Since $\|(x_1' - \lambda, y_1' - \gamma) - (x_2' - \lambda, y_2' - \gamma)\| = \|(x_1', y_1') - (x_2', y_2')\| < \delta$

we can deduce that:

$$\begin{aligned}|f(x_1', y_1') - f(x_2', y_2')| &\le |f(x_1', y_1') - f(x_1' - \lambda, y_1' - \gamma)| \\ &\quad + |f(x_1' - \lambda, y_1' - \gamma) - f(x_2' - \lambda, y_2' - \gamma)| \\ &\quad + |f(x_2', y_2') - f(x_2' - \lambda, y_2' - \gamma)| \\ &< \epsilon.\end{aligned}$$

Now to prove 1, suppose, if possible, that f is a Bohr function that is not a Bochner function. It follows that there exists a sequence with a subsequence that is not uniformly convergent. Therefore, for some $\epsilon > 0$ we have a sequence $((\lambda_k, \gamma_k))_{k=1}^{\infty}$ such that

$$\|R_{(\lambda_j,\gamma_j)}f - R_{(\lambda_k,\gamma_k)}f\|_\infty > \epsilon \text{ for all } j \neq k.$$

But f is a Bohr function which implies that there exists an $L > 0$ such that every L-square contains an $\epsilon/4$-translation vector of f.

Write $(\lambda_k, \gamma_k) = (\tau_k + \delta_k, \tau_k' + \delta_k')$ where (τ_k, τ_k') is an $\epsilon/4$-translation vector of f and (δ_k, δ_k') lies in the square $[0, L]^2$. Thus $\|R_{(\lambda_k,\gamma_k)}f - R_{(\delta_k,\delta_k')}f\|_\infty \le \epsilon/4$ for all k. By passing to a subsequence and relabelling, we may suppose that $((\delta_k, \delta_k'))$ converges with limit (δ, δ'), say. Therefore $\|(\delta_k, \delta_k') - (\delta, \delta')\| \to 0$ and by uniform continuity, there exists a $k_0 \in \mathbb{N}$ such that $\|R_{(\delta_k,\delta_k')}f - R_{(\delta,\delta')}f\|_\infty \le \epsilon/4$ for all $k \ge k_0$.

Thus for $j, k \geq k_0$ we have:

$$\begin{aligned}\|R_{(\lambda_j,\gamma_j)}f - R_{(\lambda_k,\gamma_k)}f\|_\infty &\leq \|R_{(\lambda_j,\gamma_j)}f - R_{(\delta_j,\delta'_j)}f\| \\ &+ \|R_{(\delta_j,\delta'_j)}f - R_{(\delta,\delta')}f\| \\ &+ \|R_{(\delta,\delta')}f - R_{(\delta_k,\delta'_k)}f\| \\ &+ \|R_{(\delta_k,\delta'_k)}f - R_{(\lambda_k,\gamma_k)}f\| \\ &\leq \epsilon\end{aligned}$$

which is a contradiction and shows that "Bohr implies Bochner".

2. Now we will prove that "Bochner implies Bohr":

Assume that f is not a Bohr function, then there exists an $\epsilon > 0$ for which the set S_ϵ of ϵ-translation vectors of f is not relatively dense. Take $(c_1, c'_1) = (1, 1)$ and let $(a_2, b_2) \times (a'_2, b'_2)$ be a square such that

$$|b_2 - a_2| > 2|c_1| \text{ and } |b'_2 - a'_2| > 2|c'_1|$$

and contains no ϵ-translation vector of f. Let $(c_2, c'_2) = (\frac{a_2 + b_2}{2}, \frac{a'_2 + b'_2}{2})$ be the central point of the square, then,

$$(c_2, c'_2) - (c_1, c'_1) = (\frac{a_2 + b_2}{2} - c_1, \frac{a'_2 + b'_2}{2} - c'_1)$$

lies in the square $(a_2, b_2) \times (a'_2, b'_2)$ and therefore cannot be an ϵ-translation vector of f. Then there exists a square $(a_3, b_3) \times (a'_3, b'_3)$ which does not contain an ϵ-translation vector of f such that

$$|b_3 - a_3| > 2(|c_1| + |c_2|) \text{ and } |b'_3 - a'_3| > 2(|c'_1| + |c'_2|).$$

Proceeding in this way we may define the squares $(a_n, b_n) \times (a'_n, b'_n)$ such that

$$|b_n - a_n| > 2(|c_1| + \cdots + |c_{n-1}|) \text{ and } |b'_n - a'_n| > 2(|c'_1| + \cdots + |c'_{n-1}|).$$

and containing no element of S_ϵ where

$$(c_k, c'_k) = (\frac{a_k + b_k}{2}, \frac{a'_k + b'_k}{2}), \ 1 \leq k < n.$$

If $1 \leq k < n$, then $(c_n, c'_n) - (c_k, c'_k) \notin S_\epsilon$. Now

$$\|R_{(c_n,c'_n)}f - R_{(c_k,c'_k)}f\|_\infty = \|R_{(c_n,c'_n)-(c_k,c'_k)}f - f\| > \epsilon$$

and so f does not satisfy the Bochner condition. $\square$

Corollary 6.2.8. *The Bohr functions form a closed linear subspace of* $C_b(\mathbb{R}^2)$ *and hence every function in* $AP(\mathbb{R}^2)$ *is a Bohr function.*

Proof. If f_1, f_2 are Bohr functions, and c_1, c_2 are complex constants, and $((\lambda_k, \gamma_k))$ is a sequence in $\mathbb{R}^2$, then, by passing to a subsequence and relabelling, we may suppose without loss of generality that $(R_{(\lambda_k,\gamma_k)}f_1)$ and $(R_{(\lambda_k,\gamma_k)}f_2)$ are convergent sequences in $C_b(\mathbb{R}^2)$. It now follows easily that $(R_{(\lambda_k,\gamma_k)}(c_1f_1 + c_2f_2))$ is a convergent sequence in $\mathbb{R}^2$, and therefore $c_1f_1 + c_2f_2$ is a Bohr function.

Moreover, the class of Bohr functions is closed, in other words, if (f_n) is a sequence of Bohr functions converging to f, then f is a Bohr function too. To prove this let $f_n \to f$ then

$$\forall\, \epsilon > 0, \ \exists n_0 \in \mathbb{N} \text{ such that } \|f_n - f\|_\infty < \epsilon \text{ whenever } n \geq n_0.$$

In particular, the above statement is true for $\epsilon/3$ in place of ϵ and some $n = n_0$:

$$\|f_{n_0} - f\| < \epsilon/3.$$

Let (λ, γ) be an ϵ-translation vector of f_{n_0}, then

$$\begin{aligned}\|R_{(\lambda,\gamma)}f - f\|_\infty &< \|R_{(\lambda,\gamma)}f - R_{(\lambda,\gamma)}f_{n_0}\| \\ &+ \|R_{(\lambda,\gamma)}f_{n_0} - f_{n_0}\| \\ &+ \|f_{n_0} - f\| \\ &< \epsilon\end{aligned}$$

and so (λ, γ) is an ϵ-translation vector of f. Thus, if g is in the closure of the set of Bohr functions, we can find a relatively dense set of ϵ-translation vectors for g by taking a relatively dense set of $\epsilon/3$-translation vectors for any Bohr function f with $\|f - g\|_\infty < \epsilon/3$. Thus g is also a Bohr function. □

Corollary 6.2.9. *The Bohr functions form a closed subalgebra of $C_b(\mathbb{R}^2)$.*

Proof. To prove this all it remains to show is that the point-wise product of two Bohr functions $f_1.f_2$ is a Bohr function. If f_1, f_2 are Bohr functions, and $((\lambda_k, \gamma_k))$ is a sequence in $\mathbb{R}^2$, then, by passing to a subsequence and relabelling, we may suppose without loss of generality that $(R_{(\lambda_k,\gamma_k)}f_1)$ and $(R_{(\lambda_k,\gamma_k)}f_2)$ are convergent sequences in $C_b(\mathbb{R}^2)$. It now follows easily that $(R_{(\lambda_k,\gamma_k)}(f_1.f_2))$ is a convergent sequence in $\mathbb{R}^2$, and therefore $f_1.f_2$ is a Bohr function. □

We will now prove that every Bohr function f is a limit of trigonometric polynomials, that is, finite linear combinations of the functions $e_{(\lambda,\gamma)}$. To do this, we need to determine a countable set of frequencies that are present in f.

Proposition 6.2.10. *Let $f \in C_b(\mathbb{R}^2)$ be a Bohr function. Then*

$$[f,1] = \lim_{T\to\infty} [f,1]_T = \lim_{T\to\infty} \frac{1}{4T^2} \int\limits_{[-T,T]^2} f(x,y)\, dx\, dy$$

exists. Hence

$$[f,g] = \lim_{T\to\infty} [f,g]_T = \lim_{T\to\infty} \frac{1}{4T^2} \int\limits_{[-T,T]^2} f(x,y)\overline{g(x,y)}\, dx\, dy$$

is well defined for all Bohr functions f and g.

Proof. Suppose that $T > 0$ and $M = \|f\|_\infty$. Given $\epsilon > 0$ let L be such that every square of edge length L contains an ϵ-translation vector of f. Write

$$\begin{aligned}\frac{1}{4n^2T^2} &\int\limits_{[-nT,nT]^2} f(x,y)\, dx\, dy \\ &= \frac{1}{4n^2T^2} \sum_{k=-n}^{n-1} \sum_{k'=-n}^{n-1} \int\limits_{[kT,(k+1)T]} \int\limits_{[k'T,(k'+1)T]} f(x,y)\, dx\, dy\end{aligned}$$

and suppose that $T > \max\{L, ML/\epsilon\}$ then if (λ,γ) is an ϵ-translation vector

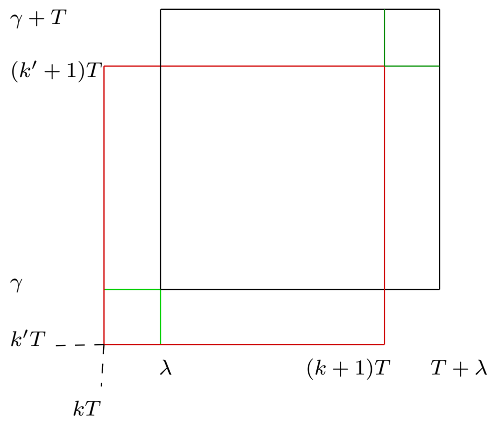

Figure 6.1: The area of integration (proof of Proposition 6.2.10)

of f in $[kT, kT+L] \times [k'T, k'T+L]$ we have

$$\begin{aligned}
\int\limits_{[kT,(k+1)T]} \int\limits_{[k'T,(k'+1)T]} f(x,y)\,dx\,dy &= \int\limits_{[0,T]^2} f(x,y)\,dx\,dy \\
&+ \int\limits_{[0,T]^2} (f(x+\lambda, y+\gamma) - f(x,y))\,dx\,dy \\
&- \int\limits_{[(k+1)T,T+\lambda]\times[\gamma,(k'+1)T]} f(x,y)\,dx\,dy \\
&- \int\limits_{[(k+1)T,T+\lambda]\times[(k'+1)T,T+\gamma]} f(x,y)\,dx\,dy \\
&- \int\limits_{[\lambda,(k+1)T]\times[(k'+1)T,T+\gamma]} f(x,y)\,dx\,dy
\end{aligned}$$

$$+\int_{[kT,\lambda]\times[\gamma,(k'+1)T]} f(x,y)\,dx\,dy$$
$$+\int_{[kT,\lambda]\times[k'T,\gamma]} f(x,y)\,dx\,dy$$
$$+\int_{[\lambda,(k+1)T]\times[k'T,\gamma]} f(x,y)\,dx\,dy.$$

Starting from the second integral, we may bound the terms in the last equality using the "size of the function" times the "length of the interval" rule. Hence adding up the $4n^2$ terms in the sum we obtain

$$|[f,1]_{nT} - [f,1]_T| \leq \epsilon + 6\frac{ML^2}{T^2} < 7\epsilon.$$

In particular, $[f,1]_{nT}$ remains bounded. For $U > 0$ sufficiently large, choose n such that $n^2T^2 < U^2 < (n^2+1)T^2$. Then

$$\begin{aligned}
|[f,1]_U - [f,1]_{nT}| &\leq |[f,1]_U - \frac{n^2T^2}{U^2}[f,1]_{nT}| + |\frac{n^2T^2}{U^2} - 1||[f,1]_{nT}| \\
&\leq \frac{1}{4U^2}4MT^2 + \frac{1}{n^2}|[f,1]_{nT}| \\
&\leq \frac{M}{n^2} + \frac{1}{n^2}|[f,1]_{nT}| \\
&< \epsilon
\end{aligned}$$

if U is sufficiently large. Thus

$$|[f,1]_U - [f,1]_T| < 8\epsilon$$

if U is sufficiently large. This gives $|[f,1]_U - [f,1]_V| < 16\epsilon$ when U and V are sufficiently large, implying the existence of the limit $[f,1]$.

Note also that $[f, g] = [f\overline{g}, 1]$ is defined for all Bohr functions f and g, since the product $f.\overline{g}$ is also a Bohr function. $\square$

Definition 6.2.11. Let f_1 and f_2 be Bohr functions then the *correlation or covariance function* of f_1 and f_2 is defined for $(x, y) \in \mathbb{R}^2$ by

$$\phi_{f_1,f_2}(x, y) = [R_{(-x,-y)}f_1, f_2] = \lim_{T\to\infty} \frac{1}{4T^2} \int_{[-T,T]^2} f_1(x + s, y + t)\overline{f_2(s, t)}\, ds\, dt.$$

Proposition 6.2.12. *If f_1 and f_2 are Bohr functions, then so is the covariance function $\phi = \phi_{f_1,f_2}$. Moreover, $[R_{(-x,-y)}f_1, f_2]_T \to \phi_{f_1,f_2}(x, y)$ uniformly in (x, y) as $T \to \infty$. Also, $[R_{(-x,-y)}f_1, f_2] = [f_1, R_{(x,y)}f_2]$.*

Proof. Since $\phi(x - \lambda, y - \gamma) - \phi(x, y)$ is equal to

$$\lim_{T\to\infty} \frac{1}{4T^2} \int_{[-T,T]^2} (f_1(x + s - \lambda, y + t - \gamma) - f_1(x + s, y + t))\, \overline{f_2(s, t)}\, ds\, dt$$

we have

$$\|R_{(\lambda,\gamma)}\phi - \phi\|_\infty \leq \|R_{(\lambda,\gamma)}f_1 - f_1\|_\infty \|f_2\|_\infty$$

from which we conclude that ϕ is also a Bohr function, since a δ-translation vector of f_1 is an ϵ-translation vector of ϕ as soon as $\delta\|f_2\|_\infty \leq \epsilon$.

For any fixed (x, y), the convergence of $[R_{(-x,-y)}f_1, f_2]_T$ is clear from Proposition 6.2.10. Now, given $\epsilon > 0$ we may use the Bochner property of f_1 to find $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$ such that for each $(x, y) \in \mathbb{R}^2$ there is a k with

$$\|R_{(-x,-y)}f_1 - R_{(-x_k,-y_k)}f_1\|_\infty < \epsilon.$$

Thus

$$|[R_{(-x,-y)}f_1, f_2]_T - [R_{(-x_k,-y_k)}f_1, f_2]_T| < \epsilon\|f_2\|_\infty$$

for all $T > 0$. Moreover, there is a number T_0 such that

$$|[R_{(-x_k,-y_k)}f_1, f_2]_T - [R_{(-x_k,-y_k)}f_1, f_2]| < \epsilon$$

for all $T \geq T_0$, for all the finite collection $k = 1, \ldots, n$.

By the triangle inequality, if $T \geq T_0$, then

$$\begin{aligned}
|[R_{(-x,-y)}f_1, f_2]_T - [R_{(-x,-y)}f_1, f_2]| &\leq |[R_{(-x,-y)}f_1, f_2]_T - [R_{(-x_k,-y_k)}f_1, f_2]_T| \\
&\quad + |[R_{(-x_k,-y_k)}f_1, f_2]_T - [R_{(-x_k,-y_k)}f_1, f_2]| \\
&\quad + |[R_{(-x_k,-y_k)}f_1, f_2] - [R_{(-x,-y)}f_1, f_2]| \\
&< \epsilon\|f_2\|_\infty + \epsilon + \epsilon\|f_2\|_\infty \\
&= \epsilon(1 + 2\|f_2\|_\infty)
\end{aligned}$$

for all (x, y). Note that

$$\begin{aligned}
\left[R_{(-x,-y)}f_1, f_2\right]_T &= \frac{1}{4T^2} \int\limits_{[-T,T]^2} f_1(x+s, y+t)\overline{f_2(s,t)}\, ds\, dt \\
&= \frac{1}{4T^2} \int\limits_{[-T+y,T+y]} \int\limits_{[-T+x,T+x]} f_1(u,v)\overline{f_2(u-x, v-y)}\, du\, dv \\
&= [f_1, R_{(x,y)}f_2]_T + \delta(T)
\end{aligned}$$

where $|\delta(T)| \leq \|(x,y)\|\|f_1\|_\infty\|f_2\|_\infty/T^2$, and so

$$[R_{(-x,-y)}f_1, f_2] = [f_1, R_{(x,y)}f_2].$$

$\square$

Note that the mapping $(f,g) \mapsto [f,g]$ satisfies all the axioms for an inner product on the class of Bohr functions except possibly the positive definiteness condition. In fact, it satisfies that too, we will prove this in Theorem 6.2.14. This enables us to develop an inner product space theory of almost periodic functions in a simple manner. Recall that, in any inner product space, if $u_1, \ldots, u_n$ is an orthonormal sequence and x any vector, then setting

$$u = \sum_{k=1}^{n} \langle x, u_k \rangle u_k$$

we have that $x - u$ is orthogonal to every u_k and hence it is orthogonal to u itself. By Pythagoras's theorem

$$\|x\|^2 = \|x-u\|^2 + \|u\|^2 \geq \|u\|^2 = \sum_{k=1}^{n} |\langle x, u_k \rangle|^2 \tag{6.1}$$

which is Bessel's inequality.

Theorem 6.2.13. *Let f be a Bohr function. Then Bessel's inequality holds, in the following form*

$$[f,f] \geq \sum_{k=1}^{n} \left|[f, e_{(\lambda_k,\gamma_k)}]\right|^2$$

for all distinct $(\lambda_1, \gamma_1), \ldots, (\lambda_n, \gamma_n) \in \mathbb{R}^2$*. Hence* $[f, e_{(\lambda,\gamma)}] \neq 0$ *for at most a countable set of* $(\lambda, \gamma) \in \mathbb{R}^2$*.*

Proof. Bessel's inequality can be obtained directly from 6.1, writing f for x and $e_{(\lambda_k,\gamma_k)}$ for u_k. This implies that, for any $N > 1$, we have $[f, e_{(\lambda,\gamma)}] > 1/N^2$ for at most finitely many (λ, γ). Hence the total number of non-zero Fourier coefficients $[f, e_{(\lambda,\gamma)}]$ is at most countable. □

We can now prove that $(f, g) \mapsto [f, g]$ is a genuine inner product on the Bohr functions, that is positive definite.

Theorem 6.2.14. *Let* f *be a Bohr function that is not identically zero. Then* $[f, f] > 0$*.*

Proof. If f is not identically zero, then there is an $\epsilon > 0$ and an $(a, b) \in \mathbb{R}^2$ such that $|f(a, b)| > \epsilon$. By continuity, we may find $\delta > 0$ such that $|f| > \epsilon/2$ on the square $(a - \delta, a + \delta) \times (b - \delta, b + \delta)$. Let L be such that every square of side length L contains an $\epsilon/4$-translation vector of f. Then for $n \geq 1$ we have

$$\frac{1}{4n^2L^2} \int_{[-nL,nL]^2} |f(x, y)|^2 \, dx \, dy \geq \frac{\delta^2\epsilon^2}{16L^2}.$$

To see this note that each square $[kL-a, (k'+1)L-a] \times [kL-b, (k'+1)L-b]$ contains an $\epsilon/4$-translation vector (μ, τ) of f, implying that $|f| > \epsilon/4$ on

$$(a + \mu - \delta, a + \mu - \delta) \times (b + \tau - \delta, b + \tau - \delta) \cap [kL, (k' + 1)L]^2;$$

this has area at least δ^2 since $(a+\mu_k, b+\tau_k) \in [kL,(k+1)L]^2$. Hence $[f,f] \geq \dfrac{\delta^2\epsilon^2}{16L^2}$ as required. $\square$

Remark 6.2.15. Suppose that (f_k) is a sequence of Bohr functions for which $[f_k - f, f_k - f] \to 0$ for some Bohr function f. If an additional property holds, namely, that for each $\epsilon > 0$ every ϵ-translation vector of f is an ϵ-translation vector of all the functions f_k, then it follows by the same argument as above that f_k tends to f *uniformly*.

Lemma 6.2.16. *Suppose that f is a Bohr function such that $[f, e_{(\lambda,\gamma)}] = 0$ for all $(\lambda,\gamma) \in \mathbb{R}^2$. Then $[f, e_{(\lambda,\gamma)}]_T \to 0$ uniformly as $T \to \infty$ uniformly in (λ,γ).*

Proof. Suppose the contrary, so that $\left|[f, e_{(\lambda_n,\gamma_n)}]_{T_n}\right| \geq \epsilon > 0$ for sequences $(\lambda_n,\gamma_n) \subset \mathbb{R}^2$ and T_n tending to ∞. Observe that, for all $(\lambda,\gamma) \neq (0,0)$,

$$\left[f, e_{(\lambda,\gamma)}\right]_T = \frac{1}{4T^2} \int\limits_{[-T,T]^2} f(x,y)e^{-i(\lambda x+\gamma y)}\, dx\, dy$$

which is equal to

$$\frac{1}{4T^2} \int\limits_{[-T+\frac{\pi}{\gamma},T+\frac{\pi}{\gamma}]} \int\limits_{[-T+\frac{\pi}{\lambda},T+\frac{\pi}{\lambda}]} f(x-\tfrac{\pi}{\lambda}, y-\tfrac{\pi}{\gamma}).e^{-i(\lambda(x-\frac{\pi}{\lambda})+\gamma(y-\frac{\pi}{\gamma}))}\, dx\, dy.$$

It follows that $[f, e_{(\lambda,\gamma)}]_T$ is equal to

$$\frac{1}{8T^2} \int\limits_{[-T,T]^2} (f(x,y) - f(x-\tfrac{\pi}{\lambda}, y-\tfrac{\pi}{\gamma}))e^{-i(\lambda x+\gamma y)}\, dx\, dy + \delta(\lambda,\gamma),$$

where the first term tends to zero uniformly (in T) as $\|(\lambda,\gamma)\| \to \infty$, by the uniform continuity of f, and

$$\delta(\lambda,\gamma) = \frac{-1}{8T}\Big(\int\limits_{-T+\frac{\pi}{\gamma}}^{-T}\int\limits_{-T+\frac{\pi}{\lambda}}^{-T} + \int\limits_{T}^{T+\frac{\pi}{\gamma}}\int\limits_{T}^{T+\frac{\pi}{\lambda}}\Big)(f(x-\frac{\pi}{\lambda}, y-\frac{\pi}{\gamma}))e^{-i(\lambda x+\gamma y)}\,dx\,dy$$

$$= O(\frac{1}{T}(\frac{1}{\lambda},\frac{1}{\gamma})).$$

It follows that the given sequence (λ_n,γ_n) must remain bounded and has a convergent subsequence. By relabelling we may suppose without loss of generality that $(\lambda_n,\gamma_n)\to(\lambda,\gamma)$. Write $(\lambda_n,\gamma_n)=(\lambda_n+\delta_n,\gamma_n+\mu_n)$ where $(\delta_n,\mu_n)\to(0,0)$. By Theorem 6.2.12,

$$[R_{(-s,-t)}f, e_{(\lambda,\gamma)}]_U \to [R_{(-s,-t)}f, e_{(\lambda,\gamma)}] = [f, R_{(s,t)}e_{(\lambda,\gamma)}] = 0$$

uniformly in (s,t) as $U\to\infty$, so we can find a number $U_0>0$ such that $\big|[R_{(-s,-t)}f, e_{(\lambda,\gamma)}]_U\big| < \epsilon/2$ for all $U\geq U_0$ and all $(s,t)\in\mathbb{R}^2$. Now, given $T_n>U_0$, we may write $T_n=NU$ for some U with $U_0<U<2U_0$ and $N\in\mathbb{N}$, both depending on n. Therefore

$$\begin{aligned}\big[f, e_{(\lambda_n,\gamma_n)}\big]_{T_n} = \frac{1}{N^2}&\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} e^{-i(\lambda_n(2j+1-N)U+\gamma_n(2k+1-N)U)}\\ &.\frac{1}{4U^2}\int\limits_{[-U,U]^2} f(x+(2j+1-N)U, y+(2k+1-N)U)\\ &.e^{-i(\lambda_n x+\gamma_n y)}e^{-i(\delta_n x+\mu_n y)}\,dx\,dy.\end{aligned}$$

But $e^{-i(\delta_n x+\mu_n y)}\to 1$ as $n\to\infty$ uniformly for $(x,y)\in[-2U_0,2U_0]^2$ so that $\big|[f,e_{(\lambda_n,\gamma_n)}]_{T_n}\big|<\epsilon$ for sufficiently large n. This is a contradiction, and the result follows. $\square$

Theorem 6.2.17 (Uniqueness theorem)**.** *Let f be a Bohr function such that $[f, e_{(\lambda,\gamma)}] = 0$ for all $(\lambda, \gamma) \in \mathbb{R}^2$. Then f is identically zero.*

Proof. We begin by defining for each $T > 0$ an auxiliary function f_T that equals f on the square $(-T, T)^2$ and is $(2T, 2T)$-periodic. Thus f has a Fourier series

$$f_T(x, y) \sim \sum_{j,k=-\infty}^{\infty} a_{j,k} e^{i(\pi jx/T + \pi ky/T)},$$

and Parseval's identity gives us

$$\frac{1}{4T^2} \int_{[-T,T]^2} |f(x, y)| \; dx \, dy = \sum_{j,k=-\infty}^{\infty} |a_{j,k}|^2.$$

The proof now proceeds by working with the quantity $\sum_{j,k=-\infty}^{\infty} |a_{j,k}|^4$, which depends on T; we note that, given $\epsilon > 0$, we have for sufficiently large T that $|a_{j,k}| = \left|[f, e_{(\frac{\pi j}{T}, \frac{\pi k}{T})}]_T\right| < \epsilon$ for all j, k by Lemma 6.2.16 Thus

$$\sum_{j,k=-\infty}^{\infty} |a_{j,k}|^4 < \epsilon^2 \sum_{j,k=-\infty}^{\infty} |a_{j,k}|^2 \leq \epsilon^2 \, \|f\|_\infty^2 \, . \tag{6.2}$$

We now construct a new $(2T, 2T)$-periodic function g_T (an *autocorrelation* function) defined by

$$g_T(x, y) = \frac{1}{4T^2} \int_{[-T,T]^2} f_T(x + s, y + t) \overline{f_T(s, t)} \, ds \, dt. \tag{6.3}$$

We can verify that the Fourier coefficients of g_T are equal to $|a_{j,k}|^2$ by a simple change of order of integration or by an approximation argument based on the relation for finite sums:

$$\begin{aligned} \frac{1}{4T^2} \int\limits_{[-T,T]^2} & \Big(\sum_{j,k=-N}^{N} a_{j,k} e^{i(\pi j(x+s)/T + \pi k(y+t)/T)} \Big) \\ & . \Big(\sum_{l,m=-N}^{N} \overline{a_{l,m}} e^{-i(\pi l s/T + \pi m t/T)} \Big) \, ds \, dt \\ & = \sum_{j,k=-N}^{N} |a_{j,k}|^2 \, e^{i(\pi j x/T + \pi k y/T)}. \end{aligned}$$

To obtain a function with coefficients $|a_{j,k}|^4$, it is enough to repeat the construction and define

$$h_T(x,y) = \frac{1}{4T^2} \int\limits_{[-T,T]^2} g_T(x+s, y+t) \overline{g_T(s,t)} \, ds \, dt. \tag{6.4}$$

Now $h_T(0,0) = \sum\limits_{j,k=-\infty}^{\infty} |a_{j,k}|^4$, because the Fourier series of h_T converges absolutely and hence pointwise. This tends to zero as $T \to \infty$ by 6.2, and so $[g_T, g_T]_T \to 0$ as $T \to \infty$ by 6.4. Now take $T_n \to \infty$ such that (T_n, T_n) is an $1/n$-translation vector of f and note that, for $(x,y) \in [0,T_n]^2$, we have

$$\begin{aligned}
g_T(x,y) &= \frac{1}{4T^2} \int\limits_{[-T_n,T_n-y]} \int\limits_{[-T_n,T_n-x]} f(x+s,y+t)\overline{f(s,t)}\,ds\,dt \\
&+ \frac{1}{4T^2} \int\limits_{[-T_n-y,T_n]} \int\limits_{[-T_n-x,T_n]} f(x+s-T_n,y+t-T_n)\overline{f(s,t)}\,ds\,dt \\
&= \frac{1}{4T^2} \int\limits_{[-T_n,T_n]^2} f(x+s,y+t)\overline{f(s,t)}\,ds\,dt + \delta_n
\end{aligned}$$

where $|\delta_n| \le \|f\|_\infty^2 / n^2$ and the same estimate holds for $(x,y) \in [-T_n,0]^2$. Recall from Proposition 6.2.12 that the function $g : \mathbb{R}^2 \to \mathbb{C}$ defined by

$$g(x,y) = [R_{(-x,-y)}f, f] = \lim_{U\to\infty} \frac{1}{4U^2} \int\limits_{[-U,U]^2} f(x+s,y+t)\overline{f(s,t)}\,ds\,dt$$

is also a Bohr function and the convergence of the right hand side to $g(x,y)$ as $U \to \infty$ is uniform in (x,y). We see therefore that

$$\eta_n = \sup\{|g_{T_n}(x,y) - g(x,y)| : (x,y) \in [-T_n,T_n]^2\} \to 0 \text{ as } n \to \infty.$$

Moreover,

$$\begin{aligned}
|[g_{T_n}, g_{T_n}]_{T_n} - [g,g]_{T_n}| &\le \frac{1}{4T^2} \int\limits_{[-T_n,T_n]^2} (|g_{T_n}(x,y)| + |g(x,y)|) \\
&\quad .(|g_{T_n}(x,y) - g(x,y)|)\,dx\,dy \\
&\le 4\,\|f\|_\infty^4\,\eta_n^2
\end{aligned}$$

which tends to zero as n tends to infinity, Thus $[g,g] = 0$, and so g is identically zero by Theorem 6.2.14. But $g(0,0) = [f,f]$, and we conclude that f is identically zero. $\square$

Theorem 6.2.18 (Parseval's identity)**.** *Let f be a Bohr function. Then* $[f,f] = \sum_{(\lambda,\gamma)\in\mathbb{R}^2} \left|[f, e_{(\lambda,\gamma)}\right|^2$.

Proof. Working again with the almost periodic covariance function $g = \phi_{f,f}$ note that

$$[g, e_{(\lambda,\gamma)}] = \lim_{T\to\infty} \lim_{X\to\infty} \frac{1}{4X^2} \int_{[-X,X]^2} e^{-i(\lambda x+\gamma y)} [R_{(-x,-y)}f, f]_T \, dx \, dy$$

since $[R_{(-x,-y)}f, f]_T \to g(x,y)$ uniformly on $\mathbb{R}^2$, by Proposition 6.2.12. Using Fubini's theorem, this gives

$$\begin{aligned}\left[g, e_{(\lambda,\gamma)}\right] = \lim_{T\to\infty} \lim_{X\to\infty} \frac{1}{4X^2} \int_{[-X,X]^2} \frac{1}{4T^2} \int_{[-T,T]^2} e^{-i(\lambda(x+s)+\gamma(y+t))} e^{i(\lambda s+\gamma t)} \\ f(x+s, y+t)\overline{f(s,t)} \, ds \, dt \\ = \frac{1}{4T^2} \int_{[-T,T]^2} [f, e_{(\lambda,\gamma)}] e^{i(\lambda s+\gamma t)} \overline{f(s,t)} \, ds \, dt \\ = \left|[f, e_{(\lambda,\gamma)}]\right|^2 .\end{aligned}$$

Now $\sum_{(\lambda,\gamma)\in\mathbb{R}^2} \left|[f, e_{(\lambda,\gamma)}]\right|^2 < \infty$, by Theorem 6.2.13, and so the series

$$\sum_{(\lambda,\gamma)\in\mathbb{R}^2} \left|[f, e_{(\lambda,\gamma)}]\right|^2 e_{(\lambda,\gamma)}$$

converges to a Bohr function h whose Fourier coefficients satisfy $[h, e_{(\lambda,\gamma)}] = \left|[f, e_{(\lambda,\gamma)}]\right|^2$ for all (λ,γ) because of the uniform convergence, and hence $g = h$ by the uniqueness theorem. Evaluating at $(x,y) = (0,0)$, we find that $\sum_{(\lambda,\gamma)\in\mathbb{R}^2} \left|[f, e_{(\lambda,\gamma)}]\right|^2 = g(0,0) = [f,f]$ as required. $\square$

It remains to show that every Bohr function is in $AP(\mathbb{R}^2)$, the closed linear span of the functions $e_{(\lambda,\gamma)}$. To do this, we take an arbitrary Bohr function f and consider the set $\Lambda = \{(\lambda,\gamma) \in \mathbb{R}^2 : [f, e_{(\lambda,\gamma)}] \neq 0\}$. If this set is finite, then there is no problem, since we form the trigonometric polynomial $h = \sum\limits_{(\lambda,\gamma)\in\Lambda} [f, e_{(\lambda,\gamma)}]e_{(\lambda,\gamma)}$. Now h has the same Fourier coefficients as f, and so, by the uniqueness theorem, $f = h$. We may therefore suppose without loss of generality that Λ is countably infinite, say

$$\Lambda = \{(\lambda_1,\gamma_1),(\lambda_2,\gamma_2),\dots\}.$$

The first step in the approximation procedure is to reduce Λ to a maximal subset $B = \{(\alpha_1,\beta_1),(\alpha_2,\beta_2),\dots\}$ (possibly finite) that is linearly independent over $\mathbb{Q}$. This can be done recursively, by successively deleting (λ_k,γ_k) if it is a linear combination with rational coefficients of (λ_j,γ_j) for $j < k$. We shall assume without loss of generality that B is countably infinite (if not, we extend it to a countably independent set by adding in new members). For a fixed positive integer n, let E_n be the finite set consisting of members of the form

$$(\lambda,\gamma) = (\sum_{k=1}^{n} \frac{m_k}{n!}\alpha_k, \sum_{k=1}^{n} \frac{m'_k}{n!}\beta_k)$$

where $m_k, m'_k \in \mathbb{Z}$ and $|m_k| \leq n.n!$, $|m'_k| \leq n.n!$ for each k. Since any rational coefficient p/q can be written as $m/n!$ and $m'/n!$ with $|m| \leq n.n!$, $|m'| \leq n.n!$ for a sufficiently large n, then $\Lambda \subseteq E = \cup_{n=1}^{\infty} E_n$.

Definition 6.2.19. Given a countable set $B = \{(\alpha_1,\beta_1),(\alpha_2,\beta_2),\dots\}$, linearly independent over $\mathbb{Q}$, and a positive integer n, the *Fejér-Bochner kernel*

K'_n corresponding to B is given by

$$K'_n(s,t) = \prod_{k=1}^{n} K_{n.n!-1}(\frac{\alpha_k s}{n!}, \frac{\beta_k t}{n!}),$$

where K denotes the "standard" Fejér kernel, defined for $p \in \mathbb{Z}_+$ and $(\rho, \omega) \in \mathbb{R}^2$ by the formula

$$K_p(\rho,\omega) = K_p(\rho)K_p(\omega) = \sum_{m=-p-1}^{p+1} \sum_{m'=-p-1}^{p+1} (1 - \frac{|m|}{p+1})(1 - \frac{|m'|}{p+1})e^{i(m\rho + m'\omega)}.$$

We may write $K'_n(s,t) = \sum_{(\lambda,\gamma)\in E_n} k_n(\lambda,\gamma)e^{i(\lambda s + \gamma t)}$, in which case the following holds

1. $K'_n \geq 0$ for all $(s,t) \in \mathbb{R}^2$,since K'_n is a product of "standard" Fejér kernels;

2. $k_n(\lambda,\gamma) = k_n(-\lambda,-\gamma)$ for each (λ,γ), and hence $K'_n(s,t) = K'_n(-s,-t)$ for all $(s,t) \in \mathbb{R}^2$; also $k_n(0,0) = 1$;

3. for each $(\lambda,\gamma) \in E$, we have $0 \leq k_n(\lambda,\gamma) \leq 1$ for all n, and $k_n(\lambda,\gamma) \to 1$ as $n \to \infty$.

Part 3 holds because, if

$$(\lambda,\gamma) = (\sum_{k=1}^{r} \frac{m_k}{r!}\alpha_k, \sum_{k=1}^{r} \frac{m'_k}{r!}\beta_k) \in E_r,$$

then

$$k_n(\lambda,\gamma) = \prod_{k=1}^{r}(1 - \frac{|m_k|}{n.n!}) \prod_{k=1}^{r}(1 - \frac{|m'_k|}{n.n!})$$

if $n \geq r$, and this tends to 1 as $n \to \infty$.

Using this kernel, we are now able to prove following fundamental approximation theorem. It is the explicit formula aspect, for the approximants, that we shall find particularly useful for almost periodic flexes in Chapter 7.

Theorem 6.2.20. *Given a Bohr function f, let $(f_n)_{n=1}^{\infty}$ denote the sequence of trigonometric polynomials defined by*

$$\begin{aligned} f_n(x,y) &= [R_{(-x,-y)}f, K'_n] \\ &= [f, R_{(x,y)}K'_n] \\ &= \lim_{T\to\infty} \frac{1}{4T^2} \int\limits_{[-T,T]^2} f(s,t)K'_n(s-x,t-y)\,ds\,dt, \ \ (x,y)\in\mathbb{R}^2. \end{aligned}$$

Then $\|f-f_n\|\to 0$ as $n\to\infty$ and thus f is almost periodic.

Proof. Observe that for $(x,y)\in\mathbb{R}^2$ we have

$$\begin{aligned} f_n(x,y) &= \sum_{(\lambda,\gamma)\in E_n} [f, k_n(\lambda,\gamma)R_{(x,y)}e_{(\lambda,\gamma)}] \\ &= \sum_{(\lambda,\gamma)\in E_n} k_n(\lambda,\gamma)[f,e_{(\lambda,\gamma)}]e_{(\lambda,\gamma)}(x,y). \end{aligned}$$

Using Parseval's identity, it is clear by the dominated convergence theorem that $[f-f_n,f-f_n]\to 0$, since $[f-f_n,e_{(\lambda,\gamma)}]\to 0$ for all (λ,γ) and $\big|[f-f_n,e_{(\lambda,\gamma)}]\big| \le \big|[f,e_{(\lambda,\gamma)}]\big|$ for each (λ,γ).

We observe that any ϵ-translation vector (κ,τ) pertaining to f is also an

ϵ-translation vector of f_n, since

$$\begin{aligned}|f_n(x-\kappa, y-\tau) - f_n(x,y)| &= [R_{(\kappa,\tau)}f - f, R_{(x,y)}K_n'] \\ &\leq \epsilon \lim_{T\to\infty} \frac{1}{4T^2} \int\limits_{[-T,T]^2} K_n'(s-x, t-y)\, ds\, dt = \epsilon,\end{aligned}$$

using the positivity of K_n' and the fact that $k_n(0,0) = 1$. The proof is now completed by using Remark 6.2.15. $\square$

We have now completed the circle of ideas that identifies the bi-variable Bohr functions with the almost periodic functions, that is, the uniform limits of trigonometric polynomials. In fact the theory of a single variable almost periodic function can be deduced from that of a two variable function when fixing one of the variables, as such an almost periodic function will be almost periodic with respect to each one of those variables. Although one has to be careful, since the converse is not true. For example $f(x,y) = \cos xy$ is almost periodic in each variable separately when the other variable is fixed but f is not an almost periodic function.

As mentioned previously, the next step of taking the theory of almost periodic functions to higher dimensions is very much similar to the two dimensional case, we state some of the main definitions and statements:

Definition 6.2.21 (Approximation by trigonometric polynomials)**.** Let $f : \mathbb{R}^d \to \mathbb{C}$ we say that f is *(uniformly) almost periodic u.a.p.* if for all $\epsilon > 0$

there exists a trigonometric polynomial $P : \mathbb{R}^d \to \mathbb{C}$ such that:

$$|f(x) - P(x)| < \epsilon \text{ for all vectors } x \in \mathbb{R}^d$$

we denote the set of all functions f satisfying the above condition by $AP(\mathbb{R}^d)$.

Definition 6.2.22. Let $f : \mathbb{R}^d \to \mathbb{C}$ be any function on $\mathbb{R}^d$, the *right shift* R_λ of f is defined by

$$(R_\lambda f)(x) = f((x_k - \lambda_k)_{k=1}^d).$$

Definition 6.2.23. Let $f : \mathbb{R}^d \to \mathbb{C}$ be a continuous function, a vector $\lambda \in \mathbb{R}^d$ is called an *ϵ-translation vector* of f if

$$\|R_\lambda f - f\|_\infty \leq \epsilon.$$

Definition 6.2.24. A set $S \subseteq \mathbb{R}^d$ is said to be *relatively dense* if there exists an $L > 0$ such that every d-dimensional box of side length L contains an element of S.

Definition 6.2.25. Let f be a function $f : \mathbb{R}^d \to \mathbb{C}$.

i. We say that f is a Bohr function if f is continuous and for every $\epsilon > 0$ the set of ϵ-translation vectors of f is relatively dense.

ii. We say that f satisfies Bochner's condition if it is continuous and bounded and the set of translates $\{R_\lambda f,\, \lambda \in \mathbb{R}^d\}$ is relatively compact in $C_b(\mathbb{R}^d)$.

As for an almost periodic function of two variables, the above definitions of almost periodicity of functions on $\mathbb{R}^d$ are equivalent.

Chapter 7

Almost Periodic Rigidity

We now consider infinitesimal flexes of a crystal framework which are almost periodic in the sense of Bohr. We prove that a crystal framework is almost periodically infinitesimally rigid if and only if for some choice of translation group, and hence for every choice of translation group, it is periodically infinitesimally rigid, and the corresponding RUM spectrum is the minimal set $\{(1, 1, \ldots, 1)\}$. More generally, we show how the almost periodic infinitesimal flexes of $\mathcal{C}$ are determined in terms of the matrix function $\Phi_{\mathcal{C}}(z)$.

7.1 Almost Periodic Sequences

In this section we state some of the basic definitions regarding almost periodic sequences in $\mathbb{Z}^d$. In the following section these definitions will be used to introduce the notions of *almost periodic velocities and flexes*.

Definition 7.1.1. The class $AP(\mathbb{Z})$ of *uniformly almost periodic sequences* is the closed linear span in $\ell^\infty(\mathbb{Z})$ of finite linear combinations of $\{e_\lambda(n) : n \in \mathbb{Z}\}$.

Definition 7.1.2. Let $f : \mathbb{Z} \to \mathbb{C}$ be any function on $\mathbb{Z}$, the *right shift* R_k of f is defined by

$$(R_k f)(n) = f(n-k).$$

Definition 7.1.3. Let $f : \mathbb{Z} \to \mathbb{C}$ be a function on $\mathbb{Z}$. An integer $k \in \mathbb{Z}$ is called an ϵ-*translation number* of f if

$$\|R_k f - f\|_\infty \leq \epsilon.$$

Definition 7.1.4. A set $S \subseteq \mathbb{Z}$ is said to be *relatively dense* if there exists an integer $L > 0$ such that among the integers in any interval of length L there is an element of S.

Definition 7.1.5. Let f be a function $f : \mathbb{Z} \to \mathbb{C}$. Then

i. f is a *Bohr function* if for every $\epsilon > 0$ the set of ϵ-translation numbers of f is relatively dense.

ii. f satisfies *Bochner's condition* if any sequence of translates $R_{m_k} f$ has a subsequence that is uniformly convergent.

Theorem 7.1.6. *Every Bohr sequence is bounded.*

Definition 7.1.7. A sequence $f = (f_m)_{m=1}^{\infty}$ of almost periodic sequences $f_m = (f_m(n))_{n\in\mathbb{Z}}$ converges to a sequence $g = (g(n))$ if $\lim_{m\to\infty}\|f_m - g\|_\infty = 0$, where $\|f\|_\infty = \sup_{n\in\mathbb{Z}}\|f(n)\|_2$.

Theorem 7.1.8. *A necessary and sufficient condition for $f \in AP(\mathbb{Z})$ is the existence of a function $F \in AP(\mathbb{R})$ such that $f(n) = F(n)$, $n \in \mathbb{Z}$.*

Theorem 7.1.9. *Let $f = (f(n))$ be an almost periodic sequence. Then the mean value*

$$[f, 1] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} f(k)$$

exists.

Definition 7.1.10. Let f be a function $f : \mathbb{Z} \to \mathbb{C}$. The *Bohr spectrum* of f is the set

$$\Lambda_f = \{e^{i\lambda} : \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} f(k)e^{-i\lambda k} \neq 0\}.$$

In the theory of almost periodic functions, the approximating polynomials, are given by the covariance function of f and the corresponding Fejér-Bochner kernel K_n'. In the discrete case, following the same discussion as in continuous functions, for the Bohr sequence f, the Fejér-Bochner kernel K_n' can be written

$$K_n'(m) = \sum_{\lambda\in E_n} k_n(\lambda)e^{i\lambda m},\ m \in \mathbb{Z}.$$

Using this kernel, we can state the approximation theorem.

Theorem 7.1.11. *Given a Bohr sequence f, let (g_n) denote the sequence of trigonometric polynomials defined by*

$$g_n(l) = [f, R_l K_n'] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} f(k) K_n'(k-l),\ l \in \mathbb{Z}.$$

Then $\|f - g_n\|_\infty \to 0$ as $n \to \infty$ and thus f is almost periodic.

The following Theorems are the "single variable version" of the corresponding Theorems for velocity sequences and flexes of a crystal framework. Understanding these cases will make it clear how the generalization in the next section is obtained.

Theorem 7.1.12. *Let f be a Bohr sequence and $R : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ a continuous linear operator that commutes with shifts. If $f \in \ker R$, then $g \in \ker R$ where g is an approximating trigonometric sequence of f.*

Proof. From the approximation theorem, f can be approximated by a sequence of linear combinations of "pure frequencies" e_λ, $\lambda \in \mathbb{R}$. Explicitly, the sequence f is the uniform limit of the sequence $g = (g_n)$, where for $l \in \mathbb{Z}$:

$$\begin{aligned}
g_n(l) &= [f, R_l K'_n] \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} f(k) K'_n(k-l) \\
&= \sum_{\lambda\in E_n} k_n(\lambda) \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} f(k) e^{-i\lambda(k-l)} \\
&= \sum_{\lambda\in E_n} k_n(\lambda) \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} f(k) e^{i\lambda(l-k)} \\
&= \sum_{\lambda\in E_n} k_n(\lambda) \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k'=l+N}^{l-N} f(l-k') e^{i\lambda k'}.
\end{aligned}$$

Now we obtain

$$\begin{aligned}
(R(g_n))_l &= R\Big(\sum_{\lambda\in E_n} k_n(\lambda) \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k'=l+N}^{l-N} f(l-k') e^{i\lambda k'}\Big) \\
&= \sum_{\lambda\in E_n} k_n(\lambda) \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k'=l+N}^{l-N} R(f(l-k')) e^{i\lambda k'} \\
&= \sum_{\lambda\in E_n} k_n(\lambda) \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k'=l+N}^{l-N} R R_{k'}(f(l) e^{i\lambda k'}).
\end{aligned}$$

But $f \in \ker R$ and R commutes with the shift from which it follows that

$$RR_{k'}(f(l)) = R_{k'}R(f(l)) = 0 \text{ for all } l \in \mathbb{Z}.$$

Thus $R(g_n)$ is equal to the zero sequence and it follows from Theorem 7.1.11 that g lies in the kernel of R. $\square$

Theorem 7.1.13. *Let $f = (f(k))$ be a trigonometric polynomial sequence where*

$$f(k) = \sum_{j=0}^{r} a_j e_{\lambda_j}(k),\ e_{\lambda_j}(k) = e^{ik\lambda_j},$$

and let R be a linear operator $R : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ that commutes with the shift operator. If $f \in \ker R$, then for all j the pure frequency sequences lie in $\ker R$ too, i.e.

$$(e_{\lambda_j}(k))_{k\in\mathbb{Z}} \in \ker R \textit{ for all } j.$$

Proof. Note that for each j

$$(e_{\lambda_j}(k))_{k\in\mathbb{Z}} = (e^{ik\lambda_j})_{k\in\mathbb{Z}} = (\dots, e^{-i2\lambda_j}, e^{-i\lambda j}, 1, e^{i\lambda_j}, e^{i2\lambda_j}, \dots).$$

Define

$$T_N : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$$

such that

$$T_N(f) = \frac{1}{N+1} \sum_{r=1}^{N} W^r f$$

where W^r is the backward shift of the sequence by r steps. Since R is a linear transformation and f is in the kernel of R and R commutes with shifts, it follows that $T_N(f)$ lies in the kernel of R.

Suppose $\lambda_0 = 0$, then $e_{\lambda_0} = (\dots, 1, 1, \dots)$. It follows that $T_N(e_{\lambda_0}) = e_{\lambda_0}$ for all N and the sequence $(T_N(a_0 e_{\lambda_0})) = (a_0 e_{\lambda_0})$ which converges to $a_0 e_{\lambda_0}$ as $N \to \infty$. When $j = 1$ note that

$$(e_{\lambda_1}(k))_{k\in\mathbb{Z}} = (\dots, e^{-i2\lambda_1}, e^{-i\lambda 1}, 1, e^{i\lambda_1}, e^{i2\lambda_1}, \dots)$$

and

$$\begin{aligned}
T_N(e_{\lambda_1}) =&(\dots, \frac{e^{-i\lambda_1} + \dots + e^{-i(N+1)\lambda_1}}{N+1},\\
&\frac{1 + e^{-i\lambda_1} + \dots + e^{-iN\lambda_1}}{N+1},\\
&\frac{e^{i\lambda_1} + 1 + e^{-i\lambda_1} + \dots + e^{-i(N-1)\lambda_1}}{N+1}, \dots)\\
=&(\dots, e^{-i\lambda_1}(\frac{1 + e^{-i\lambda_1} + \dots + e^{-iN\lambda_1}}{N+1}),\\
&\frac{1 + e^{-i\lambda_1} + \dots + e^{-iN\lambda_1}}{N+1},\\
&e^{i\lambda_1}(\frac{1 + e^{-i\lambda_1} + \dots + e^{-iN\lambda_1}}{N+1}), \dots).
\end{aligned}$$

This is equal to the sequence $(e^{in\lambda_1}(\frac{1}{N+1}\sum_{k=0}^{N} e^{-ik\lambda_1}))_{n\in\mathbb{Z}}$ which for $\lambda_j \neq 0$, converges to the zero sequence as $N \to \infty$. From all the above, we conclude that

$$T_N(f) = \sum_{i=0}^{r} a_{\lambda_j} T_N(e_{\lambda_j}) \to a_0 e_{\lambda_0} + 0 + \dots + 0.$$

Therefore $T_N(f)$ converges to the sequence $(a_0 e_{\lambda_0})$ and $(a_0 e_{\lambda_0})$ is in the kernel of R.

Furthermore, if we write $T_N(f, e_{i\lambda_1}) = \frac{1}{N+1}\sum_{r=1}^{N} e^{-ir\lambda_1} W^r f$ then this sequence converges to $(a_1 e_{\lambda_1})$, but $T_N(f, e_{i\lambda_1})$ is in the kernel of R too, which makes $(a_1 e_{\lambda_1})$ in the kernel as required.

Proceeding in the same way, we find that in general, for each j, the sequences $(a_j e_{\lambda_j})$ can be obtained as limits of the corresponding sequences (T_N) where $T_N(f, e_{i\lambda_j}) = \dfrac{1}{N+1}\sum_{r=1}^{N} e^{-ir\lambda_j} W^r f$, and since (T_N) is in the kernel of R, it follows that the limit sequences $(a_j e_{\lambda_j})$ are in the kernel of R. $\square$

7.2 Almost Periodic Velocities and Flexes

This section is dedicated to the development of the theory of almost periodic infinitesimal velocities and flexes. Furthermore, we show that almost periodic infinitesimal flexes can be determined in terms of the matrix function $\Phi_{\mathcal{C}}(z)$.

Note that the sequence $e_\lambda = (e_\lambda(k))_{k\in\mathbb{Z}} = (e^{i\lambda k})_{k\in\mathbb{Z}}$ in $AP(\mathbb{Z})$, for $\lambda \in \mathbb{R}$, has a natural analogue for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ in $\mathbb{R}^d$. Also, for $v = (v_\kappa)$ in $\mathbb{C}^{d|F_v|}$ we have phase-periodic (multi phase-periodic) velocity sequences in $AP(\mathbb{Z}^d, \mathbb{C}^{d|F_v|})$ given by $k \to e^{i\langle\lambda,k\rangle} v_\kappa$. We denote this sequence as ve_λ or $(ve_\lambda(k))$.

Definition 7.2.1. Let $\mathcal{C} = (F_v, F_e, \mathcal{T})$ be a crystal framework in $\mathbb{R}^d$. The class $AP(\mathbb{Z}^d, \mathbb{C}^{d|F_v|})$ of *uniformly almost periodic velocity sequences* is the closed linear span (with respect to the ∞-norm: $\|v_{\kappa,k}\|_\infty = \sup\|v_{\kappa,k}\|_2$) of the set of phase periodic velocity sequences.

Definition 7.2.2. The *right shift* for any velocity sequence $v = (v_{\kappa,k})$ of the crystal framework $\mathcal{C}$ in $\mathbb{R}^d$ is defined by

$$R_l(v_{\kappa,k}) = v_{\kappa,k-l},\ l \in \mathbb{Z}^d.$$

The right shift operators on the domain and co-domain of the rigidity matrix $R(\mathcal{C})$ for $l \in \mathbb{Z}^d$ are denoted by R_l^V and R_l^E respectively. We note that

$$R(\mathcal{C}) \circ R_l^V = R_l^E \circ R(\mathcal{C}).$$

Definition 7.2.3. Let $\epsilon > 0$ and a velocity sequence $v = (v_{\kappa,k})$ of the crystal framework $\mathcal{C}$ be given. A vector $l \in \mathbb{Z}^d$ is called an *ϵ-translation vector* of v if

$$\|R_l^V(v) - v\|_\infty < \epsilon.$$

A set $S \subseteq \mathbb{Z}^d$ is said to be relatively dense if there exists an integer $L > 0$ such that every d-dimensional box with side length L in $\mathbb{Z}^2$ intersects S.

Definition 7.2.4. Let $v = (v_{\kappa,k})$ be a velocity sequence of the crystal framework $\mathcal{C}$. Then

i. v is said to be *Bohr almost periodic* if for every $\epsilon > 0$ the set of ϵ-translation vectors of $v = (v_{\kappa,k})$ is relatively dense.

ii. v satisfies *Bochner's condition* if it is bounded and the set of translates $\{R_l^V(v) : l \in \mathbb{Z}^d\}$ is relatively compact.

Definition 7.2.5. Let $v = (v_{\kappa,k})$ be a Bohr velocity sequence of $\mathcal{C}$, the *mean value* of v is defined to be

$$[v, 1] = \lim_{N_1 \to \infty} \cdots \lim_{N_d \to \infty} \prod_{i=1}^{d} \frac{1}{2N_i + 1} \sum_{k_1=-N_1}^{N_1} \cdots \sum_{k_d=-N_d}^{N_d} v_{\kappa,k}.$$

The *Bohr transform* of v is a function of λ defined by $\lambda \to [v, e_\lambda]$ where

$$[v, e_\lambda] = \lim_{N_1 \to \infty} \cdots \lim_{N_d \to \infty} \prod_{i=1}^{d} \frac{1}{2N_i + 1} \sum_{k_1=-N_1}^{N_1} \cdots \sum_{k_d=-N_d}^{N_d} v_{\kappa,k} e^{-i\langle \lambda, k \rangle}.$$

The *Bohr spectrum* of v is the set $\Lambda(v, \mathcal{C}) \subseteq \mathbb{T}^d$ of points $(e^{i\lambda_1}, \ldots, e^{i\lambda_d})$ such that $[v, e_\lambda] \neq 0$.

Note that $\Lambda(v, \mathcal{C})$ is a non-empty, at most countable set, and that this is the uni-modular form of the spectrum.

In the case of Bohr almost periodic velocity sequences, the approximating polynomials, as before, are given by the covariance function of v and the corresponding Fejér-Bochner kernel K'_n is given by

$$K'_n(k) = \sum_{\lambda \in E_n} k_n(\lambda) e^{i\langle \lambda, k \rangle},\ k \in \mathbb{Z}^d.$$

Using this kernel, we can state the approximation theorem.

Theorem 7.2.6. *Given a Bohr velocity sequence $v = (v_{\kappa,k})$ of a crystal framework $\mathcal{C}$ in $\mathbb{R}^d$, let $g = (g^{(n)})$ denote the sequence of trigonometric polynomials defined for $l \in \mathbb{Z}^d$ by*

$$\begin{aligned} g^{(n)}_{\kappa,l} &= [g, R_l K'_n] \\ &= \lim_{N_1 \to \infty} \cdots \lim_{N_d \to \infty} \prod_{i=1}^{d} \frac{1}{2N_i + 1} \sum_{k_1=-N_1}^{N_1} \cdots \sum_{k_d=-N_d}^{N_d} v_{\kappa,k} K'_n(k - l). \end{aligned}$$

Then $\|v - g^{(n)}\|_\infty \to 0$ as $n \to \infty$ and in particular v is an almost periodic velocity sequence of $\mathcal{C}$.

The following theorem is key in understanding the relation between Bohr flexes and phase periodic flexes. For notational simplicity we assume that $d = 2$. However, the theorems below hold for $d \geq 3$ with the same proof.

Theorem 7.2.7. *Let $v = (v_{\kappa,k})$ be a Bohr velocity sequence for the crystal framework $\mathcal{C}$ in $\mathbb{R}^2$ and $g = (g^{(n)})$ the approximating almost periodic velocity sequence of v. If v is a flex of $\mathcal{C}$, then each $g^{(n)}$ is also a flex of $\mathcal{C}$.*

Proof. Let $v = (v_{\kappa,k})$ be a Bohr velocity sequence of $\mathcal{C}$. Then it follows from the approximation theorem that v is the norm limit of the sequence $g = (g^{(n)})$ of finite "vector" combinations of the pure frequencies $\{e_\lambda\}$, $\lambda \in \mathbb{R}^2$. Moreover, v is the uniform limit of the explicit sequence $(g^{(n)})$ where

$$\begin{aligned}
g^{(n)}_{\kappa,l} &= [g, R_l K'_n] \\
&= \lim_{N_1\to\infty} \lim_{N_2\to\infty} \frac{1}{2N_1+1} \frac{1}{2N_2+1} \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} v_{\kappa,k} K'_n(k-l) \\
&= \sum_{\lambda\in E_n} k_n(\lambda) \lim_{N_1\to\infty} \lim_{N_2\to\infty} \frac{1}{2N_1+1} \frac{1}{2N_2+1} \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} v_{\kappa,k} e^{-i\langle\lambda,k-l\rangle} \\
&= \sum_{\lambda\in E_n} k_n(\lambda) \lim_{N_1\to\infty} \lim_{N_2\to\infty} \frac{1}{2N_1+1} \frac{1}{2N_2+1} \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} v_{\kappa,k} e^{i\langle\lambda,l-k\rangle} \\
&= \sum_{\lambda\in E_n} k_n(\lambda) \lim_{N_1\to\infty} \lim_{N_2\to\infty} \frac{1}{2N_1+1} \frac{1}{2N_2+1} \sum_{k'_1=l_1+N_1}^{l_1-N_1} \sum_{k'_2=l_2+N_2}^{l_2-N_2} v_{\kappa,l-k'} e^{i\langle\lambda,k'\rangle}.
\end{aligned}$$

Now for $k = (k_1, k_2) \in \mathbb{Z}^2$, let S_k be the shift $S_k : v_{\kappa,l} \mapsto v_{\kappa,l-k}$ and as before, let S^V_k and S^E_k be the shift operators defined on the domain and codomain of the rigidity matrix $R(\mathcal{C})$ respectively. Using the fact that the rigidity matrix is a continuous linear transformation that commutes with these shifts, the l^{th} entry of $R(g^{(n)})$ can be written as

$$
\begin{aligned}
&R(g^{(n)})_l \\
&= R\Big(\sum_{\lambda \in E_n} k_n(\lambda) \lim_{N_1\to\infty} \lim_{N_2\to\infty} \frac{1}{2N_1+1} \frac{1}{2N_2+1} \sum_{k_1'=l_1+N_1}^{l_1-N_1} \sum_{k_2'=l_2+N_2}^{l_2-N_2} v_{\kappa,l-k'} e^{i\langle\lambda,k'\rangle}\Big) \\
&= \sum_{\lambda \in E_n} k_n(\lambda) \lim_{N_1\to\infty} \lim_{N_2\to\infty} \frac{1}{2N_1+1} \frac{1}{2N_2+1} \sum_{k_1'=l_1+N_1}^{l_1-N_1} \sum_{k_2'=l_2+N_2}^{l_2-N_2} R(v_{\kappa,l-k'}) e^{i\langle\lambda,k'\rangle} \\
&= \sum_{\lambda \in E_n} k_n(\lambda) \lim_{N_1\to\infty} \lim_{N_2\to\infty} \frac{1}{2N_1+1} \frac{1}{2N_2+1} \sum_{k_1'=l_1+N_1}^{l_1-N_1} \sum_{k_2'=l_2+N_2}^{l_2-N_2} RS^V_{k'}(v_{\kappa,l}) e^{i\langle\lambda,k'\rangle} \\
&= \sum_{\lambda \in E_n} k_n(\lambda) \lim_{N_1\to\infty} \lim_{N_2\to\infty} \frac{1}{2N_1+1} \frac{1}{2N_2+1} \sum_{k_1'=l_1+N_1}^{l_1-N_1} \sum_{k_2'=l_2+N_2}^{l_2-N_2} S^E_{k'} R(v_{\kappa,l}) e^{i\langle\lambda,k'\rangle}.
\end{aligned}
$$

But v is a flex, from which it follows that

$$R(g^{(n)})_l = \underline{0}.$$

Thus the approximating trigonometric sequence $g = (g^{(n)})$ is a sequence of flexes of $\mathcal{C}$. $\square$

To conclude our effort in understanding the relation between Bohr flexes and phase periodic flexes we will prove that the pure frequencies in an almost periodic flex are precisely those in the RUM spectrum of the crystal framework. The proof of the following theorem follows the style of the proof of Theorem 7.1.13.

Theorem 7.2.8. *Let $u = (u_{\kappa,k})$ be a Bohr flex of the crystal framework $\mathcal{C}$ in $\mathbb{R}^2$. Then the pure components of an approximating trigonometric flex of u are flexes of $\mathcal{C}$.*

Proof. From the previous theorem it follows that the approximating trigonometric sequences are in fact flexes of $\mathcal{C}$. And to prove that the pure component sequences are flexes assume that u is approximated by the trigonometric flex

$$w_k = \sum_{j=0}^{r} a_j e_{\lambda_j}(k)$$

where $a_j \in \mathbb{C}^2$, $k \in \mathbb{Z}^2$ and $\lambda \in \mathbb{R}^2$. Note that for each j

$$e_{\lambda_j} = (e^{i\langle \lambda_j, k\rangle})_{k\in\mathbb{Z}^2}.$$

Define

$$T_{N_1,N_2} : \ell^\infty(\mathbb{Z}^2, \mathbb{C}^{2|F_v|}) \to \ell^\infty(\mathbb{Z}^2, \mathbb{C}^{2|F_v|})$$

such that

$$T_{N_1,N_2}(u) = \frac{1}{N_1+1}\frac{1}{N_2+1}\sum_{r_1=1}^{N_1}\sum_{r_2=1}^{N_2} W_1^{r_1} W_2^{r_2} u$$

where $W_1^{r_1}$ is the backward shift of the x coordinates by r_1 steps and $W_2^{r_2}$ is the backward shift of the y coordinates by r_2 steps. Since u is a flex and the rigidity matrix commutes with shifts, it follows that $T_{N_1,N_2}(u)$ lies in the kernel of the rigidity matrix R. But $(T_{N_1,N_2}(u)) \to (a_0 e_{\lambda_0})$ as $N_1, N_2 \to \infty$ which implies that the sequence $(a_0 e_{\lambda_0})$ is a flex. In general, for each j, the sequences $(a_j e_{\lambda_j})$ can be obtained as limits of corresponding sequences

$$T_{N_1,N_2}(u, e_{\lambda_j}) = (\frac{1}{N_1+1}\frac{1}{N_2+1}\sum_{r_1=1}^{N_1}\sum_{r_2=1}^{N_2} e^{-ir_1\lambda_j^x} W_1^{r_1} e^{-ir_2\lambda_j^y} W_2^{r_2})u,$$

these sequences are in the kernel of R, from which it follows that the limit sequences $(a_j e_{\lambda_j})$ are flexes and the pure components are flexes. □

Lemma 7.2.9. *Let $v = (v_{\kappa,k}) \in \mathcal{K}_v$ be an almost periodic velocity sequence of the crystal framework $\mathcal{C}$ in $\mathbb{R}^2$. If v is a flex of $\mathcal{C}$ and $\omega = (e^{i\lambda_1}, e^{i\lambda_2}) \in \Lambda(v, \mathcal{C})$, then $\omega \in \Omega(\mathcal{C})$, the RUM spectrum of $\mathcal{C}$.*

Proof. The lemma follows immediately from Theorems 7.2.7 and 7.2.8. □

Remark 7.2.10. As noted before, a framework that admits a special kind of flex is said to be flexible in that sense, and rigid otherwise. Here, a framework that admits no almost periodic flex, is said to be *almost periodically rigid* and *almost periodically flexible* otherwise.

Theorem 7.2.11. *The following are equivalent for a crystal framework $\mathcal{C} \in \mathbb{R}^2$ in Maxwell counting equilibrium.*

i. $\mathcal{C}$ is almost periodically rigid,

ii. $\mathcal{C}$ is strictly periodically rigid and $\Omega(\mathcal{C}) = \{(1,1)\}$.

Proof. To prove that i implies ii: Let $\mathcal{C}$ be almost periodically rigid, then $\mathcal{C}$ is periodically rigid since every periodic flex is almost periodic. Also, since phase periodic flexes are almost periodic, the RUM spectrum will be the singleton set $\{(1,1)\}$ corresponding to translations.

To prove that ii implies i: Assume that ii holds, and that u is a nonzero almost periodic flex for $\mathcal{C}$. From almost periodicity theory, there exists $\omega = (e^{i\lambda_1}, e^{i\lambda_2})$ in $\Lambda(u, \mathcal{C})$. It follows from ii and Lemma 7.2.9 that $\omega = (1, 1)$ and so u is periodic. By ii u is not of translation type, so i holds. $\square$

It follows from the proof of the theorem that any almost periodic infinitesimal flex can be approximated by a sequence of finite linear combinations of phase periodic flexes. Using Theorem 7.2.8 we have the following theorem.

Theorem 7.2.12. *Let $\mathcal{C}$ be a crystal framework in $\mathbb{R}^d$. Then, the space of supercell periodic infinitesimal flexes for n-fold periodicity with $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$ is equal to the linear span of*

$$\{\omega^k u_{\kappa,0} : \Phi(\overline{\omega}) u_{\kappa,0} = 0,\ \omega \in \Omega_n(\mathcal{C})\}$$

where $\Omega_n(\mathcal{C})$ is the finite subset of the RUM spectrum given by the multiphase ω, whose k-th component is an n_{k-th} root of unity. In particular, every supercell periodic infinitesimal flexes for $\mathcal{C}$ is an almost periodic infinitesimal flex for $\mathcal{C}$.

It follows from Theorems 5.3.1, 7.2.8 and Lemma 7.2.9, that the Bohr spectrum of an almost periodic infinitesimal flex u is contained in the RUM spectrum of $\mathcal{C}$. Furthermore, since phase periodic flexes are almost periodic, it follows that the RUM spectrum is the union of the Bohr spectra of all almost periodic infinitesimal flexes.

7.3 Examples

In this section we look back at some of the basic examples of crystal frameworks. The first two show the extreme cases: firstly, where the RUM spectrum is a singleton, and secondly, where the RUM spectrum is $\mathbb{T}^d$.

The triangulated grid $\mathcal{C}_{\text{tri}}$ (Figure 7.1). We noted earlier (Section 5.3) that the RUM spectrum for $\mathcal{C}_{\text{tri}}$ is the singleton $(1,1) \in \mathbb{T}^2$. Furthermore, there are no non-trivial strictly periodic infinitesimal flexes of $\mathcal{C}_{\text{tri}}$, and so, by Theorem 7.2.11, $\mathcal{C}_{\text{tri}}$ is almost periodically infinitesimally rigid.

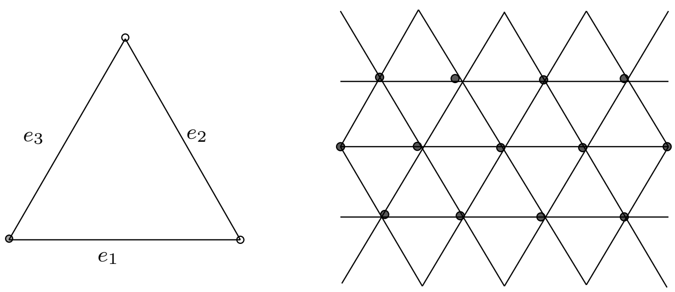

Figure 7.1: The triangulated grid $\mathcal{C}_{\text{tri}}$

The double squares framework $\mathcal{C}_{\mathbf{2sq}}$ (Figure 7.2). The determinant of $\Phi_{\mathcal{C}_{2\text{sq}}}(z,w)$ vanishes identically, and so, the RUM spectrum is $\mathbb{T}^d$. A local flex infinitesimally rotating the inner square can be defined by

$u_{2,0} = (-1,1)$, $u_{3,0} = (1,1)$, $u_{4,0} = (1,-1)$, $u_{5,0} = (-1,-1)$ and $u_{1,0} = (0,0)$.

and with zero velocities elsewhere. A phase periodic infinitesimal flex of $\mathcal{C}_{2\text{sq}}$ for $\omega = (\omega_1, \omega_2)$ is obtained by taking

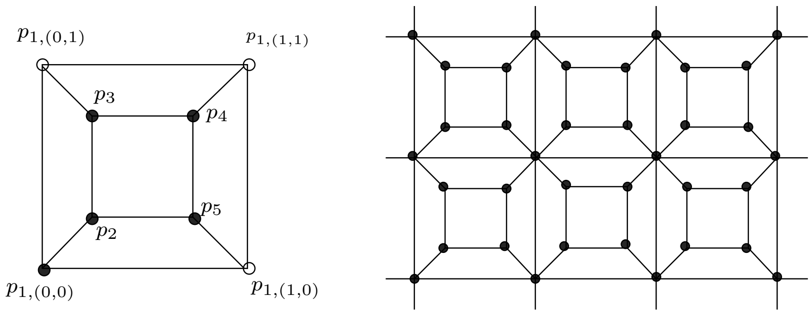

Figure 7.2: The double-squares framework $\mathcal{C}_{2\text{sq}}$

$$u_{2,k} = \omega_1^{k_1}\omega_2^{k_2}(-1,1),\ u_{3,k} = \omega_1^{k_1}\omega_2^{k_2}(1,1),\ u_{4,k} = \omega_1^{k_1}\omega_2^{k_2}(1,-1),$$
$$u_{5,k} = \omega_1^{k_1}\omega_2^{k_2}(-1,-1)$$
$$\text{and } u_{1,k} = \omega_1^{k_1}\omega_2^{k_2}(0,0).$$

In particular, any finite linear combination of such phase periodic flexes is an almost periodic flex for $\mathcal{C}_{2\text{sq}}$.

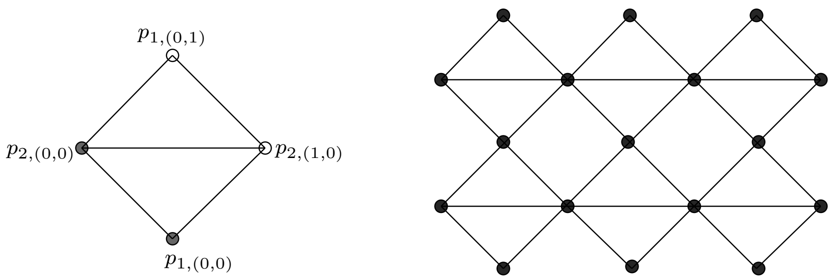

Figure 7.3: The squares framework $\mathcal{C}_{\text{sq}}$

The squares framework $\mathcal{C}_{\text{sq}}$ (Figure 7.3). Recall that the RUM spectrum for $\mathcal{C}_{\text{sq}}$ is the finite set $\{(1,1),(-1,-1)\}$. With respect to the minimal motif, $\mathcal{C}_{\text{sq}}$ does not admit any non-trivial strictly periodic infinitesimal flexes.

However, $\mathcal{C}_{\mathrm{sq}}$ does admit $(2,2)$-fold periodic infinitesimal flexes, which may be constructed from the motif and RUM spectrum. Assign velocity vectors $u_{1,0}$ and $u_{2,0}$ to the motif vertices p_1 and p_2 respectively and consider the multi-phase $\omega = (\omega_1, \omega_2) = (-1,-1) \in \Omega(\mathcal{C}_{\mathrm{sq}})$. Define for each $k = (k_1, k_2) \in \mathbb{Z}^2$,

$$u_{1,k} = \omega^k u_{1,0} = (-1)^{k_1}(-1)^{k_2} u_{1,0}$$
$$u_{2,k} = \omega^k u_{2,0} = (-1)^{k_1}(-1)^{k_2} u_{2,0}.$$

Then, u is supercell $(2,2)$-fold periodic. If for example we set $u_{1,0} = (1,0)$ and $u_{2,0} = (0,-1)$ then u is an infinitesimal alternating rotational flex. In the notation of Theorem 7.2.12, u has $(2,2)$-fold periodicity; the multi-phase $\omega = (-1,-1)$ is contained in

$$\Omega_n(\mathcal{C}_{\mathrm{sq}}) = \{\omega \in \Omega(\mathcal{C}_{\mathrm{sq}}) : \omega_1^2 = 1, \omega_2^2 = 1\}$$

and u is the ω-phase periodic velocity vector, $(\omega^k u_{1,0}, \omega^k u_{2,0})$, where $(u_{1,0}, u_{2,0}) \in \ker \Phi_{\mathcal{C}_{\mathrm{sq}}}(\overline{\omega})$.

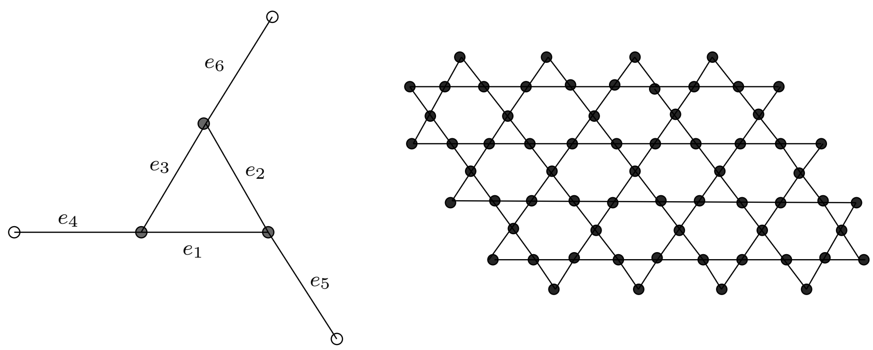

Figure 7.4: The kagome framework $\mathcal{C}_{\mathrm{kag}}$

The kagome framework $\mathcal{C}_{\mathbf{kag}}$ (Figure 7.4). Recall that the RUM spectrum $\mathcal{C}_{\mathrm{kag}}$ for the kagome framework is the union of the three curves in $\mathbb{T}^2$ defined by $z = 1$, $w = 1$ and $z = w$ or the points (s, t) of the unit square in the line segments given by

$$s = 0,\ t = 0 \text{ and } s = t$$

In particular, $\mathcal{C}_{\mathrm{kag}}$ is almost periodically infinitesimally flexible, but has no local infinitesimal flexes. In this case, every almost periodic infinitesimal flex decomposes as a sum $u_1+u_2+u_3$ corresponding to this ordered decomposition. Furthermore, u_1, with the Bohr spectrum in th line $s = 0$ is periodic in the direction of the period vector $\underline{a} = (1, 0)$, while u_2, with the Bohr spectrum in the line $t = 0$, is periodic in the direction of the period vector $\underline{b} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, and u_3, with the Bohr spectrum in the line $s = t$ is periodic in the direction $\underline{a} - \underline{b}$.

Chapter 8

Bases For The Flexes Of Crystal Frameworks

Here we consider the infinite linear decomposition of infinitesimal flexes of general infinite bar-joint frameworks in terms of a countable basis. Also, we define and identify *crystal flex bases* in the case of crystal frameworks. In previous chapters we have limited attention to infinitesimal flexes with some form of periodicity. Our main concern in this chapter is with the structure of the real vector space $\mathcal{H}_{\mathrm{fl}}(\mathcal{C})$ which is often infinite dimensional.

We say that the vector space $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$ of infinitesimal flexes of an infinite bar-joint framework $\mathcal{G}$ has a *generalized basis* if there is a finite or infinite sequence $u_1, u_2, \ldots$ in $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$ such that each flex u in $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$ admits a unique representation

$$u = \sum_{k=1}^{\infty} \alpha_k u_k,\ \alpha_k \in \mathbb{R},$$

where the series converges coordinatewise.

We shall see that such generalized bases always exist. Furthermore, we define crystal flex bases which require that the set of basis elements has a certain symmetry property. A main result is Theorem 8.3.6 which obtains such a basis for the regular octahedron net framework. We also construct crystal flex bases (or spanning sets) for nine other diverse examples.

8.1 Free Bases For Infinitesimal Flexes

In this section we consider some properties of infinite bar-joint frameworks related to decomposition possibilities for infinitesimal flexes.

Let (G, p) be a locally finite countable bar-joint framework in $\mathbb{R}^d$ and let $\mathcal{H}_v(\mathcal{C})$ be the vector space of all velocity vectors of $\mathcal{C}$. Any velocity vector $w \in \mathcal{H}_v(\mathcal{C})$ has the form

$$w = \sum_{k=1}^{\infty} \sum_{\sigma=1}^{d} \alpha_{k,\sigma} e^{k,\sigma}$$

where the coefficients $\alpha_{k,\sigma}$ are real numbers, each $e^{k,\sigma}$ indicates a basic velocity sequence in $\mathcal{H}_v(\mathcal{C})$, and where convergence of the series is coordinatewise convergence. The basic sequence $e^{k,\sigma}$ is zero except for the k^{th} velocity which is the unit velocity in the direction of the σ-axis.

Note that any choice of coefficients defines a velocity vector since for each vertex p_j there are only finitely many basic velocity vectors for which the j-th component velocity, $e_j^{k,\sigma}$ is nonzero. The following notion of a *free basis* extends this idea. Such a basis has also been referred to as a product type basis in Sait [63] and Power [58].

Definition 8.1.1. Let $\mathcal{G} = (G, p)$ be a countably infinite bar-joint framework in $\mathbb{R}^d$ and $\mathcal{M}$ be a subspace of the velocity vector space $\mathcal{H}_v(\mathcal{G})$.

(a) A *free spanning set* for $\mathcal{M}$ is a finite or countable set $\mathcal{S} = \{w^1, w^2, \dots\}$ of vectors in $\mathcal{M}$ such that every vector u in $\mathcal{M}$ has a representation:

$$u = \sum_{n=1}^{\infty} \alpha_n w^n \text{ where } \alpha_n \in \mathbb{R}$$

and for each index k the component w_k^n is non-zero for only finitely many of the vectors w^n.

(b) A *free basis* for $\mathcal{M}$ is a free spanning set for $\mathcal{M}$ such that the infinite sum representations are unique.

Let P_N be the canonical vector space projection

$$P_N : \mathcal{H}_v(\mathcal{G}) \to \prod_{k=1}^{N} \mathbb{R}^d.$$

We say that a sequence $w^1, w^2, \dots$ of velocity sequences *tends to zero weakly* if for each j and $N \in \mathbb{N}$ the sequence $(P_N w^k)_j$ is zero for all large enough k.

Lemma 8.1.2. *Let $\mathcal{S} = \{w^1, w^2, \dots\}$ be a countable set of vectors in $\mathcal{H}_v(\mathcal{G})$. Then the following are equivalent* *(i)* $\sum_{n=1}^{\infty} \alpha_n w^n$ *converges coordinatewise for every sequence* (α_n).

(ii) The velocity vectors $w^1, w^2, \ldots$ *tend to zero weakly.*

Proposition 8.1.3. *Let* $\mathcal{G}$ *be a locally finite countable bar-joint framework and let* $b_1, b_2, \ldots$ *be a generalized basis for* $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$ *in the sense that every velocity vector* u *in* $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$ *admits a unique representation*

$$u = \sum_{k=1}^{\infty} \alpha_k w^k \text{ where } \alpha_k \in \mathbb{R}$$

where the series converges coordinatewise. Then $\mathcal{G}$ *has a free basis.*

Proof. For notational simplicity view $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$ as a vector subspace of $\prod_{i=1}^{\infty} \mathbb{R}$. Let b_{k_1} be the first vector in the basis in the basis with nonzero coordinate $b_k(i_1)$ where $b_k(i) = 0$ for all $i \leq i_1$ for all k and i_1 is the first such index. Let $\mathcal{B} = \{b_1, b_2, \ldots\}$. Construct a new basis $\mathcal{B}_2$ where the first vector is b_{k_1} and the subsequent vectors are $b_i - \alpha_i b_{k_1}$, for $i \neq k_1$, in order, where α_i is chosen so that the i_1 coordinate is 0. Evidently $\mathcal{B}_2$ is also a generalized basis which we write as $\{b_1^2, b_2^2, \ldots\}$. We may repeat this process with the tail sequence $b_2^2, b_3^2, \ldots$ to obtain a basis $\mathcal{B}_3$, and then, with the subsequent tails, similarly obtain generalized bases $\mathcal{B}_3, \mathcal{B}_4, \ldots$. Finally, select the velocity vectors for the "main diagonal" to obtain the set

$$\mathcal{S} = \{b_1^2, b_2^3, b_3^4, \ldots\}.$$

In the case there is no coordinate indices with $b_k(i) = 0$ for all k this sequence has the triangular form

$$(a_1, *, *, \dots), (0, a_2, *, *, \dots), (0, 0, a_3, *, *, \dots), \dots$$

where each a_i is non-zero. Since the sequence tends to zero weakly the set $\mathcal{S}$ is a free spanning set for the associated space $\mathcal{M}(\mathcal{S})$ of arbitrary countable linear combinations. Since $\mathcal{S}$ has a triangular form it is a free basis for the space $\mathcal{M}(\mathcal{S})$. It remains to show that $\mathcal{M}(\mathcal{S}) \subseteq \mathcal{H}_{\mathrm{fl}}(\mathcal{G})$ and that every infinitesimal flex $u \in \mathcal{H}_{\mathrm{fl}}(\mathcal{G})$ has a free representation in terms of $\mathcal{S}$. This inclusion is an elementary consequence of the definition of an infinitesimal flex and the fact that $\mathcal{S}$ is a weakly null set of infinitesimal flexes. That every flex $u \in \mathcal{H}_{\mathrm{fl}}(\mathcal{G})$ has a free representation follows from the usual back substitution algorithm. $\square$

Let us say that a countable set $\mathcal{S}$ is a *generalized spanning set* for $\mathcal{H}_{\mathrm{fl}}(\mathcal{C})$ if every $u \in \mathcal{H}_{\mathrm{fl}}(\mathcal{C})$ can be written as an infinite linear combination

$$u = \sum_{k=1}^{\infty} \alpha_k u_k, \ \alpha_k \in \mathbb{R},$$

where the series converges coordinatewise.

Proposition 8.1.4. *Let $\mathcal{G}$ be a locally finite countable bar-joint framework such that $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$ has a countable generalized spanning set. Then $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$ has a free basis.*

Proof. The algorithm in the previous proof applies. $\square$

In fact we have the following general result.

Proposition 8.1.5. *The infinitesimal flex space of a locally finite countable bar-joint framework has a free basis.*

Proof. Let $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \dots$ be a complete tower of finite subframeworks for $\mathcal{G}$. Let $\mathcal{H}^0_{\mathrm{fl}}(\mathcal{G}_k)$ be the subspace of restrictions of flexes in $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$. We may sequentially choose velocity vectors $b_1, b_2, \dots$ in $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$ so that

(i) for each $k = 1, 2, \dots$ the restrictions of the velocity vectors

$$b_1, b_2, \dots, b_{d_k},$$

to $\mathcal{G}_k$, where $d_k = \dim \mathcal{H}^0_{\mathrm{fl}}(\mathcal{G}_k)$, give a basis for $\mathcal{H}^0_{\mathrm{fl}}(\mathcal{G}_k)$

(ii) for $j > d_k$ the restriction of b_j to $\mathcal{G}_k$ is the zero infinitesimal flex.

It follows that the set $\mathcal{B} = \{b_1, b_2, \dots\}$ is a free basis for $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$. Since $\mathcal{B}$ has triangular form it follows, as we noted before, that $\mathcal{B}$ is a free basis for the vector space $\mathcal{M}(\mathcal{B})$ of arbitrary infinite linear combinations of vectors in $\mathcal{B}$. It remains to show that $\mathcal{M}(\mathcal{B}) = \mathcal{H}_{\mathrm{fl}}(\mathcal{G})$. This follows, as before, by the back substitution algorithm. $\square$

We simply say that $\mathcal{G}$ has a free basis $\mathcal{B}$ when $\mathcal{B}$ is a free basis for $\mathcal{H}_{\mathrm{fl}}(\mathcal{G})$.

Definition 8.1.6. Let $\mathcal{B}$ be a free basis for $\mathcal{G}$. Then $\mathcal{B}$ is a *bounded free basis* if there is a uniform bound for the joint velocities $b(i)$ for all i and all vectors $b \in \mathcal{B}$.

Definition 8.1.7. A countably infinite bar-joint framework has a *local free flex basis* if there exists a free flex basis for which every basis vector is finitely non-zero.

8.2 Crystal Flex Bases And Spanning Sets

In this section we define crystallographic flex bases, crystallographic spanning sets and some associated properties.

Recall that the *crystallographic group* or *space group* is the group $\mathfrak{C}(\mathcal{C})$ of isometries T of $\mathbb{R}^d$ such that the map $p_i \to T(p_i)$ give a bijection on the set of framework vertices and a bijection $[p_i, p_j] \to [T(p_i), T(p_j)]$ of the set of framework edges.

The space group acts on the vector space of infinitesimal flexes $\mathcal{H}_{\mathrm{fl}}(\mathcal{C})$ in a natural way; if $u = (u_1, u_2, \dots)$ is in $\mathcal{H}_{\mathrm{fl}}(\mathcal{C})$ and $X \in \mathfrak{C}(\mathcal{C})$ then the velocity vector $Xu = (Xu_1, Xu_2, \dots)$ lies in $\mathcal{H}_{\mathrm{fl}}(\mathcal{C})$.

Let $\mathcal{H}_v(\mathcal{C})$ be the vector space of velocity vectors and let $\mathcal{S} = \{b_1, b_2, \dots\}$ be a countable subset of $\mathcal{H}_v(\mathcal{C})$. Then $\mathfrak{C}(\mathcal{C})$ is said to act on $\mathcal{S}$ if for every vector b_k and space group isometry X there is a vector b_j such that 1-dimensional spaces $\mathbb{R}Xb_k$ and $\mathbb{R}b_j$ are equal. Also we say that $\mathcal{S}$ is finitely generated if there is a finite subset such that the 1-dimensional flex spaces for $\mathcal{S}$ are generated by this finite subset and the action of $\mathfrak{C}(\mathcal{C})$.

Definition 8.2.1. Let $\mathcal{C}$ be a crystal framework in $\mathbb{R}^d$.

(i) A *crystal flex basis* for $\mathcal{C}$ is a free basis $\mathcal{B}$ for $\mathcal{H}_{\mathrm{fl}}(\mathcal{C})$ such that $\mathfrak{C}(\mathcal{C})$ acts on $\mathcal{B}$.

(ii) $\mathcal{C}$ has property CB if there exists a crystal flex basis and has property BCB (respectively LCB) if there exists a crystal flex basis consisting of bounded flexes (respectively local flexes).

(iii) A *crystal flex spanning set* if for $\mathcal{C}$ is a free spanning set $\mathcal{S}$ for $\mathcal{H}_{\mathrm{fl}}(\mathcal{C})$ such that $\mathfrak{C}(\mathcal{C})$ acts on $\mathcal{S}$ and $\mathcal{S}$ is finitely generated.

(iv) $\mathcal{C}$ has property CS if there exists a crystal flex spanning set and has property BCS (respectively LCS) if there exists a crystal flex spanning set consisting of bounded flexes (respectively local flexes).

8.3 Bases And Spanning Sets

In this section we identify crystal flex bases and spanning sets for a selection of examples.

The triangulated grid $\mathcal{C}_{\mathrm{tri}}$. As we have seen before, this framework is infinitesimally rigid and in fact is sequentially infinitesimally rigid. A basis for the finite dimensional flex space can be given by any finite set of 3 linearly independent rigid motion infinitesimal flexes. For example we can choose two infinitesimal translations in the direction of the main axes and one infinitesimal rotation about the origin. This is not a crystal basis or a crystal spanning set. However, we can choose instead three translational infinitesimal flexes in the three directions of the edges of an equilateral triangle. The resulting set of four vectors is a crystal flex spanning set.

The basic grid $\mathcal{C}_{\mathbb{Z}^2}$. Let $\mathcal{L}_u^0$ be the linear subframework of $\mathcal{C}_{\mathbb{Z}^2}$ determined by the x-axis and let $\mathcal{L}_u^k$, $k \in \mathbb{Z}$, be the parallel linear subframeworks upwards and downwards. Let u^k be the velocity of infinitesimal translation of $\mathcal{C}_{\mathbb{Z}^2}$ to the right restricted to the corresponding linear subframework $\mathcal{L}_u^k$ (Figure 8.1). Strictly speaking, $u^k = (u_{j=(j_1,j_2)})_{j\in\mathbb{Z}^2}$ where

$$u_j = \begin{cases} (0,0) & \text{if} \quad j_1 \neq k, \\ (1,0) & \text{if} \quad j_1 = k \end{cases}.$$

Similarly, let $\mathcal{L}_v^0$ be the linear subframework determined by the y-axis and let $\mathcal{L}_v^k$, $k \in \mathbb{Z}$, be the parallel linear subframeworks to the right and left. Let $v^k = (v_{j=(j_1,j_2)})_{j\in\mathbb{Z}^2}$ be the velocity of infinitesimal translation of $\mathcal{C}_{\mathbb{Z}^2}$ towards the positive y-direction which is restricted to the corresponding $\mathcal{L}_v^k$, such that

$$v_j = \begin{cases} (0,0) & \text{if} \quad j_2 \neq k, \\ (0,1) & \text{if} \quad j_2 = k \end{cases}.$$

Proposition 8.3.1. *The set $\mathcal{B} = \{u^k, v^k : k \in \mathbb{Z}\}$ is a bounded crystal basis for the space of all infinitesimal flexes $\mathcal{H}_{\mathrm{fl}}(\mathcal{C}_{\mathbb{Z}^2})$.*

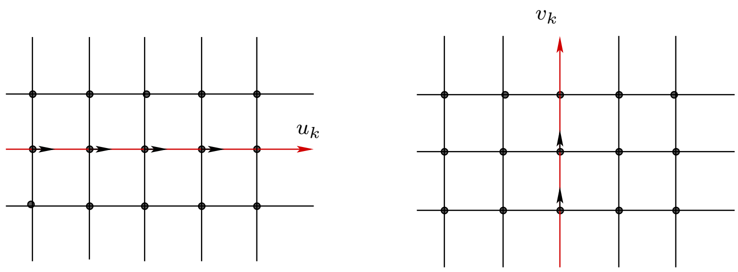

Figure 8.1: The basic grid: crystal basis elements u^k and v^k

Proof. It is clear that the crystal group acts on $\mathcal{B}$ and it remains to prove that $\mathcal{B}$ is a free basis. To prove this, let z be an arbitrary flex of $\mathcal{C}_{\mathbb{Z}^2}$, $z = (z_j)_{j\in\mathbb{Z}^2}$ where z_j is the flex at the corresponding vertex j of $\mathcal{C}_{\mathbb{Z}^2}$. Subtracting $z^x_{(0,0)}u^0 + z^y_{(0,0)}v^0$ results in a new flex, say z^1, with

$$z^1_{(0,0)} = (0,0),\ z^1_{(0,1)} = (z^{1,x}_{(0,1)}, 0) \text{ and } z^1_{(1,0)} = (0, z^{1,y}_{(1,0)}).$$

Again, subtracting $z^{1,x}_{(0,1)}u^1 + z^{1,y}_{(1,0)}v^1$, results in the flex z^2, such that

$$z^2_{(0,0)} = z^2_{(1,0)} = z^2_{(0,1)} = (0,0).$$

Also, by the flex condition it follows that $z^2_{(1,1)} = (0,0)$. Proceeding in the same manner, and downwards and to the left, it follows that

$$z - \sum_{k\in\mathbb{Z}} (z^{k,x}_{(0,k)}u^k + z^{k,y}_{(k,0)}v^k)$$

is the zero flex of $\mathcal{C}_{\mathbb{Z}^2}$ and the required representation for z follows. Note also that the coefficients are uniquely determined by z. $\square$

The squares framework $\mathcal{C}_{\mathrm{sq}}$. This framework admits a non-trivial "alternating rotation" infinitesimal flex, a^{sq} (Figure 8.2), where the rigid squares undergo infinitesimal rotations that are equal in magnitude and differ in sign. Let $\vec{x}$, $\vec{y}$ be the infinitesimal translation flexes for the x, y directions and let $\vec{r}$ be an infinitesimal rotation. Then the set $\mathcal{B} = \{\vec{x}, \vec{y}, \vec{r}, a^{\mathrm{sq}}\}$is a crystal flex basis for $\mathcal{C}_{\mathrm{sq}}$.

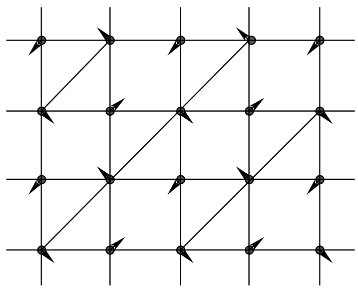

Figure 8.2: The alternating rotation infinitesimal flex of $\mathcal{C}_{\mathrm{sq}}$

The 5-regular grid $\mathcal{C}_{5\mathrm{grid}}$. Let u^0 be an infinitesimal flex restricted to the subframework $\mathcal{G}_u^0$ consisting of the lines $\mathcal{L}^0$, $\mathcal{L}^1$ and $\mathcal{L}^2$ together with the vertical edges and diagonal in between. We define u^0 to be the velocity of infinitesimal translation restricted to the vertices of both $\mathcal{L}^0$, $\mathcal{L}^2$ by 1 to the right, which is zero for the vertices of $\mathcal{L}^1$. Also, let v_0 be the infinitesimal flex restricted to the subframework $\mathcal{G}_v^0$ consisting of the lines $\mathcal{L}^1$, $\mathcal{L}^2$ and $\mathcal{L}^3$ together with the vertical edges and diagonal in between. This time, it is the velocity of infinitesimal translation restricted to the vertices of both $\mathcal{L}^1$, $\mathcal{L}^3$ by 1 towards the right, and keeping the vertices of $\mathcal{L}^2$ fixed. Figure 8.3 shows the flexes u^0 and v^0. Following the same setting upwards and downwards we obtain infinitesimal flexes u^k and w^k, for $k \in \mathbb{Z}$. Also, consider the two rigid body motions, $\vec{y}$, a rigid body translation by 1 in the positive y direction and $\vec{r}$, the rigid body rotation about the origin.

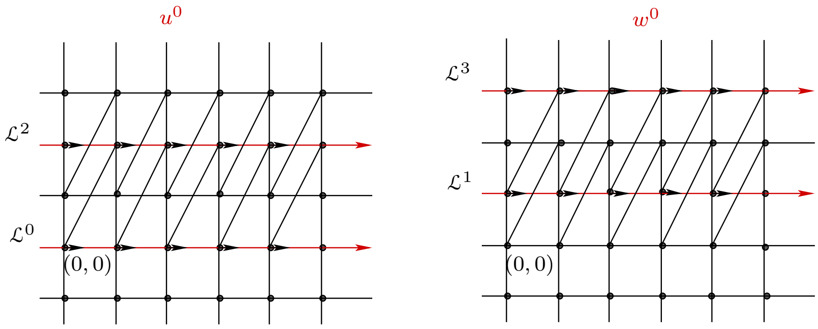

Figure 8.3: The 5-regular grid: crystal basis elements u^0 and v^0

Proposition 8.3.2. *The set* $\mathcal{B} = \{u^k,\, v^k\,, \vec{y}, \vec{r} : k \in \mathbb{Z}\}$ *is a crystal flex basis for the space of all infinitesimal flexes* $\mathcal{H}_{\mathrm{fl}}(\mathcal{C}_{5\mathrm{grid}})$ *of* $\mathcal{C}_{5\mathrm{grid}}$.

Proof. It is clear that the crystal group acts on $\mathcal{B}$ and it remains to prove that $\mathcal{B}$ is a free basis. Let z be an arbitrary flex of $\mathcal{C}_{5\mathrm{grid}}$, $z = (z_j)_{j\in\mathbb{Z}^2}$ with z_j being the flex at the corresponding vertex j. Subtracting $z^x_{(0,0)}u^0 + z^y_{(0,0)}\vec{y}$ results in a new flex, say z^1, with $z^1_{(0,0)} = (0,0)$. Now, the origin being fixed, the only option for the lines $\mathcal{L}^0$ and $\mathcal{L}^2$ is a rigid body infinitesimal rotation and for $\mathcal{L}^1$, the rotation and the x-translation. Indeed, note that the velocities on $\mathcal{L}^1$ must be horizontal and they play no role in affecting the velocities on $\mathcal{L}^0$ and $\mathcal{L}^2$ and so $\mathcal{L}^0$ and $\mathcal{L}^2$ are effectively connected by a sequence of triangles. Subtracting an appropriate multiple of $\vec{r}$ cancels the rotation. Also, subtracting $z^1_{(0,1)}v^0$ cancels the possibility of $\mathcal{L}^1$ translation. With the latter lines fixed, the next level of triangles (although they admit an infinitesimal flex on their own) are now rigid and the resulting flex, say z^2, imparts the zero flex of the strip-subframework $\mathcal{G}_u^0 \cup \mathcal{G}_v^0$. Proceeding in the same manner, upwards and downwards, the infinite linear representation of z follows. It follows that the original flex, z, is an infinite linear combination of the basis vectors and it also follows that such representation is unique and the proof is complete. $\square$

The local-flex grid. Let $\mathcal{C}$ be the framework in Figure 8.4. A crystal basis for this framework consists of two translations (for the x and y axes), a rigid body rotation and the set of all local flexes for individual squares. The proof for this is straightforward. One might think that the crystal basis of the basic grid is a subset of this one, but in fact adding the diagonal edges, although flexible, cancels the band limited flexes of the basic grid.

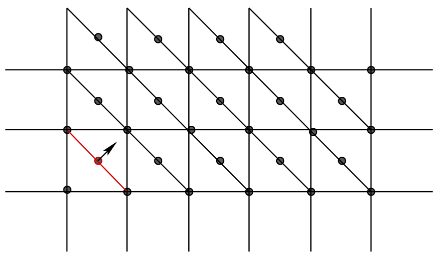

Figure 8.4: The local flex grid: crystal basis elements

The double-squares framework $\mathcal{C}_{\mathbf{2sq}}$. Let u_k, for $k \in \mathbb{Z}$ be the horizontal band limited flex of the basic grid "extended" to the double-squares framework and let v_k be the analogous vertical band limited flex. Together with r_j, the local infinitesimal rotation of the inner squares, the set $\mathcal{B} = \{u_k, v_k : k \in \mathbb{Z}\} \cup \{r_j : j \in \mathbb{Z}^2\}$ is a crystal basis for $\mathcal{C}_{2\mathrm{sq}}$ (Figure 8.5). The proof follows the usual pattern.

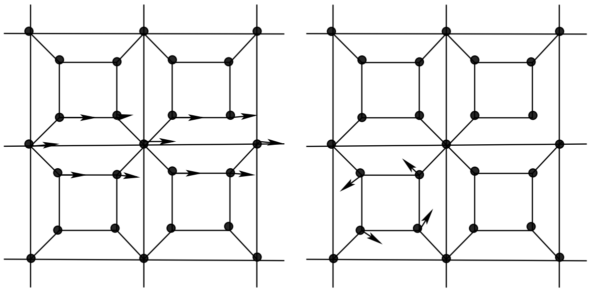

Figure 8.5: Crystal basis elements for $\mathcal{C}_{2\mathrm{sq}}$

The kagome framework $\mathcal{C}_{\mathrm{kag}}$. Identify a "central triangle subframework" with horizontal base edge $[a, b]$, and let $\mathcal{L}_u^0$ be the linear subframework of $\mathcal{C}_{\mathrm{kag}}$ containing this edge. There is an evident one dimensional subspace of infinitesimal flexes of $\mathcal{C}_{\mathrm{kag}}$ each of which vanishes off this linear subframework. It is spanned by the velocity vector u^0 such that

$$u_a^0 = (1, -1/\sqrt{3}),\ u_b^0 = (1, 1/\sqrt{3})$$

with repetition of these vectors at the triangles' bases to the left and right of abc. Let u^k, $k \in \mathbb{Z}$, be the parallel translates of u^0, with u^1 supported by the first linear subframework $\mathcal{L}_u^1$ above $\mathcal{L}_u^0$ and so on (Figure 8.6). Also, let $\{v_k : k \in \mathbb{Z}\}$ and $\{w_k : k \in \mathbb{Z}\}$ be obtained from $\{u_k : k \in \mathbb{Z}\}$ by a $2\pi/3$ and $4\pi/3$ rotation. Write $\mathcal{L}_u^k$, $\mathcal{L}_v^k$ and $\mathcal{L}_w^k$, $k \in \mathbb{Z}$, for the supporting linear subframeworks of u^k, v^k and w^k respectively. Note that

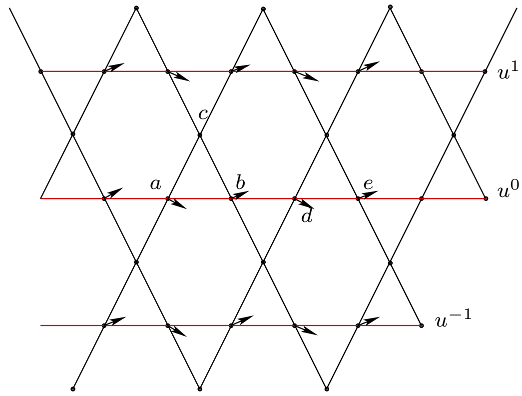

Figure 8.6: The kagome framework: crystal basis elements u^k

$$v_a^0 = (0,1),\ v_c^0 = (\sqrt{3}/2, 1/2)$$

and

$$w_b^0 = (0,1),\ w_c^0 = (-\sqrt{3}/2, 1/2).$$

The following theorem is due to A. Sait [63].

Theorem 8.3.3. *The set $\mathcal{B} = \{u^k, v^k, w^k : k \in \mathbb{Z}\}$ is a crystal basis for the space of all infinitesimal flexes $\mathcal{H}_{\mathrm{fl}}(\mathcal{C}_{\mathrm{kag}})$.*

Proof. Since the space group acts on $\mathcal{B}$ it will be sufficient to show that $\mathcal{B}$ is a free basis. Let z be an infinitesimal flex of $\mathcal{C}_{\mathrm{kag}}$. Subtracting $z_a^x u_a^0$ results in a flex z^1, such that $z_a^{1,x} = 0$. Subtracting $z_a^{1,y} v_a^0$ results in z^2, with $z_a^2 = (0,0)$ and $z_b^{2,x} = 0$. Subtracting $z_b^{2,y} w_a^0$ results in a new flex, z^3, $z_a^3 = z_b^3 = z_c^3 = (0,0)$. The triangle subframework abc being fixed, implies that $z_d^{3,x} = 0$ where d is the next vertex in the direction from a to b. Subtracting $z_d^{3,y} v^1$ results in z^4 with $z_d^4 = (0,0)$ and $z_e^{4,x} = 0$ for the next vertex. Subtracting $z_e^{4,y}w^{-1}$ results in e, the next vertex, being fixed. Continuing in this way, subtracting appropriate multiples of v^k and w^k, we obtain an infinite linear combination

$$z' = \sum_{k\in\mathbb{Z}} (\alpha_k v^k + \beta_k w^k)$$

such that the infinitesimal flex $z'' = z - z'$ is the zero flex at $\mathcal{L}_u^0$. This fact together with the rigidity of triangles implies that the flex velocities are also zero on the apex vertices for the upward triangle subframeworks based on $\mathcal{L}_u^0$. Also, z' on $\mathcal{L}_u^1$ must be a constant multiple of u^1. Subtracting this results in both of $\mathcal{L}_u^0$ and $\mathcal{L}_u^1$ being fixed. Following the same argument, upwards and downwards, we obtain an infinite linear representation for z'' in terms of the infinitesimal flexes u^k, it follows that the original flex z is an infinite linear combination of the basis vectors and it also follows that such representation is unique and the proof is complete. □

The augmented grid+strip framework $\mathcal{C}_{\mathbb{Z}^2}^+$. This framework is derived by joining countably many copies of the strip framework $\mathcal{C}_{\text{strip}}$ to the basic square grid $\mathcal{C}_{\mathbb{Z}^2}$ as shown in Figure 8.7. The strip framework admits a one dimensional space of "base fixed" infinitesimal flexes with interesting input-output (or geometric) behaviour. We start by identifying a free basis for $\mathcal{H}_{\text{fl}}(\mathcal{C}_{\text{strip}})$ and this will lead to the identification of a free basis for $\mathcal{H}_{\text{fl}}(\mathcal{C}_{\mathbb{Z}^2}^+)$.

Let u^0 be the infinitesimal flex of $\mathcal{C}_{\text{strip}}$ such that the origin has downwards velocity 1, $u^0|_{(0,0)} = (0, -1)$, and all the other x-axis vertices together with

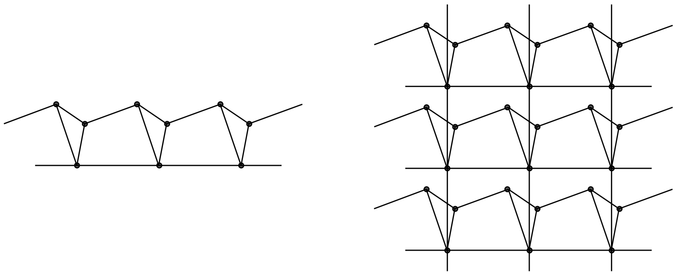

Figure 8.7: The strip framework $\mathcal{C}_{\text{strip}}$ and the augmented grid framework $\mathcal{C}^{+}_{\mathbb{Z}^2}$

all the vertices to the left of the origin having zero velocities. Similarly, let v^0 be the infinitesimal flex of $\mathcal{C}_{\text{strip}}$ such that the origin has downwards velocity 1, $v^0|_{(0,0)} = (0, -1)$, and all the other x-axis vertices together with all the vertices to the right of the origin having zero velocities, these flexes are shown in Figure 8.8.

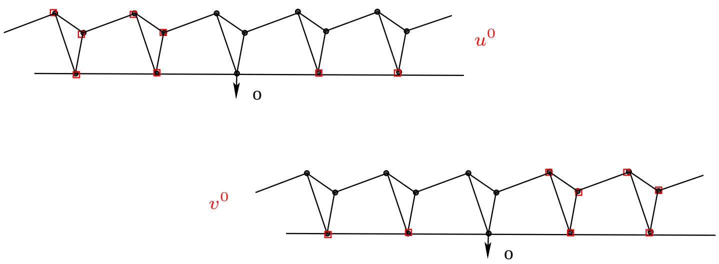

Figure 8.8: Infinitesimal flexes u^0 and v^0 for the strip framework $\mathcal{C}_{\text{strip}}$

Let u^n for $n \geq 0$ be the right translates (by 1) of u^0 and let v^n for $n < 0$ be the left translates (by 1) of v^0.

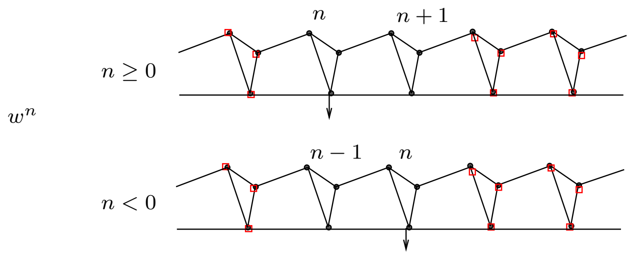

Figure 8.9: Basis elements w^n for the strip framework $\mathcal{C}_{\text{strip}}$

From the flexes u^n and v^n specified above, we can identify local flexes w^n (Figure 8.9) of $\mathcal{C}_{\text{strip}}$ as follows:

$$w^n = \begin{cases} u^n - \alpha u^{n+1} & \text{if} \quad n \geq 0, \\ \\ v^n - \beta v^{n-1} & \text{if} \quad n < 0 \end{cases} .$$

with constants α and β depending on the geometry.

For simplicity, label the triangles according to their x-axis joints, that is, triangle n is the triangle with x-axis joint $(n, 0)$ and so on. Due to the "geometric growth" of the flexes u^n and v^n, we find that the restriction of the velocity vectors of u^n to the triangle $n+2$ is equal to a constant multiple of the restriction of u^{n+1} to the same triangle $(n+2)$. Together with the fact that both u^n and u^{n+1} assign zero velocities to joints of all the triangles to the left of triangle n, for $n \geq 0$, w^n is in fact a local flex of $\mathcal{C}_{\text{strip}}$ such that

- w^n has zero velocity vectors at all joints of the triangles k, $k < n$,
- w^n has zero velocity vectors at all joints of the triangles j, $j \geq n+2$.

In other words, w^n only has non zero velocity vectors for joints of the triangles n and $n+1$.

Similarly, for $n < 0$, w^n is a local flex of $\mathcal{C}_{\text{strip}}$ such that w^n only has non zero velocity vectors for joints of the triangles n and $n-1$.

The flexes w^n together with the basic flexes $\vec{x}$, the rigid body translation by 1 unit to the right, and u^{st}, a base fixed infinitesimal flex of $\mathcal{C}_{\text{strip}}$, we have

Proposition 8.3.4. $\{\vec{x}\} \cup \{u^{\text{st}}\} \cup \{w^n : n \in \mathbb{Z}\}$ *is a free basis for* $\mathcal{H}_{\text{fl}}(\mathcal{C}_{\text{strip}})$.

Proof. Let z be an arbitrary flex of $\mathcal{C}_{\text{strip}}$. We will proceed by subtracting appropriate multiples of the flexes above until we achieve a zero flexing of $\mathcal{C}_{\text{strip}}$.

1. Subtracting $(z^x|_{(0,0)}\vec{x} - z^y|_{(0,0)}w^0)$ results in a new flex, z^1, such that z^1 has zero velocity at the origin. This implies that z^1 is only allowed to have a non-zero y velocity component at the next joint to the right, (1,0).

2. By subtracting $(-z^{1,y}|_{(1,0)}w^1)$ (this is allowed since w^1 has zero velocity at the origin) we can arrange that the new flex, z^2, has zero velocities at both joints $(0,0)$ and $(1,0)$ (similarly, joint $(2,0)$ is only allowed a non-zero y velocity component).

3. Continuing in the same way, we can subtract appropriate multiples of w^n, $n \geq 2$, until we arrive at a flex, z^3, such that z^3 has zero velocities

at the origin and all the x-axis joints to the right of the origin.

4. Now, we move on to the first vertex to the left of the origin. Subtracting $(-z^{3,y}|_{(-1,0)}w^{-1})$, we can arrange for the new flex z^4 to have zero velocity at joint $(-1, 0)$ and all the x-axis joints to the right of $(-1, 0)$.

5. Similarly, we can subtract multiples of w^n, $n \leq -2$, to achieve a flex z^5 with all vertices of the x-axis having zero velocities.

With the base fixed, z^5 must be a constant multiple of the flex u^{st} and by subtracting that we achieve a zero flex of $\mathcal{C}_{\mathrm{strip}}$. From all the above it follows that the set $\{\vec{x}\} \cup \{w^n : n \in \mathbb{Z}\} \cup \{u^{\mathrm{st}}\}$ is a free basis for $\mathcal{H}_{\mathrm{fl}}(\mathcal{C}_{\mathrm{strip}})$. $\square$

Moving on to the augmented grid, we can make use of the above argument to identify a free basis for $\mathcal{H}_{\mathrm{fl}}(\mathcal{C}^{+}_{\mathbb{Z}^2})$ as follows:

- Let u^n, $n \in \mathbb{Z}$, be the flex of infinitesimal translation of individual horizontal strips by one unit to the right labelled in a natural way such that u^0 acts on the x-axis and its augmented strip and so on (Figure 8.10).

- Let w^n be the band limited infinitesimal flex obtained by the extension of the local flexes of $\mathcal{C}_{\mathrm{strip}}$ such that w^n only has non zero velocity vectors for the framework triangles k such that triangles k have a "base joint" (k, y), $n \leq k \leq n + 1$.

- Let s^n be the infinitesimal flex of individual "base fixed" horizontal strips (Figure 8.10).

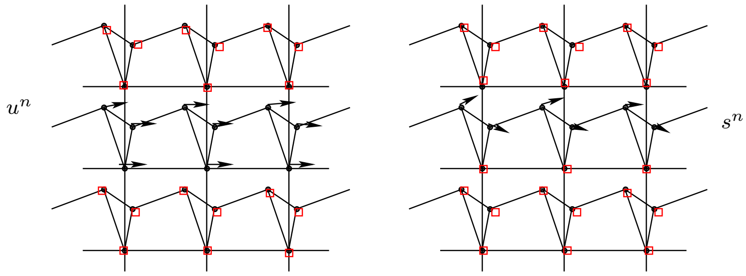

Figure 8.10: Basis elements u^n and s^n for the the augmented grid+strip framework $\mathcal{C}^+_{\mathbb{Z}^2}$

Proposition 8.3.5. *The set* $\{u^n, w^n, s^n : n \in \mathbb{Z}\}$ *is a free basis for* $\mathcal{H}_{\text{fl}}(\mathcal{C}^+_{\mathbb{Z}^2})$.

Proof. Follow the same steps as in the case of $\mathcal{C}_{\text{strip}}$, except substitute u^0 for $\vec{x}$. We first achieve a flex with zero velocities at all the vertices on the x-axis. This implies that when we go up to the next strip supported by the line $y = 1$, any flex of the vertices on $y = 1$ has to be a multiple of u^1. Subtracting this, we have a flex with zero velocities on the vertices of both lines, $y = 0$ and $y = 1$. Proceeding in the same manner, subtracting appropriate multiples of the u^n flexes upwards and downwards we can fix all the vertices lying on the basic grid.

With every "base" fixed for all the strips, each strip can only admit a flex which is a multiple of the flexes s^n. Subtracting these we arrive at a zero flex of $\mathcal{C}^+_{\mathbb{Z}^2}$. □

The regular octahedron net framework $\mathcal{C}_{\mathrm{Oct}}$. Viewing $\mathcal{C}_{\mathrm{Oct}}$ as countably many copies of the planar framework $\mathcal{C}_{\mathrm{sq}}$ one can identify a crystal flex basis for $\mathcal{C}_{\mathrm{Oct}}$ as follows: Let $\mathcal{C}_z$ be the grid framework in the xy-plane which

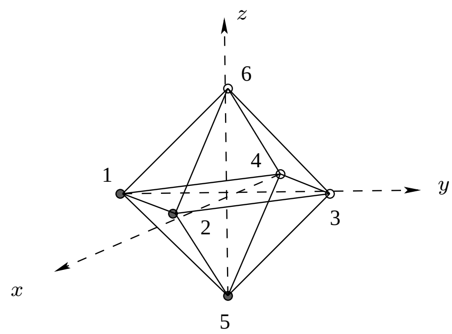

Figure 8.11: The framework $\mathcal{C}_{\mathrm{Oct}}$

contains the vertices p_1, p_2, p_3, p_4 (Figure 8.11). Similarly, let $\mathcal{C}_x$ denote the grid framework in the yz-plane which contain the vertices p_1, p_5, p_3, p_6 and let $\mathcal{C}_y$ denote the grid framework parallel to the zx-plane which contain the vertices p_2, p_5, p_4, p_6. Let $\mathcal{C}_x^n$ be the translated frameworks $\mathcal{C}_x + (2n, 0, 0)$, for $n \in \mathbb{Z}$, and similarly define $\mathcal{C}_y^n$ and $\mathcal{C}_z^n$. Then $\mathcal{C}_{\mathrm{Oct}}$ is the union of all these frameworks, that is, $\mathcal{C}_{\mathrm{Oct}}$ is the framework whose vertex set is is the union of all the vertices (without multiplicity) and whose set of edges is the union of all the edges. Also, define $\mathcal{C}_{\mathrm{Oct}}^+$ as the augmented framework in which each regular octagon is augmented by 3 edges parallel to the coordinate axis. Since the convex octagon is infinitesimally rigid, it follows that the vector spaces $\mathcal{H}_{\mathrm{fl}}(\mathcal{C}_{\mathrm{Oct}})$ and $\mathcal{H}_{\mathrm{fl}}(\mathcal{C}_{\mathrm{Oct}}^+)$ are isomorphic. Let $\mathcal{C}_{\mathrm{sq}}^+$ be the framework obtained from $\mathcal{C}_{\mathrm{sq}}$ by augmenting an edge to each rigid square, in this way the alternating rigid squares will have both cross diagonals. We may thus view $\mathcal{C}_{\mathrm{Oct}}^+$ as the union of copies of $\mathcal{C}_{\mathrm{sq}}^+$ where these copies are the augmentations of the frameworks $\tilde{\mathcal{C}}_x^n$, $\tilde{\mathcal{C}}_y^n$ and $\tilde{\mathcal{C}}_z^n$ of $\mathcal{C}_x^n$, $\mathcal{C}_y^n$ and $\mathcal{C}_z^n$ respectively. It follows that the alternation flex a of $\tilde{\mathcal{C}}_x^n$ extends to a flex a_n^x of $\mathcal{C}_{\mathrm{Oct}}^+$ with zero velocities at all the other vertices. We similarly define the alternating flexes a_n^y and a_n^z for $n \in \mathbb{Z}$. Let r^x, r^y and r^z be the infinitesimal rotations about the rotational axis of the central octahedron. We assume that, up to signs, these flexes are permuted by the action of the spatial symmetry group of $\mathcal{C}_{\mathrm{Oct}}$ We also assume the normalization such that for $\sigma = x, y, z$ the restrictions of a_n^σ agree with the restriction of r^σ. Finally, let $\vec{x}$, $\vec{y}$ and $\vec{z}$ be the velocities of infinitesimal translation by 1 unit in the axis directions.

In the next proof we make use of the following flex projection principle. If the edge $[p_a, p_b]$ lies in a plane $\mathcal{P}$ of $\mathbb{R}^3$ and if the vertex velocity vectors v_a, v_b in $\mathbb{R}^3$ give an infinitesimal flex of $[p_a, p_b]$ then the $\mathcal{P}$ components v_a', v_b' of v_a, v_b also give an infinitesimal flex of that edge. We say such a flex is *in-plane* when the plane in question is understood.

Theorem 8.3.6. *The set*

$$\mathcal{B} = \{\vec{x}, \vec{y}, \vec{z}, r^x, r^y, r^z\} \cup \{a_n^x, a_n^y, a_n^z : n \in \mathbb{Z}\}$$

is a crystal basis for $\mathcal{H}_{\mathrm{fl}}(\mathcal{C}_{\mathrm{Oct}})$.

Proof. The set $\mathcal{B}$ satisfies the crystal property and so it will suffice to show that it is a free basis for $\mathcal{C}_{\mathrm{Oct}}^+$. Let z be a vector in $\mathcal{H}_{\mathrm{fl}}(\mathcal{C}_{\mathrm{Oct}}^+)$. There is a linear combination z_{rig} of $\vec{x}, \vec{y}, \vec{z}, r^x, r^y, r^z$ which agrees with z on the vertices $p_1, \ldots, p_6$. Replacing z by $z - z_{\mathrm{rig}}$ we may assume that these velocities for z are zero. Now we make use of the flex projection principle. Note that the velocity vector z_{xy} given by the xy-plane projection of the velocities $z(p)$, for vertices in $\tilde{C}_z^0$. This in-plane flex is equal to the restriction of a scalar multiple $a_0^z - r^z$. In this way we obtain scalar multiples $\alpha_0(a_0^z - r^z)$, $\beta_0(a_0^x - r^x)$ and $\gamma_0(a_0^y - r^y)$ which provide the in-plane flexes of z for the planes $z = 0$, $x = 0$ and $y = 0$. Consider now the tower subframework given by the tower of octahedra whose connecting vertices lie on the z-axis. Since z is zero on the central octahedron supported by $p_1, \ldots, p_6$ denoted $O_{(0,0,0)}$, it follows that the z component of the of the velocity vector for a vertex on this line is zero. It also follows that there is a flex

$$\beta_0(a_0^x - r^x) + \gamma_0(a_0^y - r^y)$$

with velocity vectors agreeing with those of z for the vertices on the axial line. It follows similarly that there is a flex

$$w = \alpha_0(a_0^z - r^z) + \beta_0(a_0^x - r^x) + \gamma_0(a_0^y - r^y)$$

with this agreement property for the three axial lines through $O_{(0,0,0)}$. Replacing z by $z - w$ we may assume that z is zero on $O_{(0,0,0)}$ and all the vertices on the three axial lines of $O_{(0,0,0)}$. Note that the restriction of such a flex z to any other octahedron O with an axis on the coordinate axis must be an infinitesimal rotation flex of the octahedron about this axis. Also each such flex of an individual octahedron O, on the σ-axis say, agrees with the restriction of a scalar multiple of the local alternation flex a_n^σ, for some $n \neq 0$. Evidently these flexes act on distinct octahedra on the axial lines. It follows that there is an infinite linear combination of these flexes, w_2 say, whose restriction to any octahedron on a coordinate axis is equal to the restriction of z. Replacing z by $z - w_2$ we may assume that z is zero on this triple tower $\mathcal{T}$. The entire framework can built from the triple tower subframework by successively identifying the joints of attachment. It follows that z must be identically zero. Thus it follows that every velocity vector z in $\mathcal{H}_{\mathrm{fl}}(\mathcal{C}_{\mathrm{Oct}})$ is an infinite linear combination of the vectors in the set $\mathcal{B}$. Note that $\mathcal{B}$ is a weakly null sequence of velocity vectors and a free spanning set for the vector space of infinitesimal flexes. Moreover, the scalar coefficients in the identification above are determined uniquely by the vertex velocity vectors of the flex z. Thus $\mathcal{B}$ is a free infinitesimal flex basis as required. $\square$

Chapter 9

Further Developments And Related Work

In this chapter we suggest further developments related to some of the areas developed in the thesis. In Section 9.1 we define the class of *dilation periodic* bar-joint frameworks. In Section 9.2 we suggest new *convex polyhedron crystal frameworks* that can be obtained from familiar planar frameworks. Finally, in Section 9.3 we define *almost periodic* bar-joint frameworks.

9.1 Dilation Periodic Frameworks

In this section we define a class of infinite frameworks with special periodicity. Similar to crystal frameworks, dilation periodic frameworks can be identified by a finite motif and *dilation group* $\mathcal{D}$ such that $\mathcal{D} = \{D_k : k \in \mathbb{Z}\}$ and

$$D_k : (x, y) \to \alpha^k(x, y), \text{ where } 1 < \alpha.$$

Definition 9.1.1. A *dilation periodic framework* $\mathcal{G} = (F_v, F_e, \mathcal{T})$ in $\mathbb{R}^d$, with *motif* (F_v, F_e) and dilation group $\mathcal{D} = \{D_k : k \in \mathbb{Z}\}$ is a countable bar-joint framework with framework points $p_{\kappa,k}$, for $1 \leq \kappa \leq t, k \in \mathbb{Z}$, such that

(i) F_v is a finite set of framework vertices, $\{p_{\kappa,0} : 1 \leq \kappa \leq t\}$ in $\mathbb{R}^d$, and F_e is a finite set of framework edges,

(ii) for each κ and k the point $p_{\kappa,k}$ is $D_k p_{\kappa,0}$,

(iii) the set $\mathcal{G}_v$ of framework points is the disjoint union of the sets $D_k(F_v)$ for $k \in \mathbb{Z}$,

(iv) the set $\mathcal{G}_e$ of framework edges is the disjoint union of the sets $D_k(F_e)$ for $k \in \mathbb{Z}$.

A model example for a dilation periodic framework is the "two way" infinite cobweb $\mathcal{G}_{\text{cob}}$ (Figure 9.1). Let $\mathcal{D}$ be the dilation group $\mathcal{D} = \{D_k : k \in \mathbb{Z}\}$

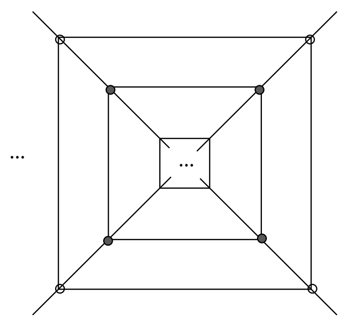

Figure 9.1: The infinite cobweb

such that

$$D_k(x, y) = 2^k(x, y).$$

Then $\mathcal{G}_{\text{cob}}$ is dilation periodic with motif vertex set $F_v = \{p_1, p_2, p_3, p_4\}$,

$$p_1 = p_{1,0} = (-1, -1),\ p_2 = p_{2,0} = (-1, 1),\ p_3 = p_{3,0} = (1, 1)$$
$$p_4 = p_{4,0} = (1, -1).$$

The motif edges are:

$$e_1 = [p_{1,0}, p_{2,0}],\ e_2 = [p_{2,0}, p_{3,0}],\ e_3 = [p_{3,0}, p_{4,0}],\ e_4 = [p_{4,0}, p_{1,0}],$$
$$e_5 = [p_{1,0}, p_{1,1}],\ e_6 = [p_{2,0}, p_{2,1}],\ e_7 = [p_{3,0}, p_{3,1}],\ e_8 = [p_{4,0}, p_{4,1}].$$

View the infinite cobweb as an increasing sequence of the "continuously infinitesimally rigid" double square finite frameworks. Then it follows that the infinite cobweb is continuously infinitesimally rigid.

Free basis for the infinite cobweb. A property not "exclusive" to crystal frameworks, it is possible to completely understand the flexibility and identify a free basis for the space of all infinitesimal flexes for the infinite cobweb. To do this let

1. u: the infinitesimal flex such that all the vertices of the base edges receive a velocity of magnitude 1 in the x-direction and with zero velocities elsewhere (Figure 9.2).

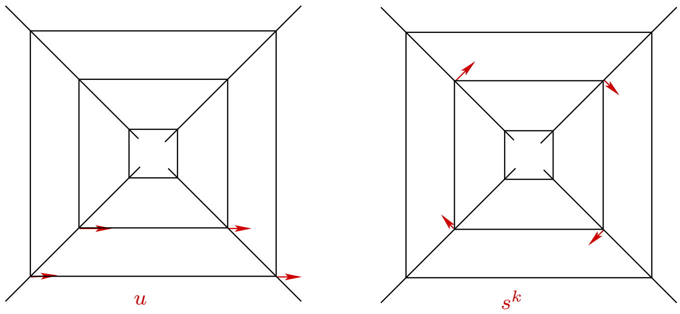

Figure 9.2: Basis elements u and s^k for the infinite cobweb $\mathcal{G}_{\text{cob}}$

2. v: the infinitesimal flex such that all the vertices of the edges to the left receive a velocity of magnitude 1 in the y-direction and with zero velocities elsewhere.

3. w: the infinitesimal flex such that all the vertices of the edges to the right receive a velocity of magnitude 1 in the y-direction and with zero velocities elsewhere.

4. r: the infinitesimal flex such that all the vertices of the top edges receive a velocity of magnitude 1 in the x-direction and with zero velocities elsewhere.

5. s^k, $k \in \mathbb{Z}^*$: the infinitesimal rotation of the individual corresponding squares with zero velocities elsewhere (Figure 9.2).

Note that infinitesimal flexes v, w and r can be obtained from u by a $\pi/2$, π and $3\pi/2$ rotations respectively.

Proposition 9.1.2. *The set* $\{u,\ v,\ w,\ r,\ s^k\ : k \in \mathbb{Z}\}$ *is a free basis for* *the space $\mathcal{H}_{\mathrm{fl}}(\mathcal{G}_{\mathrm{cob}})$ of all infinitesimal flexes of the infinite cobweb framework $\mathcal{G}_{\mathrm{cob}}$.*

Proof. Let z be an arbitrary flex of $\mathcal{G}_{\mathrm{cob}}$. Subtracting $z^x_{(-1,-1)}u + z^y_{(-1,-1)}v$, the resulting flex, z^1, satisfies $z^1_{(-1,-1)} = (0,0)$. To achieve a zero velocity at vertex $(1,-1)$, subtract $z^{1,y}w$ and the resulting flex z^2 cannot be a rigid body infinitesimal flex. Subtracting $z^{2,x}_{(-1,1)}r$ implies that the new flex ,z^3, satisfies

$$z^3_{(-1,-1)} = z^3_{(1,-1)} = z^3_{(1,1)} = z^3_{(-1,1)} = (0,0).$$

With all the vertices of one square now having zero velocities, subtracting appropriate multiples of s^k's inwards and outwards results in the zero flex of the cobweb. □

We expect that one could define a matrix function as in the case for crystal frameworks and use it to identify the space of *dilation phase periodic velocities and flexes*, for example.

9.2 Convex Polyhedra Spatial Crystals

Using the same layer construction of spatial crystal frameworks such as the bipyramid and the regular octahedron net, one can identify new crystal frameworks derived from familiar planar frameworks by placing different "bipyramids". The rigidity of convex polyhedra implies that these new frameworks can be analysed by viewing them as infinitely many copies of the planar frameworks depending on their geometry.

For example, one can form a corner connected square bipyramid framework similar to the octahedron net but with non-equal pyramids joined at their square faces. In this way, the resulting framework can be viewed as the union of countably many infinite copies of the planar frameworks $\mathcal{C}_{\text{sq}}$ and $\mathcal{C}_{\text{kite}}$, where $\mathcal{C}_{\text{kite}}$ is identified in [4] by the period vectors $(1, 0)$, $(0, 1)$ together with the motif given by Figure 9.3 and with motif vertices $(0, 0)$ and $(\frac{-1}{2}, \alpha)$, for some choice $0 < \alpha < \frac{1}{2}$. Similar to the framework $\mathcal{C}_{\text{sq}}$, $\mathcal{C}_{\text{kite}}$

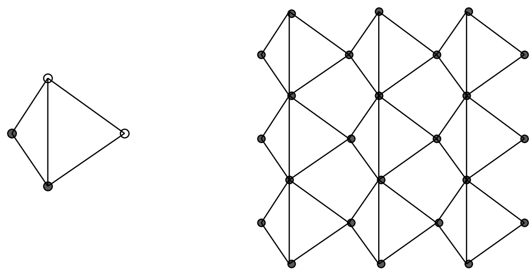

Figure 9.3: The kite framework $\mathcal{C}_{\text{kite}}$

admits a non-trivial infinitesimal flex, a^{kite}, where the individual kites undergo an infinitesimal rotation about the midpoints of the cross-bar edges. The rotation speeds are constant in the y direction and form a geometrically increasing sequence in any positive x direction. In [4], the set $\{\vec{x}, \vec{y}, \vec{r}, a^{\text{kite}}\}$ has been identified as a crystal flex basis for $\mathcal{C}_{\text{kite}}$. These facts can lead to the identification of a crystal flex basis for the new bipyramid framework (Figure 9.4).

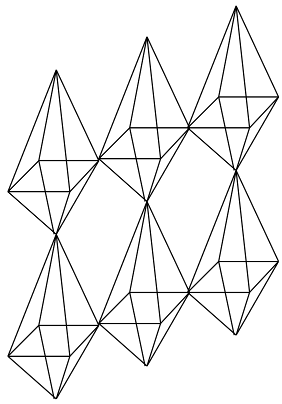

Figure 9.4: The kite-bipyramid framework

9.3 Almost Periodic Bar-Joint Frameworks

From the almost periodicity definitions, we can introduce the class of almost periodic bar-joint frameworks. Such frameworks can be obtained from existing crystal frameworks and it would be natural to investigate whether they can admit almost periodic infinitesimal flexes. Formally, let a crystal framework $\mathcal{C}$ be given, $\mathcal{C} = (G, p)$, $p = (p_{\kappa,k})$ and with translation group $\mathcal{T} = \{T_k : k \in \mathbb{Z}\}$. Let $\mathcal{C}'$ be a perturbation of $\mathcal{C}$, $\mathcal{C}' = (G, p')$, $p' = (p'_{\kappa,k})$ where

$$p'_{\kappa,k} = p_{\kappa,k} + \delta_{\kappa,k}.$$

A vector k' is said to be an ϵ-translation vector for $\mathcal{C}'$ if

$$\|T_{k'} p'_{\kappa,k-k'} - p'_{\kappa,k}\| \leq \epsilon \text{ for all } k \text{ and all } \kappa.$$

The framework $\mathcal{C}'$ is almost periodic if for every $\epsilon > 0$, the set of ϵ-translation vectors is relatively dense.

To develop an understanding of almost periodic frameworks one can start by investigating basic "strip" frameworks which are periodic in one direction as in the following example. Here we add an almost periodic sequence to the coordinates of the vertices.

Example. Let $\mathcal{C}$ be the framework in Figure 9.5. Let $\delta_{1,k} = (0, 0)$ and $\delta_{2,k} = (0, \alpha \sin \sqrt{2}\pi k)$ (with α small). Then the framework $\mathcal{C}'$ is almost periodic.

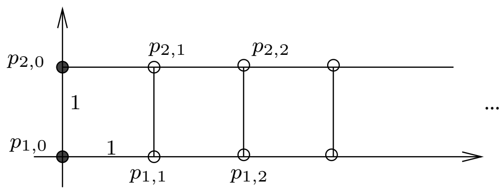

Figure 9.5: A periodic strip framework

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