**Monotonicity of the mid-point and trapezium estimates for integrals**

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*Introduction*

The "mid-point" approximation to the integral $\int_0^1 f$ is

$$M_n(f) = \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{2r-1}{2n}\right).$$

This represents the area obtained by adopting the mid-point value on each interval $[\frac{r-1}{n}, \frac{r}{n}]$, equally by taking the tangent to the curve (if there is one) at these mid-points within their respective intervals.

Meanwhile, the trapezium rule estimate using these sub-intervals is

$$T_n(f) = \frac{1}{2n}\sum_{r=0}^{n-1}\left[f\left(\frac{r}{n}\right) + f\left(\frac{r+1}{n}\right)\right].$$

For all reasonably well-behaved functions, both approximations converge to the integral as $n \to \infty$. Do they become closer to it as $n$ increases? A very simple example is enough to show that this is not always the case.

*Example 1.* Let $f(x) = |x - \frac{1}{2}|$. It is easily checked that $M_2(f) = T_2(f) = \frac{1}{4}$, coinciding with the integral (a diagram helps). However,

$$M_3(f) = \tfrac{1}{3}(\tfrac{1}{3} + 0 + \tfrac{1}{3}) = \tfrac{2}{9}, \qquad T_3(f) = \tfrac{1}{6}(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{2}) = \tfrac{5}{18}.$$

Both approximations have a particular resonance for *convex* functions. Recall that a function $f$ is convex (informally, curving upwards) if it lies below the straight-line chord between any two points on its graph. Convex functions lie above their tangents (see below for details). From these two descriptions, it is clear that $M_n(f) \le \int_0^1 f \le T_n(f)$ for such $f$. So one might expect $M_n(f)$ to increase with $n$ and $T_n(f)$ to decrease. However, Example 1 disposes of this idea, since the function there is convex.

The following partial result is true: if $n$ is a multiple of $m$ and $f$ is convex, then $M_n(f) \ge M_m(f)$ and $T_n(f) \le T_m(f)$. In both cases, this is quite easily seen from the fact that each sub-interval for $m$ divides neatly into a number of sub-intervals for $n$.

For the general case, the correct statement is that $M_n(f)$ increases with $n$, and $T_n(f)$ decreases, under the further condition that the derivative $f'$ is either convex or concave. This was established by the author and Grahame Bennett in [1], but the proofs given there are long and intricate: I would not recommend them to anyone. For $M_n(f)$, Bennett gave a more pleasant proof in [2, Theorem 1]. Actually, his proof delivers a more general result: here I will reproduce it in the slightly simplified version applying to $M_n(f)$. It must be conceded that even this version is not without work; however, it is a vast improvement on the original and (in my view) rather elegant. I will then show how the method can be adapted to prove the result for $T_n(f)$. This has not appeared anywhere in Bennett's papers, though a proof restricted to functions of the form $x^p$ was given in [3, p. 1055]. (Sadly, Bennett died in 2016, after a lifetime of pioneering work in the area of inequalities.)

Of course, these statements can be translated to a general interval $[a, b]$ by the substitution $F(x) = f[a + (b - a)x]$. Having said this once, we shall present everything for the interval $[0, 1]$.

A few more remarks about convex functions will help to prepare the way. Formally, the definition is: $f$ is convex on the interval $I$ if for $x_1$, $x_2$ in $I$ and $0 \le \lambda \le 1$, we have $f[(1-\lambda)x_1 + \lambda x_2] \le (1-\lambda)f(x_1) + \lambda f(x_2)$. It is "strictly convex" if strict inequality holds for $0 < \lambda < 1$. We say that $f$ is "concave" if $-f$ is convex. If $f$ is both convex and concave, then it is linear. Clearly, if $f$ is convex on $[0, 1]$, then so is $f(1 - x)$.

For differentiable functions, convexity is equivalent to $f'(x)$ increasing with $x$: with the mean-value theorem, it follows that the function lies above its tangents, as stated earlier. Clearly, it is sufficient if $f''(x) \ge 0$; if $f''(x) > 0$, then $f$ is strictly convex. In particular, $x^p$ is strictly convex for $x > 0$ if $p > 1$ or $p < 0$, and strictly concave if $0 < p < 1$ .

An equivalent way to state convexity is: if $m_f(x_1, x_2)$ is the gradient of the chord between $x_1$ and $x_2$, and $x_1 < x < x_2$, then $m_f(x_1, x) \le m_f(x, x_2)$. Now let $g(x)$ denote the linear function agreeing with $f$ at $x_1$ and $x_2$. By definition, if $x_1 < x < x_2$, then $f(x) \le g(x)$. The following fact, less frequently mentioned, will be important for us: if $y > x_2$, then $m_f(x_1, x_2) \le m_f(x_2, y)$, hence $f(y) \ge g(y)$, and similarly for $y < x_1$.

*An inequality for functions with convex derivative*

The method is based on an inequality for functions with convex derivative which is of interest in its own right, and has other applications. We now present it, in two versions.

*Proposition 1*: Suppose that $a_1$, $a_2$, $a_3$ and $b_1$, $b_2$, $b_3$ are real numbers such that

$$a_1 < b_1 \le b_2 < a_2 \le a_3 < b_3. \tag{1}$$

Let $f$ be a function such that $f'$ is convex on $[a_1, b_3]$. Let $p$, $q$, $r$ be positive numbers such that

$$pa_1 + qa_2 + ra_3 = pb_1 + qb_2 + rb_3. \tag{2}$$

Suppose further that either

$$pa_1^2 + qa_2^2 + ra_3^2 = pb_1^2 + qb_2^2 + rb_3^2 \tag{3}$$

or that $f$ is also convex and

$$pa_1^2 + qa_2^2 + ra_3^2 \leq pb_1^2 + qb_2^2 + rb_3^2. \tag{4}$$

Then

$$pf(a_1) + qf(a_2) + rf(a_3) \leq pf(b_1) + qf(b_2) + rf(b_3). \tag{5}$$

*Proof*: First, assume (3). Conditions (2) and (3) can be rewritten as

$$q(a_2 - b_2) = p(b_1 - a_1) + r(b_3 - a_3),$$

$$q(a_2^2 - b_2^2) = p(b_1^2 - a_1^2) + r(b_3^2 - a_3^2).$$

These identities equate, respectively, to the statements that

$$q\int_{b_2}^{a_2} g = p\int_{a_1}^{b_1} g + r\int_{a_3}^{b_3} g \tag{6}$$

for $g(x) = 1$ and for $g(x) = x$, and hence for all linear $g(x) = mx + n$. Now take $g$ to be the linear function agreeing with $f'$ at $b_2$ and $a_2$. Since $f'$ is convex, we have $f' \leq g$ on $[b_2, a_2]$, while $f' \geq g$ on $[a_1, b_1]$ and $[a_3, b_3]$. Hence

$$q\int_{b_2}^{a_2} f' \leq p\int_{a_1}^{b_1} f' + r\int_{a_3}^{b_3} f'.$$

In other words,

$$q[f(a_2) - f(b_2)] \leq p[f(b_1) - f(a_1)] + r[f(b_3) - f(a_3)],$$

which equates to (5).

Now assume that $f$ is convex (so that $f'$ is increasing) and (4) holds. Then equality is replaced by $\leq$ in (6) for $g(x) = x$, hence also for $g(x) = mx+n$ with $m \geq 0$. This condition is satisfied by the linear function agreeing with $f'$ at $b_2$ and $a_2$, since $f'(b_2) \leq f'(a_2)$. Inequality (5) follows as before.

Of course, if $f'$ is strictly convex, then strict inequality holds in (5).

Applied to $-f$, the Proposition says that if $f'$ is concave, and in the second version if $f$ is also concave, with the other conditions unchanged, then the reverse of (5) applies. Note that if $f(x) = x^p$, then $f'(x) = px^{p-1}$ is strictly convex for $p > 2$ and $0 < p < 1$, and strictly concave for $1 < p < 2$ and $p < 0$. (Apologies for the double use of $p$!)

Some further remarks on the hypotheses are in order. First, there is some redundancy in conditions (1) and (2). Given that $a_1 < b_1, \; a_3 < b_3$ and $p > 0, \;\; r > 0$, the rewritten version of (2) shows that if $q > 0$, then $a_2 > b_2$, and conversely if $a_2 > b_2$, then $q > 0$. Second, if $a_j$, $b_j$ satisfy (1), (2), (3) or (1), (2), (4), then so do the numbers $a_j + h$, $b_j + h$ for any $h$.

The special case $p = q = r = 1$ is already of interest: given (1), if $\sum_{j=1}^3 a_j = \sum_{j=1}^3 b_j$ and $\sum_{j=1}^3 a_j^2 = \sum_{j=1}^3 b_j^2$, then $\sum_{j=1}^3 f(a_j) \le \sum_{j=1}^3 f(b_j)$ when $f'$ is convex, and the opposite when $f'$ is concave. So, for instance, $\sum_{j=1}^3 a_j^p < \sum_{j=1}^3 b_j^p$ (strict inequality) for all $p > 2$. There are plenty of integer triples that that satisfy these conditions, for example $(a_j) = (1,4,4)$, $(b_j) = (2,2,5)$ and $(a_j) = (1,5,6)$, $(b_j) = (2,3,7)$. (A systematic description of such pairs of triples would be interesting, but we will not embark upon it here.) A completely different route to results of this sort, but restricted to the functions $x^p$, is by a generalisation of Descartes' rule of signs: see [4, Example 3].

We now restate Proposition 1 for the case where $a_2 = a_3$ and $b_1 = b_2$. A change of notation is appropriate. Write $a_1 = a, \;\; b_1 = b_2 = b, \;\; a_2 = a_3 = c$ and $b_3 = d$. Then (2) becomes

$$pa + (q + r)c = (p + q)b + rd,$$

equivalently $q(c - b) = p(b - a) + r(d - c)$, and (3), (4), (5) can be rewritten similarly. We make the further substitution

$$p = \alpha, \quad p + q = \beta, \quad q + r = \gamma, \quad r = \delta.$$

The conclusion now appears as follows. This was Bennett's version, given in [2, Lemma 2] and [3, Theorem 8].

*Proposition 2*: Suppose that $a < b < c < d$ and $\alpha$, $\beta$, $\gamma$, $\delta$ are positive numbers such that

$$\alpha + \gamma = \beta + \delta, \tag{7}$$

$$\alpha a + \gamma c = \beta b + \delta d. \tag{8}$$

Let $f$ be a function such that $f'$ is convex on $[a, d]$. Suppose that either

$$\alpha a^2 + \gamma c^2 = \beta b^2 + \delta d^2, \tag{9}$$

or that $f$ is convex and

$$\alpha a^2 + \gamma c^2 \le \beta b^2 + \delta d^2, \tag{10}$$

Then

$$\alpha f(a) + \gamma f(c) \le \beta f(b) + \delta f(d). \tag{11}$$

*Proof*: Let $p = \alpha, \;\; r = \delta$ and $q = \beta - \alpha = \gamma - \delta$. Then (8), (9), (10), (11) translate into (2), (3), (4), (5), rewritten as above in terms of $a$, $b$, $c$, $d$. As remarked previously, the fact that $q > 0$ follows from $p > 0$ and $r > 0$ and condition (2). So Proposition 1 translates into the statement given.

To set this result in a wider context, we divert briefly here to mention a much simpler statement of the same sort which applies when the two middle values are compared with the two outside ones. No quadratic condition or assumption about $f'$ is needed.

*Proposition 3*: Let $a$, $b$, $c$, $d$ be real numbers with $a < d$ and $b$, $c$ in $[a, d]$. Let $\alpha$, $\beta$, $\gamma$, $\delta$ be non-negative numbers such that

$$\beta + \gamma = \alpha + \delta,$$

$$\beta b + \gamma c = \alpha a + \delta d.$$

Then for any convex function $f$ on $[a, d]$,

$$\beta f(b) + \gamma f(c) \leq \alpha f(a) + \delta f(d).$$

*Proof*: The hypotheses are equivalent to the statement that for any linear function $g$, $\beta g(b) + \gamma g(c) = \alpha g(a) + \delta g(d)$. Take $g$ to be the linear function agreeing with $f$ at $a$ and $d$: then $f(b) \leq g(b)$ and $f(c) \leq g(c)$. The statement follows. (It does not matter whether $c$ is greater or less than $b$.)

By applying this twice, one can derive a companion result to Proposition 1 for the case where $a_1 > b_1$ and $a_3 < b_3$: given this, condition (2) and convexity of $f$, then (5) follows.

However, Proposition 2 is what we need for our purposes. The quadratic condition (10) can be laborious to verify in particular cases. The following Lemma sometimes serves to simplify it.

*Lemma 1*: Let $a < b < c < d$. Given (7) and (8), a sufficient condition for (10) is: $b + c \leq a + d$ and $\alpha \leq \delta$.

*Proof*: We revert to the notation $p$, $q$, $r$, so $p \leq r$. In these terms, (10) is equivalent to

$$q(c^2 - b^2) \leq p(b^2 - a^2) + r(d^2 - c^2).$$

Recall that $q(c - b) = p(b - a) + r(d - c)$. Substituting this, we have

$$\begin{aligned} p(b^2 - a^2) - q(c^2 - b^2) + r(d^2 - c^2) &= p(b^2 - a^2) - (c + b)[p(b - a) + r(d - c)] + r(d^2 - c^2) \\ &= r(d - c)(d - b) - p(b - a)(c - a) \\ &\geq 0, \end{aligned}$$

since $r \geq p$, $d - c \geq b - a$ and $d - b \geq c - a$.

*The mid-point estimate*

It will slightly simplify the formulae if we replace $f[(2r-1)/2n]$ by $f[(2r-1)/n]$ in the definition of $M_n(f)$ (this amounts to considering $f(2x)$ instead of $f(x)$). So we now take

$$M_n(f) = \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{2r-1}{n}\right).$$

*Theorem 1*: If $f$ is convex and $f'$ is either convex or concave on $(0,2)$, then $M_n(f)$ increases with $n$.

*Proof*: We use Proposition 2 to prove the statement when $f'$ is convex. The result for concave $f'$ then follows by applying it to $g(x) = f(2-x)$: then $M_n(f) = M_n(g)$, and $g$ and $g'$ are convex, since $g'(x) = -f'(2-x)$.

The statement $M_n(f) \le M_{n+1}(f)$ equates to

$$(n+1)\sum_{r=1}^{n} f\left(\frac{2r-1}{n}\right) \le n\sum_{r=1}^{n+1} f\left(\frac{2r-1}{n+1}\right). \tag{12}$$

Bennett's master stroke in [2] is the introduction of extra terms that cancel. For certain terms $J_r$ to be chosen (with $J_0 = 0$), we will prove an inequality of the form

$$(n+1)f\left(\frac{2r-1}{n}\right) + J_{r-1} \le nf\left(\frac{2r-1}{n+1}\right) + J_r \tag{13}$$

for $1 \le r \le n$. Addition then gives

$$(n+1)\sum_{r=1}^{n} f\left(\frac{2r-1}{n}\right) \le n\sum_{r=1}^{n} f\left(\frac{2r-1}{n+1}\right) + J_n.$$

To recapture (12), we require

$$J_n = nf\left(\frac{2n+1}{n+1}\right).$$

This is achieved by taking $J_r = rf(rE_n)$, where

$$E_n = \frac{2n+1}{n(n+1)} = \frac{1}{n} + \frac{1}{n+1}.$$

We prove (13), with $J_r$ defined in this way. The case $r = 1$ says

$$(n+1)f\left(\frac{1}{n}\right) \le nf\left(\frac{1}{n+1}\right) + f(E_n).$$

This follows directly from convexity of $f$, since $\frac{n+1}{n} = \frac{n}{n+1} + E_n$, hence $\frac{1}{n} = \frac{n}{n+1}\frac{1}{n+1} + \frac{1}{n+1}E_n$. For $2 \le r \le n$, we apply Proposition 2, with

$$a = (r-1)E_n, \quad b = \frac{2r-1}{n+1}, \quad c = \frac{2r-1}{n}, \quad d = rE_n,$$

$$\alpha = r - 1, \quad \beta = n, \quad \gamma = n + 1, \quad \delta = r.$$

Then $\alpha + \gamma = \beta + \delta$. Also,

$$\delta d - \alpha a = [r^2 - (r-1)^2]E_n = (2r-1)E_n,$$

$$\gamma c - \beta b = (2r-1)\left(\frac{n+1}{n} - \frac{n}{n+1}\right) = (2r-1)E_n.$$

Clearly, $b < c$. The condition $a < b$ is equivalent to $(r-1)(2n+1) < n(2r-1)$, which equates to $r - 1 < n$. The condition $c < d$ is equivalent to $(2r-1)(n+1) < r(2n+1)$, which equates to $r < n + 1$. Finally, we use Lemma 1 to verify the quadratic condition (10). We have $\alpha < \delta$ and

$$b + c = (2r-1)\left(\frac{1}{n+1} + \frac{1}{n}\right) = (2r-1)E_n = a + d.$$

So if $f$ is concave and $f'$ is convex or concave, then $M_n(f)$ decreases with $n$.

Applied to $f(x) = x^p$, the conclusion is that $\frac{1}{n^{p+1}} \sum_{r=1}^{n} (2r-1)^p$ increases with $n$ if $p \geq 1$ or $p \leq 0$, and decreases if $0 \leq p \leq 1$: in all cases, $f'$ is either convex or concave (of course, the expression is constant if $p$ is 0 or 1). However, a direct proof of this result is not difficult: see [6]. Another application of Theorem 1 is:

*Corollary*: Let $Q_n = \prod_{r=1}^{n} (2r-1)$. Then $\frac{1}{n} Q_n^{1/n}$ decreases with $n$.

*Proof*. Let $f(x) = \log x$. Then $f$ is concave and $f'$ is convex, so $M_n(f)$ decreases with $n$. But

$$M_n(f) = \frac{1}{n} \sum_{r=1}^{n} \Big(\log(2r-1) - \log n\Big) = \frac{1}{n} \log Q_n - \log n = \log\left(\frac{1}{n} Q_n^{1/n}\right).$$

We mention that $\lim_{n\to\infty} M_n(f) = \int_0^1 \log 2x \, dx = \log 2 - 1$, hence $\frac{1}{n} Q_n^{1/n} \to \frac{2}{e}$ as $n \to \infty$.

We describe another application of the reasoning in Theorem 1, rather than the Theorem itself.

*Example 2*. We show that if $f$ is convex and $f'$ is either convex or concave, then

$$2f(2) + 2f(6) \leq f(1) + 2f(4) + f(7).$$

(This Example is given in [3, p. 1047]. Its significance, for any readers familar with the concept, is that it shows that $(6, 6, 2, 2)$ is "power majorised" by $(7, 4, 4, 1)$, although it is not majorised; other readers can ignore this comment.) We prove the statement for convex $f'$; the statement for concave $f'$ then follows by considering $g(x) = f(8 - x)$. As in the proof of Theorem 1, we introduce an extra term: by convexity of $f$, we have $2f(2) \leq f(1) + f(3)$. To prove our statement, we now require

$$f(3) + 2f(6) \leq 2f(4) + f(7),$$

which follows at once from Proposition 1, applied to $(3, 6, 6)$ and $(4, 4, 7)$.

*The trapezium rule*

We now present the companion result for the trapezium rule. The steps are analogous, but the details are different.

*Theorem 2*: If $f$ is convex and $f'$ is either convex or concave on $[0,1]$, then $T_n(f)$ decreases with $n$.

*Proof.* We use Proposition 2 to prove the statement when $f'$ is convex. The result for concave $f'$ then follows, since $T_n(f) = T_n(g)$, where $g(x) = f(1-x)$.

We have to show

$$n\sum_{r=0}^{n}\left[f\left(\frac{r}{n+1}\right)+f\left(\frac{r+1}{n+1}\right)\right] \le (n+1)\sum_{r=0}^{n-1}\left[f\left(\frac{r}{n}\right)+f\left(\frac{r+1}{n}\right)\right]. \tag{14}$$

Again we introduce extra terms that cancel. We will show that

$$nf\left(\frac{r}{n+1}\right)+nf\left(\frac{r+1}{n+1}\right)+J_{r+1} \le (n+1)f\left(\frac{r}{n}\right)+(n+1)f\left(\frac{r+1}{n}\right)+J_r \tag{15}$$

for $0 \le r \le n-1$, for terms $J_r$ to be chosen (with $J_0 = 0$). Addition then gives

$$n\sum_{r=0}^{n-1}\left[f\left(\frac{r}{n+1}\right)+f\left(\frac{r+1}{n+1}\right)\right]+J_n \le (n+1)\sum_{r=0}^{n-1}\left[f\left(\frac{r}{n}\right)+f\left(\frac{r+1}{n}\right)\right].$$

To recapture (14), we require

$$J_n = n\left[f\left(\frac{n}{n+1}\right)+f(1)\right].$$

We take

$$J_r = r\left[f\left(\frac{r}{n+1}\right)+f\left(\frac{r}{n}\right)\right].$$

With this choice, (15) becomes

$$(n-r)f\left(\frac{r}{n+1}\right)+(n+r+1)f\left(\frac{r+1}{n+1}\right) \le (n+r+1)f\left(\frac{r}{n}\right)+(n-r)f\left(\frac{r+1}{n}\right).$$

The case $r=0$ says $(n+1)f[1/(n+1)] \le f(0)+nf(1/n)$, which follows from convexity of $f$. For $1 \le r \le n-1$, we apply Proposition 2 with

$$a = \frac{r}{n+1}, \quad b = \frac{r}{n}, \quad c = \frac{r+1}{n+1}, \quad d = \frac{r+1}{n},$$

$\alpha = \delta = n-r$ and $\beta = \gamma = n+r+1$. Then $a<b<c<d$ and

$$\alpha a+\gamma c = \frac{1}{n+1}\Big[(n-r)r+(n+r+1)(r+1)\Big] = \frac{(n+1)(2r+1)}{n+1} = 2r+1,$$

$$\beta b+\delta d = \frac{1}{n}\Big[(n+r+1)r+(n-r)(r+1)\Big] = \frac{n(2r+1)}{n} = 2r+1.$$

Again we use Lemma 1 to verify (10). We have $\alpha = \delta$ and

$$b - a = \frac{r}{n(n+1)} < \frac{r+1}{n(n+1)} = d - c.$$

*Note*: The choice of $J_r$ is critical. The author first tried $f(\frac{r+1}{n+1})$ instead of $f(\frac{r}{n})$ for the second term: this fails dismally!

Explicit applications of Theorem 2 can involve awkward expressions because of the half values at the end points. However, these disappear if $f(0) = f(1) = 0$, as in the following example.

*Example 3*: If $0 < p \le 1$, then

$$\frac{1}{n^{p+2}} \sum_{r=1}^{n-1} r^p (n-r)$$

increases with $n$. To show this, let $f(x) = x^p - x^{p+1}$. Then $f$ is concave, since $x^p$ is concave and $x^{p+1}$ is convex. Similarly, $f'$ is convex. We have

$$T_n(f) = \frac{1}{n} \sum_{r=1}^{n-1} \left( \frac{r^p}{n^p} - \frac{r^{p+1}}{n^{p+1}} \right) = \frac{1}{n^{p+2}} \sum_{r=1}^{n-1} r^p (n-r).$$

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