ON THE (CROSSED) BURNSIDE RING OF PROFINITE GROUPS

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Abstract. In this paper we investigate some properties of the Burnside ring of a profinite group as defined in [6]. We introduce the notion of the crossed Burnside ring of a profinite FC-group, and generalise some results from finite to profinite (FC-)groups. In our investigations, we also obtain results on profinite FC-groups which may be of independent interest.

1. Introduction

The Burnside ring $B(G)$ of a group G is a commutative ring, which encodes in some way useful information about the abstract group G (e.g. [5]). In particular, if G is a finite group, the mapping of a subgroup H of G to the underlying abelian group of the Burnside ring $B(H)$ defines a projective Mackey functor for G, and this fact is key in the study of Mackey functors for finite groups [17]. The crossed G-sets of a finite group G act on the category of Mackey functors for G, leading to a decomposition of the Mackey algebra of G into p-blocks, after extending the scalars to some suitable p-local ring for a given prime p dividing the order of G [2].

In [5], Dress proves that, for G finite, there exists a 1-1 correspondence between the connected components of the prime ideal spectrum of $B(G)$ and the conjugacy classes of perfect subgroups of G. In [7], Gluck gives a formula to calculate the primitive idempotents of $B(G)$, and he uses his result to provide an algebraic proof of Brown's result on the Euler characteristic of the simplicial complex whose vertices are the nontrivial p-subgroups of G.

In [6], the authors introduce the Burnside ring $\widehat{B}(G)$ of a profinite group G as a generalisation of the Burnside ring of a finite group. In [6, Section 5], the authors hint at certain properties of $\widehat{B}(G)$, which are similar to those of Mackey functors according to Dress [17, Section 2]. Their results have been used in [1] to study Mackey functors arising in number theory.

The main objective of the present paper is to investigate a generalisation to profinite groups of the above results on the (crossed) Burnside ring of a finite groups and its applications. Hence, in Sections 2 and 3, we review the known background on crossed Burnside rings for finite groups and on the Burnside ring of a profinite group. These lead us to take a closer look at profinite FC-groups in Section 4. An *FC-group* is a group in which every element has a finite conjugacy class. That is, a group G is FC if and only if $C_G(g)$ has finite index in G, for all $g \in G$. Properties of FC-groups are described in [18], and also in [15, Section 14.5]. In particular, finite groups, and profinite abelian groups are profinite FC. We make a few observations about the structure of profinite FC-groups, which we have not found in the literature, and which may be of independent interest. In Section 5, we use Dress and Siebeneicher's construction of the Burnside ring of a profinite group and define the crossed Burnside ring [13] of profinite FC-groups. In Section 6, we generalise in some way Dress and Gluck's results on the idempotents of the Burnside $\mathbb{Q}$-algebra and Burnside ring of a finite group to the class of profinite groups. Finally, in Section 7, we turn to Mackey functors, and review the approaches in [1, 6], before generalising Oda and Yoshida's results [2, 13] to obtain an action of almost finite crossed G-spaces on the category of Mackey functors.

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2. Background on the crossed Burnside ring of finite groups

We recall the needed background on crossed Burnside rings of finite groups from [13, Section 2]. Let G be a finite group and let $\mathbf{Set}_G$ denote the category whose objects are the finite G-sets and the morphisms are G-equivariant maps. Let S be a normal subgroup of G, which we regard as a G-set for the conjugation action: $(g,s)\mapsto {}^gs=gsg^{-1}$ for all $g\in G$ and all $S\in S$. Then S is a G-monoid. That is, S is a G-set equipped with a multiplication $S\times S\to S$ for which there is a multiplicative identity 1_S.

Definition 2.1. [13, (2.6,2.7)] Let S be a normal subgroup of G. A *crossed G-set over S* is a morphism $f:X\to S$ in $\mathbf{Set}_G$.

Given two crossed G-sets over S, say $f_i:X_i\to S$, for $i=1,2$, their sum and product are the crossed G-sets:

$$f_1+f_2\ :\ X_1\sqcup X_2\ \longrightarrow S\quad\text{and}\quad f_1\times f_2\ :\ X_1\times X_2\ \longrightarrow S,$$

where

$$(f_1+f_2)(x)=f_i(x)\quad\text{for } x\in X_i,\text{ for } i=1,2,\text{ and}\quad (f_1\times f_2)(x_1,x_2)=f_1(x_1)f_2(x_2).$$

The additive identity element is the unique morphism $\emptyset\to S$, where $\emptyset$ is the empty set (i.e. the initial object in $\mathbf{Set}_G$), and the multiplicative identity is $u:G/G\to S$, where $u(G)=1_S$. Addition and multiplication are commutative up to isomorphism. In particular,

$$\begin{array}{rcl}\big(f_1\times f_2\ :\ X_1\times X_2\ \longrightarrow S\big) & \longrightarrow & \big(f_2\times f_1\ :\ X_2\times X_1\ \longrightarrow S\big)\\ (f_1\times f_2)(x_1,x_2) & \longmapsto & (f_2\times f_1)(f_1(x_1)x_2,x_1),\end{array}$$

is an isomorphism of crossed G-sets since

$$f_2\big(f_1(x_1)x_2\big)f_1(x_1)={}^{f_1(x_1)}f_2(x_2)f_1(x_1)=f_1(x_1)f_2(x_2)\quad\text{in } S.$$

Define the category ${}^{\times}\mathbf{Set}_{(G,S)}$ to be the category whose objects are the crossed G-sets over S, and the morphisms

$$\phi:(f_1:X_1\to S)\longrightarrow(f_2:X_2\to S)\quad\text{in } {}^{\times}\mathbf{Set}_{(G,S)}$$

are the G-equivariant maps $\phi:X_1\to X_2$ such that $f_2\phi=f_1:X_1\to S$.

The category of crossed G-sets over S is a commutative monoid. The Grothendieck construction [11, Section 24.1] turns the abelian monoid of isomorphism classes of crossed G-sets over S into a commutative ring ${}^{\times}B(G,S)$, called the *crossed Burnside ring* of G over S . That is, the elements of ${}^{\times}B(G,S)$ are the isomorphism classes of virtual crossed G-sets over S, which can be written as differences

$$[f_1:X_1\to S]-[f_2:X_2\to S],$$

where $[f_i:X_i\to S]$ are isomorphism classes of crossed G-sets over S.

Unless otherwise stated, we will henceforth take $S=G$ as G-monoid with conjugation action of G, and we denote it G^c to avoid any confusion. Thus, $G^c=\displaystyle\bigsqcup_{x\in\mathrm{Cl}(G)}G/C_G(x)$, where $\mathrm{Cl}(G)$ is a set of representatives of the conjugacy classes of the elements of G. Then, we let $B^c(G)$ denote the crossed Burnside ring of G over G^c and simply call it the *crossed Burnside ring of G*.

As a group, $B^c(G)$ is free abelian with basis the isomorphism classes of transitive crossed G-sets. These have the form $[w_a:G/H\to G^c]$, where $w_a(gH)={}^ga$ for some $a\in C_G(H)$. We have $(w_a:G/H\to G^c)\cong(w_b:G/K\to G^c)$ as crossed G-sets if and only if $K={}^gH$ and $b={}^ga$ for some $g\in G$. The Burnside ring $B(G)$ of G embeds into $B^c(G)$ via the injective ring homomorphism: $[G/H]\mapsto[w_1:G/H\to G^c]$. We refer to [2, 13] for further properties of $B^c(G)$ for a finite group.

3. From finite to profinite

Let G be a profinite group. By a *subgroup* of G, we mean a *closed subgroup* of G. If U is an open (normal) subgroup of G, we write $U \leq_o G$ ($U \trianglelefteq_o G$). We refer the reader to [14, 19] for the background on profinite groups.

We recall the definition of the Burnside ring of a profinite group and the basic concepts introduced in [6, Section 2], referring the interested reader to that article for the details.

A *G-space* is a Hausdorff topological space X equipped with a continuous G-equivariant action $\rho : G \times X \to X$. For $x \in X$, its stabiliser is the closed subgroup $G_x = \{g \in G \mid gx = x\}$ of G and its orbit is the closed compact subset $Gx = \{gx \mid g \in G\}$ of X. Throughout, we denote $G\backslash X$ the set of G-orbits of X, and $[G\backslash X]$ a set of representatives.

We call X *essentially finite* if the fixed point sets $|X^U|$ are finite for all the open subgroups U of G. A G-space X is *almost finite* if X is an essentially finite discrete topological space.

Given two essentially finite G-spaces X, Y, we define an equivalence relation

$$X \sim Y \quad \text{if and only if} \quad |X^U| = |Y^U|,\ \forall\, U \leq_o G,$$

where $X^U = \{x \in X \mid ux = x,\ \forall\ u \in U\}$. It follows that two almost finite G-spaces are equivalent if and only if they are isomorphic. If X is an essentially finite G-space, then the equivalence class $[X]$ of X contains an almost finite G-space which is unique up to isomorphism. In other words, considering equivalence classes of essentially finite G-spaces is the same as considering isomorphism classes of almost finite G-spaces. Observe that if X is an essentially finite G-space, then for all $U \leq_o G$, the set of U-fixed points X^U is a finite $N_G(U)/U$-set.

Suppose that X is a discrete G-space and write $X = \sqcup_{x\in[G\backslash X]} Gx$ as the disjoint union of its G-orbits. Since G is compact, every orbit is a compact discrete G-space, i.e. finite. It follows that the bijection from the coset space G/G_x to Gx, defined by $gG_x \mapsto gx$, is a homeomorphism. Hence, a discrete G-space X is almost finite if

$$X \cong \bigsqcup_{x\in[G\backslash X]} G/G_x, \quad \text{where} \quad G_x \leq_o G, \text{ and} \tag{1}$$

for all $U \leq_o G$, there exist finitely many orbits Gx with U contained in a G-conjugate of G_x.

Definition 3.1. Let $\mathcal{A}F_G$ be the category of *almost finite G-spaces*. The objects are the almost finite G-spaces, and the morphisms $f : X \to Y$ between two almost finite G-spaces X and Y are the G-equivariant maps (necessarily continuous). We write $\mathrm{Hom}_{\mathcal{A}F_G}(X, Y)$ for the set of morphisms $X \to Y$.

Similarly to the case of finite groups, if X and Y are almost finite G-spaces, then $f : X \to Y$ can be expressed as

$$\big(f_{x,y}\big)_{x,y} \quad \text{where } (x, y) \text{ runs through } [G\backslash X] \times [G\backslash Y]$$

and $f_{x,y} : G/G_x \to G/G_y$ is of the form $f_{x,y}(G_x) = gG_y$ for some $g \in G$ such that $G_x \leq {}^gG_y$. In particular, $G/U \cong G/V$ as almost finite G-spaces if and only if U and V are G-conjugate.

The isomorphism classes of almost finite G-spaces form an abelian monoid, with addition given by disjoint unions, and multiplication given by the cartesian product. Recall that

$$(X \times Y)^U = X^U \times Y^U, \quad \forall\, U \leq_o G,\ \forall\, X, Y \in \mathbf{Ob}(\mathcal{A}F_G).$$

Definition 3.2. The *Burnside ring* $\widehat{B}(G)$ of a profinite group G is the Grothendieck ring of the category $\mathcal{A}F_G$. The elements are the isomorphism classes of virtual almost finite G-spaces. In $\widehat{B}(G)$, we have $1 = [G/G]$ and $0 = [\emptyset]$, where $[X]$ denotes the isomorphism class of an almost finite G-space X.

Every element of $\widehat{B}(G)$ can be written as a difference $[X]-[Y]$ of the isomorphism class of two almost finite G-spaces. For convenience, we will often make the abuse of notation and omit the brackets to indicate elements of $\widehat{B}(G)$.

In [6], the authors show that $\widehat{B}(G) \cong \varprojlim_{N \trianglelefteq_o G} B(G/N)$ is a complete topological commutative ring, generated by the isomorphism classes of transitive almost finite G-spaces. We now want to generalise their results to introduce a crossed Burnside ring [13] for profinite groups. In order to do so, we first want to find a suitable class of profinite groups where a similar construction works. Following the same approach as for finite groups, given a profinite group G, let G^c denote the G-space on which G acts by conjugation. We have a decomposition

$$G^c \cong \bigsqcup_{g \in \mathrm{Cl}(G)} {}^G g,$$

where $\mathrm{Cl}(G)$ denotes a set of representatives of the conjugacy classes ${}^G g = \{{}^u g \mid u \in G\}$ of G. The topology on G^c is induced by the subspace topology on each ${}^G g$. In particular,

- G^c is discrete if and only if $|{}^G g| < \infty$, i.e. if and only if $C_G(g) \leq_o G$ for all $g \in G$.
- G^c is essentially finite if and only if $|(G^c)^U| = |C_G(U)| < \infty$ for all $U \leq_o G$.

These observations lead us to focus on the class of profinite FC-groups.

4. FC-groups

Definition 4.1. An *FC-group* is a group G whose elements have finitely many conjugates. Equivalently, $|G : C_G(g)| < \infty$ for every $g \in G$.

The term FC means *finite conjugacy (classes)*. The class of FC-groups is closed under taking subgroups, finite products and intersections, and quotients. It obviously contains all the abelian groups and all the finite groups. FC-groups are a subclass of the class of groups with *restricted centralisers*, that is, groups in which the centralisers of elements are either finite or of finite index (cf. [16]).

If G is FC, the centraliser $C_G(H) = \bigcap_{1 \leq i \leq n} C_G(h_i)$ of a finitely generated subgroup $H = \langle h_1, \dots, h_n \rangle$ of G is the intersection of finitely many subgroups of finite index in G, and therefore $C_G(H)$ has finite index in G too.

From [8, Section 1], we know that if G is a torsion FC-group, then G is locally finite (i.e. every finitely generated subgroup is finite). It follows that $G/Z(G)$ and G' are locally finite for any FC-group G. In particular, if G is finitely generated then $|G/Z(G)|$ and $|G'|$ are finite. In [16, Lemma 2.6], the author proves that if G is a profinite FC-group, then G' is finite, improving on the previous result, stating that G' is a torsion group. Therefore, a profinite FC-group is finite-by-abelian. Recall that in a profinite group, $G' = \overline{[G,G]} = \bigcap_{N \trianglelefteq_o G} [G,G]N$ is the closure of the derived subgroup of G.

As observed above, the centralisers of finitely generated subgroups of FC-groups have finite index. What can we say about the centraliser of a closed subgroup of an FC-group in general?

Proposition 4.2. *Let G be an FC-group. TFAE*

(i) *$Z(G)$ has finite index in G.*
(ii) *$\forall\, U \leq G$ of finite index, $C_G(U)$ has finite index in G.*
(iii) *$\exists\, U \leq G$ of finite index such that $C_G(U)$ has finite index in G.*

By contrapositive, $Z(G)$ is a subgroup of infinite index in G, if and only if the centraliser of each subgroup of G of finite index is itself a subgroup of infinite index in G.

Note that in (iii), it is equivalent to assume that such U is a normal subgroup of finite index in G (up to replacing U with its core in G).

Proof. (i)⇒(ii)⇒(iii) are obvious. To show (iii) implies (i), we pick a transversal $\{t_1,\dots,t_n\}$ of U in G. Then,

$$Z(G)=C_G(U)\cap\bigcap_{1\le i\le n}C_G(t_i)\quad\text{has finite index in } G,$$

since it is a finite intersection of subgroups of finite index in G. The proposition follows. □

Now, if G is profinite FC, Shalev's result leads to the following.

Proposition 4.3. *Let G be a profinite FC-group. Then $Z(G)$ is an open subgroup of G. In particular, G is virtually abelian. More generally, if G is a residually finite FC-group whose derived subgroup is finite, then $Z(G)$ is a subgroup of finite index in G.*

As a consequence of Proposition 4.3, if G is profinite FC, then the centraliser of any subgroup of G is open in G.

Proof. Since G is residually finite, for each $x\in G'$, there exists a normal subgroup $U_x\lhd G$ of finite index in G such that $x\notin U_x$. Set $U=\bigcap_{x\in G'}U_x$. Then, $U\lhd G$ has finite index in G, and $U\cap G'=1$. Moreover, $[G,U]\le G'\cap U=1$ shows that U is a central subgroup. The result follows. □

Note that the profinite completion of an FC-group need not be an FC-group, as seen on a variant of P. Hall's example [18, Example 2.1].

Example 4.4. Let p be a prime. For each $n\in\mathbb{Z}$, let:

$$X_n=\langle x_n,y_n,z_n\mid x_n^p=y_n^p=z_n^p=1,\ [x_n,y_n]=z_n\rangle\cong p_+^{1+2}$$

be an extraspecial p-group of order p^3 and exponent p. For every $n\in\mathbb{Z}$, set

$$g_{2n-1}=x_{2n-1}x_{2n}\quad\text{and}\quad g_{2n}=y_{2n}y_{2n+1},\quad\text{and define}\quad G=\langle g_n\mid n\in\mathbb{Z}\rangle.$$

By definition, $[g_{2n-1},g_{2n}]=z_{2n}$ and $[g_{2n},g_{2n+1}]=z_{2n+1}^{-1}$, with $[g_i,g_j]=1$ whenever $|j-i|\ge 2$. We have $G'=Z(G)=\langle z_n\mid n\in\mathbb{Z}\rangle$ is an infinite elementary abelian p-group, and G/G' too. Moreover, G has exponent p and is nilpotent of class 2.

For $g\in G$, let $\{g^G\}$ denote its conjugacy class, and for a subgroup H of G, let $\langle H^G\rangle$ denote its normal closure. We have

$$\{g_n^G\}=\{g_nz_n^iz_{n+1}^j\mid 0\le i,j<p\}\quad\text{and}\quad\langle H^G\rangle\le HZ(G).$$

Every element of the abstract group G can be written in a unique way as a finite product $w=g_{i_1}^{a_1}\cdots g_{i_n}^{a_n}z_{j_1}^{b_1}\cdots z_{j_m}^{b_m}$ for some integers $i_1<\cdots<i_n$ and $j_1<\cdots<j_m$, and for integers $0<a_1,\dots,a_n,b_1,\dots,b_m<p$. We calculate $C_G(w)=\langle g_l\mid |l-i_s|>1\ ,\ \forall\ 1\le s\le n\rangle Z(G)$, and note that $C_G(w)$ is a normal subgroup of G of finite index. Therefore, as an abstract group, G is FC since the conjugacy class of any word $g_{i_1}^{e_1}\cdots g_{i_k}^{e_k}$ is finite. However, in the profinite completion $\hat{G}$ of G, the elements which cannot be expressed as words of finite length in the g_i's have conjugacy classes of infinite size. (For instance the conjugacy class of the element whose image in every finite quotient is the image of $g_1g_2g_3\cdots$ is infinite.)

5. The crossed Burnside ring for profinite FC-groups

Let G be a profinite group, and let G^c denote the G-space on which G acts by conjugation. Since

$$(G^c)^U = C_G(U),\ \forall\, U \leq_o G \quad \text{and} \quad G^c \cong \bigsqcup_{g\in \mathrm{Cl}(G)} {}^G g$$

as G-space, G^c is almost finite if and only if G is FC and $Z(G)$ is finite, that is, if and only if G is finite. Let us relax the requirement for G^c to be almost finite, and only ask for G to be a discrete G-space. By the above, this hold if and only if G is profinite FC, and we then have ${}^G g \cong G/C_G(g)$ for all $g \in G$.

Definition 5.1. Let G be a profinite FC-group. Define the category ${}^{\times}\!\mathcal{A}F_G$ of almost finite *crossed* G-spaces to be the category whose objects are the morphisms $f : X \to G^c$, where X is almost finite and f is G-equivariant. The morphisms $\phi : (f_1 : X_1 \to G^c) \longrightarrow (f_2 : X_2 \to G^c)$ between two objects in ${}^{\times}\!\mathcal{A}F_G$ are the morphisms $\phi \in \mathrm{Hom}_{\mathcal{A}F_G}(X_1, X_2)$ such that $f_1 = f_2\phi$.

Let X be an almost finite G-space. Then a map $w : X \to G^c$ decomposes as a sum $\sqcup w_x$, where $X = \bigsqcup_{x\in[G\backslash X]} G/G_x$ and $w_x : G/G_x \to G^c$ is G-equivariant. Explicitly, $w_x(gG_x) = {}^g w_x(G_x)$, for some element $w_x(G_x) \in C_G(G_x)$.

We define the sum and product of almost finite crossed G-spaces using disjoint unions and cartesian products, similarly to the case of finite groups. In particular, if $w_a : G/H \to G^c$ and $w_b : G/K \to G^c$ are transitive almost finite crossed G-spaces, their product is the almost finite crossed G-space

$$\bigsqcup_{g\in[H\backslash G/K]} \big(w_{a\cdot {}^g b} \ :\ (G/H \cap {}^g K) \longrightarrow G^c\big),$$

see [13, Lemma 2.13(7)]. Two morphisms $w_a : G/H \to G^c$ and $w_b : G/K \to G^c$ are isomorphic if and only if there exists $g \in G$ such that $K = {}^g H$ and $b = {}^g a$. With these operations, the isomorphism classes of almost finite crossed G-spaces form an abelian monoid.

Definition 5.2. The *crossed Burnside ring of* G is the Grothendieck ring of the category ${}^{\times}\!\mathcal{A}F_G$. The elements are the isomorphism classes of virtual almost finite crossed G-spaces. In particular, $1_{\widehat{B^c}(G)} = [w_1 : G/G \to G^c]$ and $0_{\widehat{B^c}(G)} = [\emptyset \to G^c]$, where the square brackets denote isomorphism classes (which we will omit if there is no confusion), where $w_1(G) = 1$ and $\emptyset$ is the initial object of the category $\mathcal{A}F_G$.

The following observation is immediate (cf. [4, Section IV.8]).

Lemma 5.3. *If G is finite, then $\widehat{B^c}(G) = B^c(G) = B(G^c)$, where $B(G^c)$ is the evaluation of the Burnside Green functor for G at the G-set G^c.*

As for finite groups, there is an injective ring homomorphism $\widehat{B}(G) \to \widehat{B^c}(G)$, defined by mapping a virtual almost finite G-space X to $w_1 : X \to G^c$, where $w_1(x) = 1$ for all $x \in X$.

[2, Lemma 2.2.2] extends to our context.

Lemma 5.4. *$(v : X \to G^c) \cong (w : Y \to G^c)$ in ${}^{\times}\!\mathcal{A}F_G$ if and only if*

$$|\,\mathrm{Hom}_{{}^{\times}\!\mathcal{A}F_G}\big((w_g : G/H \to G^c), (v : X \to G^c)\big)| = |\,\mathrm{Hom}_{{}^{\times}\!\mathcal{A}F_G}\big((w_g : G/H \to G^c), (w : Y \to G^c)\big)|,$$

for all $(w_g : G/H \to G^c) \in {}^{\times}\!\mathcal{A}F_G$.

Proof. Given almost finite G-spaces X and Y, then $X \cong Y$ in $\mathcal{A}F_G$ if and only if $|X^U| = |Y^U|$ for all $U \leq_o G$. Write $X = \sqcup G/G_x$ and $v = \sqcup v_{a_x}$, where $a_x \in C_G(G_x)$, and x runs through a set of representatives of the G-orbits of X. Similarly, write $Y = \sqcup G/G_y$ and $w = \sqcup w_{b_y}$. If $[w_g : G/H \to G^c] \in \widehat{B^c}(G)$, then

$$\mathrm{Hom}_{{}^{\times}\mathcal{A}F_G}\big((w_g : G/H \to G^c), (v : X \to G^c)\big) = \\ = \bigsqcup_{x\in[G\backslash X]} \mathrm{Hom}_{{}^{\times}\mathcal{A}F_G}\big((w_g : G/H \to G^c), (v_{a_x} : G/G_x \to G^c)\big)$$

and similarly for Y. Note that these are finite sets because $\mathrm{Hom}_{\mathcal{A}F_G}(G/H, X) \cong X^H$, via the correspondence $(\varphi : G/H \to X) \mapsto \varphi(H)$, is a finite set, for all $H \leq_o G$ and for all $X \in \mathcal{A}F_G$. Now, $\mathrm{Hom}_{{}^{\times}\mathcal{A}F_G}\big((w_g : G/H \to G^c), (v_{a_x} : G/G_x \to G^c)\big)$ is the subset of $\mathrm{Hom}_{\mathcal{A}F_G}(G/H, G/G_x) = \big\{m_s : H \mapsto sG_x \mid H \leq {}^{s}G_x\big\}$ formed by the almost finite crossed G-spaces such that, if $H \leq {}^{s}G_x$, then $w_g(H) = g = {}^{s}a_x = w_{a_x}(sG_x)$. The cardinality of these two sets of homomorphisms coincide if and only if the almost finite crossed G-spaces $(v : X \to G^c)$ and $(w : Y \to G^c)$ have the same number of G-orbits of the same type. □

In [6, Section 2], the authors prove that $\widehat{B}(G)$ is a complete topological ring isomorphic to $\varprojlim_{N \trianglelefteq_o G} B(G/N)$ via the ring homomorphism induced by the product of the fixed point maps $\mathrm{Fix}_N : \widehat{B}(G) \to B(G/N)$ defined below. More generally, let $U \leq_o G$ and let $X \in \mathcal{A}F_G$. Then $N_G(U)$ acts on the finite set of U-fixed points X^U. Indeed, for all $x \in X^U$, all $u \in U$ and all $g \in N_G(U)$, we have $u(gx) = g((u^g)x) = gx$. Since U acts trivially on X^U, we can regard X^U as a finite $N_G(U)/U$-set. Since $(X \sqcup Y)^U = X^U \sqcup Y^U$, $(X \times Y)^U = X^U \times Y^U$ and $X \cong Y \implies X^U \cong Y^U$, for any subgroup U of G and any G-spaces X and Y, this function extends to a ring homomorphism $\widehat{B}(G) \to B(N_G(U)/U)$, for all $U \leq_o G$. Now, let $N \trianglelefteq_o G$ and let $V \leq_o G$. Define

$$\mathrm{Fix}_N(G/V) = (G/V)^N = \begin{cases} G/V & \text{if } N \leq V. \\ \emptyset & \text{otherwise,} \end{cases} \tag{2}$$

A routine exercise shows that the maps Fix_N are surjective ring homomorphisms. Each such map has a section, called *inflation*, $\mathrm{Inf}^G_{G/N} : B(G/N) \to \widehat{B}(G)$, which sends a finite G/N-set to itself, regarded as an almost finite G-space on which N acts trivially. Define

$$\mathrm{Fix} = \prod_{N \trianglelefteq_o G} \mathrm{Fix}_N \ : \ \widehat{B}(G) \longrightarrow \prod_{N \trianglelefteq_o G} B(G/N).$$

This is the injective ring homomorphism used in [6, Section 2] to show that $\widehat{B}(G) \cong \varprojlim_{N \trianglelefteq_o G} B(G/N)$ is a complete topological ring. In this topology, a basis of open ideals is $\{\ker(\mathrm{Fix}_N) \mid N \trianglelefteq_o G\}$.

Let G be a profinite FC-group and let $(w_a : G/U \to G^c)$ be a transitive almost finite crossed G-space, where $U \leq_o G$ and $a \in C_G(U)$. For $N \trianglelefteq_o G$, the fixed point map $\mathrm{Fix}_N : \widehat{B}(G) \to B(G/N)$ induces a ring homomorphism ${}^{\times}\mathrm{Fix}_N : \widehat{B^c(G)} \to B^c(G/N)$, where

$$\mathrm{Fix}_N(w_a : G/U \to G^c) = \begin{cases} (w_{aN} : G/U \to (G/N)^c) & \text{if } N \leq U, \text{ or} \\ 0_{B^c(G/N)} & \text{otherwise.} \end{cases}$$

Note that ${}^{\times}\mathrm{Fix}_N$ is neither injective nor surjective.

If $N_2, N_1 \trianglelefteq_o G$ with $N_2 \leq (U \cap N_1)$, then

$${}^{\times}\mathrm{Fix}_{N_1/N_2} : B^c(G/N_2) \to B^c(G/N_1)$$

$${}^{\times}\mathrm{Fix}_{N_1/N_2}(w_{aN_2} : G/U \to (G/N_2)^c) = \begin{cases} (w_{aN_1} : G/U \to (G/N_1)^c) & \text{if } N_1 \leq U, \text{ or} \\ 0_{B^c(G/N_1)} & \text{otherwise.} \end{cases}$$

Hence

$$ {}^{\times}\mathrm{Fix} = ({}^{\times}\mathrm{Fix}_N)_{N\trianglelefteq_o G} : \widehat{B^c}(G) \longrightarrow \prod_{N\trianglelefteq_o G} \widehat{B^c}(G/N) \quad \text{is a ring homomorphism,} $$

and, given $N_1, N_2 \trianglelefteq_o G$ with $N_2 \le N_1$, we have

$$ {}^{\times}\mathrm{Fix}_{N_1} = {}^{\times}\mathrm{Fix}_{N_1/N_2}\,{}^{\times}\mathrm{Fix}_{N_2}. $$

We aim to show that $\widehat{B^c}(G) \cong \varprojlim_{N\trianglelefteq_o G} B^c(G/N)$ (cf. [19, Definition 1.1.3 and Proposition 1.1.4]). That is, we want to show that $\widehat{B^c}(G)$ is isomorphic to the subring of $\prod_{N\trianglelefteq_o G} B^c(G/N)$ formed by all the elements of the form $\big(w_N : x_N \to (G/N)^c\big)_{N\trianglelefteq_o G} \in \prod_{N\trianglelefteq_o G} B^c(G/N)$ with

$$ {}^{\times}\mathrm{Fix}_{N_1/N_2}(w_{N_2} : X_{N_2} \to G^c) = (w_{N_1} : X_{N_1} \to (G/N_1)^c), $$

for all $N_1, N_2 \trianglelefteq_o G$ with $N_2 \le N_1$.

By [6, Section 2], we know that $\mathrm{Fix} : \widehat{B}(G) \to \prod_{N\trianglelefteq_o G} B(G/N)$ is an injective ring homomorphism, and that $\mathrm{Fix}(\widehat{B}(G)) \cong \varprojlim_{N\trianglelefteq_o G} B(G/N)$.

For the injectivity of ${}^{\times}\mathrm{Fix}$, suppose that ${}^{\times}\mathrm{Fix}(w : X \to G^c) = (0_{B^c(G/N)})_{N\trianglelefteq_o G}$. Then $X^N = 0_{B(G/N)}$ for all $N \trianglelefteq_o G$, which forces $X = 0_{\widehat{B}(G)}$ too, by injectivity of Fix. Since $0_{\widehat{B}(G)} = [\emptyset]$ is the initial object in the category $\mathcal{AF}_G$, there is a unique almost finite crossed G-space with domain $0_{\widehat{B}(G)}$, it follows that $[w : X \to G^c] = 0_{\widehat{B^c}(G)}$.

Let now $\big(w_N : X_N \to (G/N)^c\big)_{N\trianglelefteq_o G} \in \varprojlim_{N\trianglelefteq_o G} B^c(G/N)$ be a nonzero element. The sequence of the domains produces a unique element $X \in \widehat{B}(G)$. Suppose that X is the isomorphism class of $\sum_{U\in\mathcal{O}_G} \lambda_U G/U$, where $\mathcal{O}_G$ denotes a set of representatives of the conjugacy classes of open subgroups of G, and the λ_U are integers. We can then write

$$ w_N = \sum_{\substack{U\in\mathcal{O}_G\\ N\le U}} \sum_{1\le i\le|\lambda_U|} w_{a_{U/N,i}}, \quad \text{with} \quad G/U = (G/N)\big/(U/N), $$

and where $a_{U/N,i} \in C_{G/N}(U/N)$, for all $1 \le i \le |\lambda_U|$, and all $U \in \mathcal{O}_G$ with $N \le U$, $N \trianglelefteq_o G$. By convention, if $\lambda_U = 0$, then $\sum_{1\le i\le|\lambda_U|} w_{a_{U/N,i}} = 0$.

Note that if $N_1, N_2 \trianglelefteq_o G$ with $N_2 \le N_1$, then $C_{G/N_2}(U/N_2)N_1/N_1 \le C_{G/N_1}(U/N_1)$, via the quotient map $G/N_2 \to G/N_1$. We can pick $a_{U,N,i} \in G$ such that $a_{U,N,i}N/N = a_{U/N,i}$ for $N \trianglelefteq_o G$, and our definition of ${}^{\times}\mathrm{Fix}$ implies that $a_{U,N_2,i}N_1 = a_{U,N_1,i}N_1$. The elements $a_{U,N,i}$ satisfy $[a_{U,N,i}, U] \subseteq N$. Hence, for $U \in \mathcal{O}_G$ with $\lambda_U \ne 0$, and for $1 \le i \le |\lambda_U|$, let

$$ \mathbf{a}_{U,i} = \bigcap_{\substack{N\trianglelefteq_o G\\ N\le U}} a_{U,N,i}N. $$

Then $\mathbf{a}_{U,i} \ne \emptyset$ is a closed subset of G (cf. [14, Proposition 1.1.4]), which consists of a single element $a_{U,i}$. Indeed, suppose that $a, b \in \mathbf{a}_{U,i}$. That is, $a, b \in a_{U,N,i}N$, or equivalently, $b^{-1}a \in N$ for all $N \trianglelefteq_o G$, which forces $a = b$ because $\cap_{N\trianglelefteq_o G} N = 1$. Therefore $\mathbf{a}_{U,i} = \{a_{U,i}\}$, where

$$ a_{U,i} \in \bigcap_{\substack{N\trianglelefteq_o G\\ N\le U}} \{g \in G \mid [g, U] \subseteq N\}. $$

Putting $a_{U,N,i} = 1$ if $N \not\le U$, we have $(a_{U,N,i}N)_{N\trianglelefteq_o G} \in \varprojlim_{N\trianglelefteq_o G} C_{G/N}(UN/N) = C_G(U)$, we conclude that $a_{U,i} \in C_G(U)$ (cf. [19, Exercise 0.4(2)]).

Consequently, $(w_{a_{U,i}} : G/U \to G^c) \in {}^{\times}\!\mathcal{A}F_G$, and

$$^{\times}\mathrm{Fix}(w_{a_{U,i}} : G/U \to G^c)_N = \begin{cases} [w_{a_{U/N,i}} : G/U \to (G/N)^c] \in B^c(G/N) & \text{if } N \leq U \\ 0_{B^c(G/N)} & \text{otherwise,} \end{cases}$$

saying that $^{\times}\mathrm{Fix}(w_{a_{U,i}} : G/U \to G^c) \in \varprojlim_{N \trianglelefteq_o G} B^c(G/N)$. We have thus proved the following.

Proposition 5.5. *Let G be a profinite FC-group. Then* $^{\times}\mathrm{Fix}$ *induces a ring isomorphism*

$$\widehat{B^c}(G) \xrightarrow{\cong} \varprojlim_{N \trianglelefteq_o G} B^c(G/N).$$

The crossed Burnside ring of a profinite FC-group has some of the properties similar to those of the crossed Burnside ring of a finite group. Let R be a commutative ring, and write $\widehat{B^c_R}(G) = R \otimes_{\mathbb{Z}} \widehat{B^c}(G)$. Given $U \leq_o G$ and $a \in C_G(U)$, we have $\sum_{g \in [N_G(U)/U]} {}^g a \in Z(RC_G(U))$, since ${}^g a \in C_G({}^g U) = C_G(U)$ for all $g \in N_G(U)$. As in [2, Section 2.3], we obtain a ring homomorphism:

$$z_U : \widehat{B^c_R}(G) \longrightarrow Z(RC_G(U)), \quad z_U(w : X \to G^c) = \sum_{\substack{x \in [G\backslash X] \\ U \leq_G G_x}} \sum_{g \in [N_G(G_x)/G_x]} {}^g a_x,$$

where

$$(w : X \to G^c) = \bigsqcup_{x \in [G\backslash X]} (w_{a_x} : G/G_x \to G^c)$$

is an almost finite crossed G-space. Here, $a_x \in C_G(G_x)$ for all x, and the notation $U \leq_G G_x$ means that there exists $h \in G$ such that $U \leq {}^h G_x$. The map z_U extends to virtual almost finite crossed G-spaces, and since $|G : U| < \infty$, the above sums are finite. Therefore z_U is well defined, and we obtain a ring homomorphism

$$\zeta : \widehat{B^c_R}(G) \longrightarrow \prod_{U \in \mathcal{O}_G} Z(RC_G(U)), \quad \zeta(\hat{w}) = \big(z_U(\hat{w})\big)_{U \in \mathcal{O}_G}, \ \forall\, \hat{w} \in \widehat{B^c_R}(G),$$

where $\mathcal{O}_G$ denotes a set of representatives of the conjugacy classes of open subgroups of G. The same argument as in [2, Lemma 2.3.2] shows the following (for the proof, we now use $K \leq_o G$ with $|G : K|$ minimal such that $\hat{w} = \sum_i \lambda_U(w_{a_U} : G/U \to G^c)$ has a nonzero λ_K).

Lemma 5.6. *If R is torsionfree, then ζ is injective. Consequently, we obtain a mapping* $\mathrm{Spec}(\prod_{U \in \mathcal{O}_G} Z(RC_G(U))) \to \mathrm{Spec}(\widehat{B^c_R}(G))$.

Note that the ring extension $\widehat{B}(G) \subset \widehat{B^c}(G)$ is not algebraic, and therefore the mapping in Lemma 5.6 need not be surjective.

6. Idempotents of $\widehat{B}(G)$

Let G be a profinite group. We draw on the properties of the ring homomorphisms Fix_N and $\mathrm{Inf}^G_{G/N}$ defined in Section 5 in order to investigate the relationships between the idempotents of $\widehat{B}(G)$ with those of the Burnside rings of the finite quotient groups of G.

If G is finite, Dress proved that G is soluble if and only if the prime ideal spectrum of $B(G)$ is connected, i.e. the only idempotents of $B(G)$ are 0 and 1 (cf. [9, Section 7.5, Corollary]). (By *ideal*, we mean an ideal that is closed in the topology of $\widehat{B}(G)$ defined by taking $\{\ker(\mathrm{Fix}_N) \mid N \trianglelefteq_o G\}$ as open neighbourhood basis of $0 \in \widehat{B}(G)$.) This result extends to profinite groups and $\widehat{B}(G)$ in the following way.

Proposition 6.1. *Let G be a profinite group. Then G is prosoluble if and only if the prime ideal spectrum of $\widehat{B}(G)$ is connected, i.e. the only idempotents of $\widehat{B}(G)$ are 0 and 1.*

Proof. We know that the result holds for finite soluble groups. Let G be a prosoluble profinite group, i.e. G/N is soluble for all $N \trianglelefteq_o G$. Suppose that $e = e^2 \in \widehat{B}(G)$. Since Fix is a ring homomorphism, $\mathrm{Fix}_N(e)$ is an idempotent in $B(G/N)$, and therefore $\mathrm{Fix}_N(e) \in \{0, 1\}$, for all $N \trianglelefteq_o G$.

Since Fix is injective $\mathrm{Fix}_N(e) = 0$ for all $N \trianglelefteq_o G$ if and only if $e = 0$ in $\widehat{B}(G)$, i.e. e is the isomorphism class of the empty set. So, suppose that $e \neq 0$. Then there must be some open normal subgroup N of G such that $\mathrm{Fix}_N(e) \neq 0 \in B(G/N)$. Since G/N is a finite soluble group, we must have $\mathrm{Fix}_N(e) = 1 \in B(G/N)$. That is, $\mathrm{Fix}_N(e) = [(G/N)/(G/N)] \cong [G/G] \cong [(G/M)/(G/M)]$, and it follows that $\mathrm{Fix}_M(e) = 1 \in B(G/M)$ for every open normal subgroup M of G. We conclude that $e = 1$ in $\widehat{B}(G)$.

Conversely, suppose that 0 and 1 are the only idempotents of $\widehat{B}(G)$. Let $N \trianglelefteq_o G$. Suppose that $e_N^2 = e_N \in B(G/N)$. Then $\mathrm{Inf}_{G/N}^G(e_N)$ is an idempotent of $\widehat{B}(G)$. By assumption, this idempotent must be either 0 or 1. It follows that either $e_N = 0$ of $e_N = 1$ in $B(G/N)$ for all $N \trianglelefteq_o G$, and so G/N is a finite soluble group. □

The above result leads us to investigate a possible correspondence between the (primitive) idempotents of $\widehat{B}(G)$ and those of the Burnside rings $B(G/N)$, for $N \trianglelefteq_o G$ of the finite quotients of G.

First, let us recall some elementary facts in group theory. By convention, a perfect group is a nonabelian (hence nontrivial) group.

Remark 6.2.

(1) G is a perfect group if and only if G/H is perfect for all $H \trianglelefteq G$. Indeed, G is perfect if and only if G has no nontrivial abelian quotient, if and only if no nontrivial quotient G/H of G has a nontrivial abelian quotient.

In particular, if G is profinite FC, [16, Lemma 2.6] shows that G' is finite. Since the property FC is inherited by subgroups, and since $H' \leq G'$ for all $H \leq G$, any perfect subgroup of G is finite. More generally, for an arbitrary FC-group G, any perfect subgroup is torsion (since G' is torsion).

(2) Let p be a prime, and let G be a profinite group. Recall that for a finite group H, there is a unique well-defined characteristic subgroup $O^p(H)$ which is the minimal normal subgroup of H with quotient a p-group. For all $N \trianglelefteq_o G$, let U_N the characteristic open subgroup of G such that $N \leq U_N \leq G$ and $G/U_N \cong (G/N)/O^p(G/N)$. If $N_2 \leq N_1$ are open normal subgroups of G, then $G/U_1 \cong (G/N_2)/(U_1/N_2)$ is a (finite) p-group, quotient of G/N_2, where $U_i = U_{N_i}$. Therefore, $G/U_2 \twoheadrightarrow G/U_1$, and we obtain an inverse system of finite p-groups $\{G/U_i,\ G/U_j \twoheadrightarrow G/U_i\ (\forall\ N_j \leq N_i),\ N_i, N_j \trianglelefteq_o G\}$. Let $\overline{G} = \varprojlim_{N \trianglelefteq_o G} G/U_N$ be the inverse limit, and $\theta_p : G \to \overline{G}$ the quotient map induced by the projections $G/N \to G/U_N$. Note that θ_p is well defined since the squares

$$\begin{array}{ccc} G/N_2 & \longrightarrow & G/N_1 \\ \downarrow & & \downarrow \\ G/U_2 & \longrightarrow & G/U_1 \end{array}$$

, where the maps are the quotient maps, commute for all $N_1, N_2 \trianglelefteq_o G$ with $N_2 \leq N_1$. We define

$$O^p(G) = \ker(\theta_p) = \bigcap_{N \trianglelefteq_o G} U_N.$$

Then, $O^p(G)$ is a closed characteristic subgroup of G with the property that any pro-p quotient group of G is a quotient of $G/O^p(G)$.

(3) Let $H \leq G$ be a finite subgroup of a residually finite group G. For each $x \in H$, there exists $N_x \trianglelefteq_o G$ such that $x \notin N_x$. Let $N_H = \bigcap_{x \in H} N_x$. Then $N_H \trianglelefteq_o G$ and $N_H \cap H = 1$. Hence, there are infinitely many open normal subgroups of G which do not meet H. In particular, if G is profinite FC and H is a perfect subgroup of G, then the set

$$\mathcal{N}_H = \{N \trianglelefteq_o G \mid |H \cap N| = 1\}$$

is a filter base for G, that is, $\mathcal{N}_H$ is a family of open normal subgroups of G such that:
(i) for all $N_1, N_2 \in \mathcal{N}_H$, there exists $N_3 \in \mathcal{N}_H$ with $N_3 \leq N_1 \cap N_2$, and
(ii) $\bigcap_{N \in \mathcal{N}_H} N = \{1\}$.
Indeed we note that we have the stronger condition that for any $N_1 \trianglelefteq_o G$ and for any $N_2 \in \mathcal{N}_H$, then $N_1 \cap N_2 \in \mathcal{N}_H$, from which follows that $\bigcap_{N \in \mathcal{N}_H} N = \bigcap_{N \trianglelefteq_o G} N = \{1\}$.

Remark 6.2 (2) would lean towards a definition of idempotents in $\widehat{B}_{\mathbb{Z}_p}(G)$, if we tried to extend the result from finite groups. However, as we shall shortly see, our methods only allow us to define idempotents indexed by open subgroups of G, but we cannot expect the p-perfect subgroups of a profinite group to be all open. We thus leave this question aside, referring the reader to Proposition 6.4 as a starter towards generalising further our results. We close this parenthesis on some remarks about groups with the following observation.

Let G be a profinite group, and define the set of (closed) subgroups of G

$$\mathcal{P} = \{1 < H \leq G \mid H' = H\}.$$

Write $[\mathcal{P}]$ for a set of representatives of the G-conjugacy classes of perfect subgroups of G. For $n \in \mathbb{N}$, define inductively the (closed) derived series $G = G^{(1)} \geq G^{(2)} \geq G^{(3)} \geq \dots$ for G, where we define $G^{(1)} = G$ and $G^{(i)} = \overline{[G^{(i-1)}, G^{(i-1)}]}$ for all $i \geq 2$. Since the series is monotone decreasing, if some $G^{(n)}$ is finite, then the series converges and we have a well defined subgroup $G^{(\infty)} = \bigcap_{n \in \mathbb{N}} G^{(n)}$.

Lemma 6.3. *Let G be a profinite group. Then $\mathcal{P} \neq \emptyset$ if and only if G is not prosoluble, if and only if $G^{(\infty)}$ exists and is nontrivial, that is, the derived series converges to a nontrivial subgroup of G.*

Proof. First, note that $\mathcal{P} \neq \emptyset$ if and only if G is not prosoluble, since $H \in \mathcal{P}$ if and only if for all $N \trianglelefteq_o G$ such that $H \not\leq N$, then G/N is a finite group with a perfect nontrivial subgroup HN/N. Hence, we need to show that $\mathcal{P} \neq \emptyset$ if and only if $G^{(\infty)}$ exists and is nontrivial.

If $G^{(\infty)}$ exists and is nontrivial, then $G^{(\infty)} \in \mathcal{P}$. Conversely, suppose that $\mathcal{P} \neq \emptyset$. Let $U = \overline{\langle H \mid H \in \mathcal{P}\rangle}$. Note that $1 \neq U$ is characteristic in G since $\mathcal{P}$ is closed under G-conjugation and since the image of any perfect subgroup of G by an automorphism of G is again a perfect subgroup of G. Moreover, $U' \geq \overline{\langle H' \mid H \in \mathcal{P}\rangle} = U$ shows that $1 \neq U \in \mathcal{P}$ is perfect. The assertion follows from the observation that G/U is soluble. Indeed, any perfect subgroup H/U of G/U, with $U \leq H \leq G$, satisfies $H = H'U = H'U' = H'$, where the first equality holds because $H/U = (H/U)' = H'U/U$. Hence, $H \in \mathcal{P}$, which implies that $H = U$. Therefore, G/U is profinite and soluble, and we must have $1 \neq U = G^{(\infty)}$ as required. □

The set $[\mathcal{P}]$ is useful in the description of the integral idempotents of the Burnside ring of a finite group G. Indeed, if G is a finite group, the primitive idempotents of the Burnside $\mathbb{Q}$-algebra $B_{\mathbb{Q}}(G)$ of G are indexed by the conjugacy classes of subgroups H of G, and have the form [7]:

$$e_H = \sum_{1\leq K\leq H} \frac{\mu(K,H)}{|N_G(H):K|} G/K$$

where $\mu(-,-)$ denotes the Möbius inversion formula, and the sum is over all the subgroups of H. In $B_{\mathbb{Q}}(G)$, we have $e_H^2 = e_H$ and e_H is characterised by $|(e_H)^K| = 1$ if and only if K is G-conjugate to H, and $|(e_H)^K| = 0$ otherwise.

In the (integral) Burnside ring $B(G)$, the set

$$\{f_H = \sum_K e_K \mid H \in [\mathcal{P}] \cup \{1\}\}$$

is a complete set of primitive pairwise orthogonal idempotents, where K runs through a set of representatives of the conjugacy classes of subgroups of G such that $K^{(\infty)}$ is G-conjugate to H, for all $H \in [\mathcal{P}] \cup \{1\}$.

Suppose now that G is profinite. We generalise Gluck's idempotent formula to $\widehat{B}_{\mathbb{Q}}(G)$ as follows: Let $\mathcal{O}_G$ denote a set of representatives of the conjugacy classes of open subgroups of G. For $H \in \mathcal{O}_G$, put

$$e_H = \sum_{K\leq_o H} \frac{\mu(K,H)}{|N_G(H):K|} G/K.$$

For all $N \trianglelefteq_o G$, we have $\mathrm{Fix}_N(e_H) = e_{H/N}$ if $N \leq H$ and $\mathrm{Fix}_N(e_H) = 0$ if $N \not\leq H$, as element in $B_{\mathbb{Q}}(G/N)$. Indeed,

$$\mathrm{Fix}_N(e_N) = \sum_{N\leq K\leq H} \frac{\mu(K,H)}{|N_G(H):K|} G/K = \sum_{N\leq K\leq H} \frac{\mu(K/N,H/N)}{|N_{G/N}(H/N):K/N|} (G/N)/(K/N)$$

in $B_{\mathbb{Q}}(G/N)$, since, if $N \leq H$, then $N_{G/N}(H/N) = \{gN \in G/N \mid {}^{gN}H \leq HN = H\} = N_G(H)/N$. By definition, $(e_H)^2 = e_H$, since $(\mathrm{Fix}_N(e_H))^2 = \mathrm{Fix}_N(e_H)$ for all $N \trianglelefteq_o G$. Now, if $U \leq_o G$, then $|(e_H)^U| = |(e_{H/N})^{U/N}|$ for any $N \trianglelefteq_o G$ with $N \leq H \cap U$. By the case of finite groups, this number is 1 if U/N is G/N-conjugate to H/N, and 0 otherwise. It then suffices to observe that for such N, U/N is G/N-conjugate to H/N if and only if U is G-conjugate to H. It follows that $|(e_H)^U| = 1$ if U is G-conjugate to H and 0 otherwise, for all $H, U \leq_o G$.

We have shown the following.

Proposition 6.4. *Assume the above notation.*

1. *For every $H \leq_o G$, the element $e_H = \sum_{K\leq_o H} \frac{\mu(K,H)}{|N_G(H):K|} G/K \in \widehat{B}_{\mathbb{Q}}(G)$ is an idempotent. In particular, it need not be a finite $\mathbb{Q}$-linear combination of transitive finite G-sets.*
2. *The ghost map*

$$\widehat{B}_{\mathbb{Q}}(G) \longrightarrow \mathbb{Q}^{\mathcal{O}_G}, \quad x \mapsto (|x^U|)_{U\in\mathcal{O}_G}$$

maps the set $\{e_H \mid H \in \mathcal{O}_G\}$ to a canonical basis of the ghost $\mathbb{Q}$-algebra. That is, $e_H \longmapsto (\delta_{U,H})_{U\in\mathcal{O}_G}$, where $\delta_{U,H} = 1$ if U is conjugate to H and is 0 otherwise.

Example 6.5. Let $G = \mathbb{Z}_p$ for a prime p. Then, for all $n \geq 0$,

$$e_{p^nG} = \frac{1}{p^n} G/p^nG - \frac{1}{p^{n+1}} G/p^{n+1}G \ ,$$

since for nonnegative integers $m \leq n$, we have $\mu(p^mG, p^nG) = 1$ if $n = m$, $\mu(p^mG, p^nG) = -1$ if $m + 1 = n$, and $\mu(p^mG, p^nG) = 0$ otherwise.

If $G = \widehat{\mathbb{Z}}$, then $e_G = \sum_n \frac{\mu(n\widehat{\mathbb{Z}}, \widehat{\mathbb{Z}})}{n} \widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}}$, where n runs through all the integers which factorise into a product of distinct primes.

Dress's result [5, Proposition 2] does not extend as such to obtain a complete list of the integral primitive idempotents in $\widehat{B}(G)$. If H is a perfect subgroup of G, it need not contain any open subgroup, and there may be infinitely many subgroups K of G such that $K^{(\infty)}$ is G-conjugate to H.

In particular, if G is an infinite profinite FC-group, then any perfect subgroup H of G is finite, and therefore the set $\mathcal{P}$ is a subset of the finite subgroups of G, each of which possessing finitely many conjugates. Since $Z(G)$ has finite index in G, there are infinitely many subgroups $K \leq G$ such that $K^{(\infty)}$ is G-conjugate to H. But the elements e_H introduced above are only defined for subgroups of finite index. Our methods lead, for instance to idempotents inflated from $B(G/Z(G))$. We conclude this section with an example.

Example 6.6. Let $G = A_5 \times \widehat{\mathbb{Z}}$. Then G is profinite FC and $H = A_5 \times \{1\}$ is the unique nontrivial perfect subgroup of G. Note that for any $H \leq U \leq G$ we have $U' = U^{(\infty)} = H$. Let

$$e_H = \mathrm{Inf}_{G/Z(G)}^{G}\big(A_5/A_5 - A_5/A_4 - A_5/D_{10} - A_5/D_6 + A_5/C_3 + 2A_5/C_2 - A_5/1\big),$$

where we write $A_5 = G/Z(G) \cong H$, and we use the obvious identifications of the subgroups of H. We might expect $(e_H)^2 = e_H \in \widehat{B}(G)$ to be a summand of e_G.

7. Action of almost finite crossed G-spaces on Mackey functors for profinite groups

Let G be a finite group and let R be a commutative ring. Each crossed G-set acts on the category $\mathbf{Mack}_R(G)$ of Mackey functors for G over R, producing a natural transformation of the identity morphism in $\mathbf{Mack}_R(G)$. This property has been used in [2] to obtain a ring homomorphism from the crossed Burnside ring of G to the centre of the Mackey R-algebra for G over R. Mackey functors have been extended from finite to profinite groups, taking some different perspectives depending on the objective(s) of the authors ([1, 12] and [6, Section 5]). In the present section, we show that the almost finite crossed G-spaces act on a category of Mackey functors. We follow in parallel [1] and [6], specialising their perspective to our context.

Throughout, let G be a profinite group and let R be a commutative ring. We build on Section 3.

Definition 7.1. Let $\mathcal{AF}_G^r$ be the subcategory of $\mathcal{AF}_G$ with the same objects as $\mathcal{AF}_G$, and morphisms $f : X \to Y$ are the almost finite morphisms such that the fibres $f^{-1}(y)$ are finite, $\forall\, y \in Y$.

The categories $\mathcal{AF}_G^r$ and $\mathcal{AF}_G$ of discrete G-spaces are introduced and used in [1, 6, 12]. By contrast, in [10], the authors consider G-spaces X, for a discrete group G, such that each point stabiliser is a finite subgroup of G, and such that X has finitely many G-orbits.

Let us record some useful observations.

Remark 7.2.

(1) If $f \in \mathrm{Hom}_{\mathcal{AF}_G}(X, Y)$, then $f^{-1}(y)$ is an almost finite G_y-space for all $y \in Y$, and $G_x \leq G_y$ for all $x \in f^{-1}(y)$.
(2) If G is finite, then $\mathcal{AF}_G = \mathcal{AF}_G^r$.
(3) If G is infinite, then $\mathcal{AF}_G^r$ has no terminal object, since $\mathrm{Hom}_{\mathcal{AF}_G^r}(X, G/G) \neq \emptyset$ if and only if X is finite.

We introduce two kinds of Mackey functors for a given profinite group G (compare with [1, Definition 2.6] and [6, Section 5]).

Definition 7.3. A *Mackey functor for G* is an additive functor $M = (M_*, M^*) : \mathcal{AF}_G \times \mathcal{AF}_G \to \mathcal{Ab}$ with M_* covariant and M^* contravariant, subject to the following axioms.

(MF1) $M_*(X) = M^*(X)$ for every almost finite G-space X. Thus we write simply $M(X)$.

(MF2) If $\bigsqcup_i X_i$ is almost finite, then the natural inclusions $X_i \to \bigsqcup_i X_i$ induce an isomorphism $M(\bigsqcup_i X_i) \cong \prod_i M(X_i)$ of abelian groups.

(MF3) For any pull back diagram of almost finite G-spaces

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ Z & \xrightarrow[\delta]{} & W \end{array} \quad \text{in } \mathcal{A}F_G\text{, the diagram} \quad \begin{array}{ccc} M(X) & \xleftarrow{M^*(\alpha)} & M(Y) \\ {\scriptstyle M_*(\beta)}\downarrow & & \downarrow{\scriptstyle M_*(\gamma)} \\ M(Z) & \xleftarrow[M^*(\delta)]{} & M(W) \end{array} \quad \text{commutes in } \mathcal{A}b.$$

A *restricted Mackey functor for G* is an additive functor $M = (M_*, M^*) : \mathcal{A}F^r_G \times \mathcal{A}F_G \to \mathcal{A}b$ with M_* covariant and M^* contravariant, subject to the same axioms **(MF1, MF2)**, and with a *restricted* variant of **(MF3)**, where the vertical maps in the left hand side pull back diagram are morphisms in $\mathcal{A}F^r_G$.

We let $\mathbf{Mack}(G)$ (resp. $\mathbf{Mack}^r(G)$) denote the category of (restricted) Mackey functors for G, whose objects are the (restricted) Mackey functors for G, and the morphisms are the natural transformations of functors. Given a commutative ring R, we set $\mathbf{Mack}_R(G) = R \otimes_{\mathbb{Z}} \mathbf{Mack}(G)$ and call it the category of Mackey functors for G over R.

Remark 7.4. Recall that a pull back of G-spaces encodes the Mackey formula: If

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & G/H \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ G/K & \xrightarrow[\delta]{} & G/L \end{array},$$

is a pull back with H, K, L closed subgroups of G, and, to simplify, assume that the maps γ and δ are induced by inclusions of subgroups $H, K \hookrightarrow L$, then X is of the form

$$X \cong \bigsqcup_{u \in [H\backslash L/K]} G/(H \cap {}^uK).$$

In particular, the double coset space $[H\backslash L/K]$ is a discrete space if and only if at least one of H or K is an open subgroup of L.

In [6, Section 5], the authors show that the *Burnside functor* $\widehat{B} := \widehat{B^G} : \mathcal{A}F_G \to \mathcal{A}b$, where

$$\widehat{B}(X) = \mathrm{Hom}_{\mathcal{A}F_G}(-, X) =: \{f : Y \to X \text{ in } \mathcal{A}F_G\}$$

satisfies the axioms of Mackey functors, without any finiteness assumption on the fibres of the maps. In their setting, the covariant part $\widehat{B}_*$ is given by the composition of maps: If $\phi \in \mathrm{Hom}_{\mathcal{A}F_G}(X, Z)$, then $\widehat{B}_*(\phi) : \widehat{B}(X) \to \widehat{B}(Z)$ is given by $\widehat{B}_*(\phi)\big(f : Y \to X\big) = \big(\phi f : Y \to Z\big)$. The contravariant part $\widehat{B}^*(\phi) : \widehat{B}(Z) \to \widehat{B}(X)$ is given by the pull back: For $f \in \mathrm{Hom}_{\mathcal{A}F_G}(Y, Z)$,

$$\begin{array}{ccc} U & \longrightarrow & Y \\ {\scriptstyle\widehat{B}^*(\phi)(f)}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{\phi} & Z \end{array}$$

By contrast, if V is an RG-module, for some commutative ring R, the fixed point module functor FP_V and the fixed quotient module functor FQ_V are not Mackey functors for G over R, but restricted Mackey functors. Recall that they are defined on an almost finite G-space X by

$$\mathrm{FP}_V(X) = \prod_{x\in[G\backslash X]} V^{G_x} \quad \text{and} \quad \mathrm{FQ}_V(X) = \prod_{x\in[G\backslash X]} V_{G_x},$$

where $V_H = V/\langle hv - v \mid h \in H,\ v \in V\rangle$ denotes the H-coinvariants of V. If $f \in \mathrm{Hom}_{\mathcal{A}F^r_G}(X, Y)$, then the image of

$$(\mathrm{FP}_V)_*(f) : \prod_{x\in[G\backslash X]} V^{G_x} \longrightarrow \prod_{y\in[G\backslash Y]} V^{G_y}$$

in the V^{G_y} coordinate consists of elements of the form $\displaystyle\sum_{x\in f^{-1}(y)} \sum_{g\in[G_y/G_x]} gv_x$ for elements $v_x \in V^{G_x}$, for all $x \in f^{-1}(y)$. This is well defined if and only if $f^{-1}(y)$ is a finite set. The contravariant Mackey functor $(\mathrm{FP}_V)^*$ is induced by the inclusions of fixed points $V^{G_y} \hookrightarrow V^{G_x}$ for all $x \in f^{-1}(y)$. Similarly, the image of

$$(\mathrm{FQ}_V)_*(f) : \prod_{x\in[G\backslash X]} V_{G_x} \longrightarrow \prod_{y\in[G\backslash Y]} V_{G_y}$$

in the V_{G_y} coordinate consists of elements of the form $\displaystyle\sum_{x\in f^{-1}(y)} \sum_{g\in[G_y/G_x]} \overline{v_x}$, where $\overline{v_x} \in V_{G_y}$ is the image of $v_x \in V_{G_x}$ via the quotient map $V_{G_x} \twoheadrightarrow V_{G_y}$ for all $x \in f^{-1}(y)$. Again, this is well defined if and only if $f^{-1}(y)$ is a finite set. The contravariant Mackey functor $(\mathrm{FQ}_V)^*$ is induced by the inclusions of R-modules $V_{G_y} \hookrightarrow V_{G_x}$ for all $x \in f^{-1}(y)$.

A key observation in [17], extended in [2] (referring to the original work of Yoshida), is that, if G is a finite group, then the crossed G-sets act on the category of Mackey functors. We now generalise this action to profinite FC-groups and almost finite crossed G-spaces.

Let $(f : X \to G^c) \in {}^{\times}\!\mathcal{A}F_G$ and let $Y \in \mathcal{A}F_G$. Define the mappings:

$$\begin{aligned} \pi_Y, \tau_Y^f :\ & X \times Y \longrightarrow Y, \\ & \tau_Y^f(x, y) = f(x)y \quad \text{and} \\ & \pi_Y(x, y) = y, \quad \text{for all } (x, y) \in X \times Y, \end{aligned}$$

where we have abbreviated the notation for convenience ($\pi_Y^{X\times Y}$ and $\tau_Y^{(f:X\to G^c)}$ would be more precise than π_Y and τ_Y^f, respectively). Clearly, both are continuous and π_Y is G-equivariant (G acts on $X \times Y$ diagonally). The map τ_Y^f is G-equivariant too, since for all $g \in G$ and all $(x, y) \in X \times Y$, we have

$$\tau_Y^f\big(g \cdot (x, y)\big) = \tau_Y^f(gx, gy) = f(gx)gy = {}^g f(x)gy = gf(x)y = g\tau_Y^f(x, y).$$

The fibres $\pi_Y^{-1}(y)$ and $(\tau_Y^f)^{-1}(y)$ are subsets of the almost finite G-space $X \times Y$, and therefore they are almost finite G_y-spaces for all $y \in Y$. Thus π_Y and τ_Y^f are morphisms in $\mathcal{A}F_G$. Note that, if $f(x) = 1$ for all $x \in X$, then $\tau_Y^f = \pi_Y$.

Now, let M be a Mackey functor for G over R. Consider the composition

$$\eta_Y^f = M_*(\tau_Y^f)M^*(\pi_Y) \ :\ M(Y) \longrightarrow M(Y).$$

By definition of Mackey functors, this composition is an R-module homomorphism. Given $\alpha \in \mathrm{Hom}_{\mathcal{AF}_G}(Y, Y')$, the diagrams of R-modules and homomorphisms

$$\begin{array}{ccc} M(Y) & \xrightarrow{\eta^f_Y} & M(Y) \\ {\scriptstyle M_*(\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle M_*(\alpha)} \\ M(Y') & \xrightarrow{\eta^f_{Y'}} & M(Y') \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M(Y) & \xrightarrow{\eta^f_Y} & M(Y) \\ {\scriptstyle M^*(\alpha)}\big\uparrow & & \big\uparrow{\scriptstyle M^*(\alpha)} \\ M(Y') & \xrightarrow{\eta^f_{Y'}} & M(Y') \end{array} \qquad \text{commute.}$$

It follows that [2, Proposition 4.3] holds in the present context.

Proposition 7.5. *Let* $(f : X \to G^c) \in {}^{\times}\mathcal{AF}_G$. *The map* η^f *is a natural transformation of the identity functor of the category* $\mathbf{Mack}_R(G)$. *Moreover, if* $(f' : X' \to G^c) \in {}^{\times}\mathcal{AF}_G$, *then*

$$\eta^f + \eta^{f'} = \eta^{f \sqcup f'}, \quad \textit{and}$$
$$\eta^f \cdot \eta^{f'} = \eta^{f \times f'},$$

where, for any almost finite G*-space* Y, *there are* R*-module endomorphisms of* $M(Y)$,

$$(\eta^f + \eta^{f'})_Y = \eta^f_Y \oplus \eta^{f'}_Y : \ M(Y) \longrightarrow M(X) \oplus M(X') \longrightarrow M(Y)$$

and

$$(\eta^f \cdot \eta^{f'})_Y = \eta^{f \times f'}_Y : \ M(Y) \longrightarrow M(X) \otimes_R M(X') \longrightarrow M(Y)\ .$$

Explicitly, Proposition 7.5 states that, for a profinite FC-group G, the abelian monoid of almost finite crossed G-spaces acts on the category of Mackey functors. If G is an arbitrary profinite group, the action extended from [17, Section 9] remains well defined too, where

$$(X \cdot M)(Y) = \big(M_*(\pi_Y) M^*(\pi_Y)\big)(M(Y)),$$

for all almost finite G-spaces X and Y, and for all Mackey functors M for G.

The proof is routine. For instance, for the equality $\eta^f \cdot \eta^{f'} = \eta^{f \times f'}$, let $f : X \to G^c$, let $f' : X' \to G^c$, let M be a Mackey functor for G over R, and let $Z \in \mathcal{AF}_G$. Then,

$$\begin{array}{ccccc} M(Z) & \xrightarrow{M^*(\pi_Z)} & M(X \times Z) & \xrightarrow{M_*(\tau^f_Z)} & M(Z) \\ {\scriptstyle M^*(\pi_Z)}\big\downarrow & \swarrow {\scriptstyle M^*(\pi_{X \times Z})} & & & \big\downarrow{\scriptstyle M^*(\pi_Z)} \\ M(X \times X' \times Z) & & \xrightarrow{M_*(\tau^f_{X' \times Z})} & & M(X' \times Z) \\ & \searrow & & & \big\downarrow{\scriptstyle M_*(\tau^{f'}_Z)} \\ & & {\scriptstyle M_*(\tau^{f \times f'}_Z)} & \longrightarrow & M(Z) \end{array}$$

the dotted maps make the diagram commute, and they are obtained applying M to the pull back in $\mathcal{AF}_G$,

$$\begin{array}{ccc} X \times X' \times Z & \xrightarrow{\pi_{X \times Z}} & X \times Z\ . \\ {\scriptstyle \tau^f_{X' \times Z}}\big\downarrow & & \big\downarrow{\scriptstyle \tau^f_Z} \\ X' \times Z & \xrightarrow[\pi_Z]{} & Z \end{array}$$

If instead we consider restricted Mackey functors for a profinite FC-group G, as in [1], then ${}^{\times}\mathcal{AF}_G$ does not act on $\mathbf{Mack}^r(G)$. Indeed, if $(f : X \to G^c) \in {}^{\times}\mathcal{AF}_G$ and $Y \in \mathcal{AF}_G$, then $\pi_Y : X \times Y \to Y$ is a morphism in ${}^{\times}\mathcal{AF}^f_G$ if and only if X is finite. Instead, τ^f_Y is a morphism in ${}^{\times}\mathcal{AF}^f_G$ if and only if, for all $y \in Y$, the set $\{(x, z) \in X \times Y \mid y = f(x)z\}$ is finite. Thus, only the finite crossed G-sets act on $\mathbf{Mack}^r(G)$. This observation does not come as a surprise to us, but it raises the question of the structure and purpose of the category (or categories) of Mackey functors for a profinite groups.

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References

[1] W. Bley and R. Boltje, *Cohomological Mackey functors in number theory*, J. Number Theory **105** (2004), 1–37.
[2] S. Bouc, *The p-blocks of the Mackey algebra*, Algebr. Represent. Theory 6 (2003), no. 5, 515–543.
[3] C. Curtis and I. Reiner, *Methods of representation theory: with applications to finite groups and orders*, Volume 1, Wiley, 1981
[4] T. tom Dieck, *Transformation groups*, de Gruyter Studies in Mathematics 8, Walter de Gruyter, 1987.
[5] A. Dress, *A characterization of solvable groups*, Math Z. **110** (1969), 213–217.
[6] A. Dress and C. Siebeneicher, *The Burnside ring of profinite groups and the Witt vector construction*, Adv. in Math. **70** (1988), no. 1, 87–132.
[7] D. Gluck, *Idempotent formula for the Burnside algebra with applications to the p-subgroup simplicial complex*, Ill. J. Math. 25 (1981), 63–67.
[8] P. Hall, *Periodic FC-groups*, J. LMS **34** (1959), 289–304.
[9] N. Jacobson, *Basic Algebra II*, Second Edition, Dover Publications, 2009.
[10] C. Martinez-Pérez, B. Nucinkis, *Cohomological dimension of Mackey functors for infinite groups*, J. Lond. Math. Soc., II. Ser. 74, No. 2 (2006), 379–396
[11] J. P. May, *A concise course in algebraic topology*, Chicago lectures in mathematics series, the University of Chicago Press, 1999
[12] H. Nakaoka, *Tambara functors on profinite groups and generalised Burnside functors*, Comm. Alg. 37 (2009), 3095 – 3151.
[13] F. Oda and T. Yoshida, *Crossed Burnside Rings I.*, J. Algebra **236** (2001), 29–79.
[14] L. Ribes and P. Zalesskii, *Profinite groups*, Springer, 2000.
[15] D. Robinson, *A course in the theory of groups*, Second Edition, Springer, 1996.
[16] A. Shalev, *Profinite groups with restricted centralizers*, Proc. AMS **122** number 4 (1994), 1279–1284.
[17] J. Thévenaz and P. Webb, *The structure of Mackey functors*, Trans. Amer. Math. Soc. 347, number 6 (1995), 1865–1961.
[18] M. J. Tomkinson, *FC-groups*, Research Notes in Mathematics 96, Pitman Advanced Publishing Program, 1984.
[19] J. Wilson, *Profinite groups*, London Math. Soc. Monographs New Series **19**, Oxford University Press, 1998.

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