

Influence of random fluctuations on delayed bifurcations. II. The cases of white and colored additive and multiplicative noise

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The influence of noise on the delay of a bifurcation point in the presence of a swept control parameter has been investigated theoretically, by digital simulation and by analog electronic experiment. The results obtained in an earlier paper [N. G. Stocks, R. Mannella, and P. V. E. McClintock, *Phys. Rev. A* **40**, 5361 (1989)] have thereby been extended and complemented. In particular, exact analytic expressions have been derived for the time-dependent probability densities $P(x, t)$, and these have been used to obtain the mean first-passage time t_{MFPT}^* for $x^2(t)$ to reach a threshold under the influence of Gaussian fluctuations, in several contexts: additive external white noise, additive external exponentially correlated noise, additive internal white noise, additive internal exponentially correlated noise, multiplicative white and colored noise. Based on Zeghlache, Mandel, and Van den Broeck's [*Phys. Rev. A* **40**, 286 (1989)] alternative definition of the bifurcation time t_{moment}^* in terms of the evolution of the second moment $\langle x^2(t) \rangle$, an expression is derived for t_{moment}^* for the general case of combined additive and multiplicative noises. The calculations are tested by comparison with the results of analog experiments and digital simulations, with which they are shown to be in excellent agreement.

I. INTRODUCTION

In a recent paper¹ (hereinafter referred to as I) the influence of additive white noise on the delay of a bifurcation point in the presence of a swept control parameter was studied by analog electronic experiment and by digital simulation. It was shown that the results were in good agreement with theoretical predictions, on the basis of two different kinds of comparisons. The system considered was of the form

$$\dot{x} = \mu(t)x(t) + \xi(t), \quad (1)$$

where the swept parameter

$$\mu(t) = \mu_0 + vt, \quad (2)$$

and $\xi(t)$ is a Gaussian variable with zero average and correlator

$$\langle \xi(t)\xi(s) \rangle = \frac{D}{\tau} e^{-(|s-t|)/\tau}. \quad (3)$$

In I the case of white noise ($\tau \rightarrow 0$) was considered; in the present paper we extend the investigation to the cases of both colored noise (finite τ) and of multiplicative noise where there is a noisy factor multiplying $x(t)$ in (1).

As pointed out in I (see also Refs. 2 and 3), when trying to define the value of the control parameter at which the bifurcation takes place, two alternative definitions are possible.

- (i) Average over $x^2(t)$. Successive realizations of $x(t)$ are squared and ensemble averaged together. The bifurcation point is defined^{4,5} as the value of the control parameter at which $\langle x^2(t) \rangle$ crosses a chosen threshold x_{th}^2 .
- (ii) Average over first-passage times. Each realization

$x(t)$ is squared; the value of the control parameter at which $x^2(t)$ crosses x_{th}^2 is found, and a distribution is built. The bifurcation point is defined^{6,7} as the average value of this distribution. We shall refer to this definition as the mean-first-passage-time (MFPT) criterion.

It was shown that, except for very small noise intensities, the bifurcation time defined by these two approaches was, in general, different.

In Sec. II we discuss the theory of the bifurcation time, and extending the investigations of I and of Refs. 6 and 7, we show how it may be calculated in a variety of circumstances. In Sec. III we report tests of the theory based on analog experiments and digital simulations. Finally, in Sec. IV we try to summarize the progress made and draw conclusions.

II. THEORY

A theoretical derivation of the bifurcation point based on the $\langle x^2(t) \rangle$ moment-average definition has been presented in Refs. 4 and 5, hereinafter referred to collectively as ZMV. A theory based on the mean first-passage time definition has been presented in Ref. 6 for white noise and in Ref. 7 for the case of colored noise: We will refer to these two papers together as TSM. A brief review of the various approaches was also given in I.

The theory proposed by ZMV is quite general. It is exact, and as such it ought to agree for any chosen values of parameters with analog experiments or digital simulations. The first-passage time approach, on the other hand, does contain some approximation. (The final theoretical result quoted by TSM, for example, while relevant to real systems, refers to a region of parameter space that is inaccessible to the simulations.) We now

discuss how the TSM approach may be generalized to cover as much of the parameter space as possible.

Our starting point (see also I) is the solution of (1) in the form

$$x(t) = \exp \left[\int_0^t \mu(s) ds \right] x(0) + \int_0^t \exp \left[\int_s^t \mu(y) dy \right] \xi(s) ds, \quad (4)$$

where $x(0)$ can be either a random variable, whose distribution is decided by the evolution equation (1) (“internal noise” case) or a fixed value for $x(t)$ externally set at time $t=0$ (“external noise” case). If now we are interested in deriving the distribution of the first-passage times, a possible approach would be to choose a particular realization $\{\xi(t)\}$ of the noise, on the resulting trajectory $x(t)$, to measure the time taken to reach the threshold value, and to use this information together with the statistical weight of the particular noise trajectory $\{\xi(t)\}$ to build the distribution of first-passage times. These are indeed some points of contact between this approach and approaches⁸ based on path-integral formalism.

As already discussed in I, quite a lot can be understood about the distribution function of the first-passage times if we discard terms coming from the usual (static) Kramers process. In a more formal way, we are going to substitute for the real trajectory $x(t)$ a sort of averaged (smoothed) trajectory which has the same statistical properties as $x(t)$. As pointed out, we are then discarding contributions to the distribution function of the first-passage times stemming from “bursts” in $x(t)$ due solely to random noise, and not to the intrinsic instability in the system. In effect, we are evaluating the contribution from the dominant path in a path-integral formulation. We now consider separately the two cases of internal and external noise.

A. External noise

For external noise, if ϕ_1 is a Gaussian variable with zero average and standard deviation one, we can write

$$x^2(t) = e^{2\mu_0 t + vt^2} [x(0) + \sqrt{F(t)}\phi_1]^2, \quad (5)$$

where $F(t)$ is given by (for colored noise)

$$F(t) \equiv \int_0^t \int_0^t \exp \left[-\mu_0(s+y) - \frac{v}{2}(s^2+y^2) \right] \langle \xi(s)\xi(y) \rangle ds dy = \frac{D}{\tau} \exp \left[\frac{(1/\tau - \mu_0)^2}{2v} \right] \left[\frac{2\pi}{v} \right]^{1/2} \int_0^t \exp \left[-\frac{vy^2}{2} \right] \exp \left[-y \left[\frac{1}{\tau} + \mu_0 \right] \right] \left[\operatorname{erf} \left[\frac{\sqrt{v}y + a - c}{\sqrt{2}} \right] - \operatorname{erf} \left[\frac{a - c}{\sqrt{2}} \right] \right] dy, \quad (6)$$

where erf is the usual error function⁹ and where we have adopted the definitions of Ref. 5 [t^* is the time when $x^2(t)$ reaches x_{th}^2]:

$$\alpha = \frac{x_{th}^2}{x^2(0)}, \quad b = \frac{D}{x^2(0)} \left[\frac{\pi}{v} \right]^{1/2}, \quad \beta = \frac{D}{x_{th}^2} \left[\frac{\pi}{v} \right]^{1/2}, \quad a = \frac{\mu_0}{\sqrt{v}}, \quad z = \frac{\mu_0 + vt^*}{\sqrt{v}}, \quad c = \frac{1}{\tau\sqrt{v}}. \quad (7)$$

This result follows from defining^{10,11}

$$\sqrt{F(t)}\phi_1 \equiv \int_0^t \exp \left[-\mu_0 s - \frac{vs^2}{2} \right] \xi(s) ds, \quad (8)$$

and noticing that $\sqrt{F(t)}\phi_1$, a linear combination of Gaussian variables, is itself a Gaussian variable with zero average and standard derivation given by (6).

The approach followed in I (semianalytical method) was now to generate ϕ_1 , to solve Eq. (5) for t , and to build the appropriate probability distribution from which the mean first-passage time can in turn readily be derived. However, we have now improved our previous results. Let us first see the derivation for white ($\tau \rightarrow 0$) noise.

Given Eq. (5) and knowing the equilibrium distribution of ϕ_1 , it is possible to invert both to obtain the distribution of x . We find then

$$P(x,t) = \frac{1}{[2\pi\sigma^2(t)]^{1/2}} \exp \left[-\int_0^t \mu(s) ds \right] \times \exp \left[-\frac{\left[x \exp \left[-\int_0^t \mu(s) ds \right] - x_0 \right]^2}{2\sigma^2(t)} \right], \quad (9)$$

with

$$\sigma^2(t) = D\sqrt{\pi/v} e^{a^2} [\operatorname{erf}(z) - \operatorname{erf}(a)]. \quad (10)$$

We now introduce

$$W(t) = \int_{-x_{th}}^{x_{th}} P(x,t) dx, \quad (11)$$

representing the fraction of the population still to reach the threshold value normalized to 1 at $t=0$. The distribution of the first-passage times is now given by $-\dot{W}(t)$, which is the rate at which trajectories leave from within the threshold values. The average first passage time is then given by

$$\begin{aligned} \langle t \rangle &= - \int_0^\infty \dot{W}(t) t \, dt \\ &= - \int_0^\infty t \partial_t \int_{-x_{th}}^{x_{th}} P(x, t) dx \, dt . \end{aligned} \tag{12}$$

It is now possible to write (12) in a compact form, changing the integration variable x . We find

$$\begin{aligned} \langle t \rangle &= - \int_0^\infty \frac{1}{\sqrt{\pi}} t \partial_t \int_{-x_l^-}^{x_l^+} e^{-x^2} dx \, dt \\ &= -\frac{1}{2} \int_0^\infty t \partial_t \{ \text{erf}[x_l^+(t)] + \text{erf}[x_l^-(t)] \} dt \\ &= -\frac{1}{\sqrt{\pi}} \int_0^\infty t (\dot{x}_l^+ e^{-x_l^{+2}} + \dot{x}_l^- e^{-x_l^{-2}}) dt , \end{aligned} \tag{13}$$

where

$$x_l^\pm(t) = \frac{-x_{th} \exp \left[- \int_0^t \mu(s) ds \right] \pm x_0}{\sqrt{2\sigma(t)}} . \tag{14}$$

For the case of colored noise, it is possible to derive exactly the same Eq. (13), with the same definition of $x_l^\pm(t)$, but with $\sigma^2(t)$ defined as

$$\begin{aligned} \sigma^2(t) &= Dc\sqrt{2\pi}e^{(c-a)^2/2} \\ &\times \int_0^t e^{[-vy^2/2-y(1/\tau+\mu_0)]} \\ &\times \left[\text{erf} \left[\frac{\sqrt{v}y+a-c}{\sqrt{2}} \right] - \text{erf} \left[\frac{a-c}{\sqrt{2}} \right] \right] dy . \end{aligned} \tag{15}$$

This equation follows directly from the definition of $F(t)$ in (6).

B. Internal noise

The case of internal noise is slightly more complicated. We start by solving (1) in the presence of colored noise (see also Ref. 1). If ϕ_1 and ϕ_2 are two Gaussian independent variables with zero average and of standard deviation unity, the solution can be cast in the form

$$\begin{aligned} x(t) &= e^{\mu_0 t + vt^2/2} \left\{ \phi_1 \left[\left[\frac{D}{\mu_0(\mu_0\tau-1)} \right]^{1/2} + G(t) \right] \right\} \\ &+ \phi_2 [F(t) - G^2(t)]^{1/2} , \end{aligned} \tag{16}$$

where $F(t)$ is defined in Eq. (8) and $G(t)$ is given by

$$G(t) = \left[\frac{D\mu_0}{\mu_0\tau-1} \right]^{1/2} \int_0^t e^{-\mu_0 s - vs^2/2} ds . \tag{17}$$

The forms for $F(t)$ and $G(t)$ follow from the equalities

$$\begin{aligned} \langle x^2(0) \rangle &= \frac{D}{\mu_0(\mu_0\tau-1)} , \\ \langle x(0)\xi(t) \rangle &= \frac{D}{(1-\mu_0\tau)} e^{-t/\tau} , \quad t > 0 \\ \langle \xi(0)\xi(t) \rangle &= \frac{D}{\tau} e^{-t/\tau} . \end{aligned} \tag{18}$$

In Eq. (16), we now have again a linear combination of

Gaussian variables, which is itself a Gaussian variable with the appropriate average and standard deviation.

After some algebra it is possible to derive an equation of the same form as (13), but with the definition

$$x_l^+ = x_l^- = \frac{x_{th} \exp \left[- \int_0^t \mu(s) ds \right]}{\sqrt{2\sigma(t)}} , \tag{19}$$

where

$$\begin{aligned} \sigma^2(t) &= A(t) + B(t) + C(t) , \\ A(t) &= \frac{2D}{1+\tau|\mu_0|} e^{(a+c)^2/2} \left[\frac{\pi}{2v} \right]^{1/2} \\ &\times \left[\text{erf} \left[\frac{\sqrt{v}t+c+a}{\sqrt{2}} \right] - \text{erf} \left[\frac{a+c}{\sqrt{2}} \right] \right] , \end{aligned} \tag{20}$$

$$B(t) = \frac{D}{|\mu_0|(1+\tau|\mu_0|)} ,$$

and

$$\begin{aligned} C(t) &= Dc\sqrt{2\pi}e^{(c-a)^2/2} \int_0^t e^{[-vy^2/2-y(1/\tau+\mu_0)]} \\ &\times \left[\text{erf} \left[\frac{\sqrt{v}y+a-c}{\sqrt{2}} \right] - \text{erf} \left[\frac{a-c}{\sqrt{2}} \right] \right] dy . \end{aligned}$$

The case of white internal noise is identical, with only a change in $\sigma^2(t)$, which would then need

$$\sigma^2(t) = \frac{D}{|\mu_0|} + D \left[\frac{\pi}{v} \right]^{1/2} e^{a^2} [\text{erf}(z) - \text{erf}(a)] . \tag{21}$$

C. Multiplicative noise

For the case of multiplicative noise, given the equation

$$\dot{x} = [\mu_0 + vt + \eta(t)]x ,$$

where

$$\begin{aligned} \langle \eta \rangle &= 0 , \\ \langle \eta(t)\eta(s) \rangle &= \frac{D\eta}{\tau_\eta} e^{-|t-s|/\tau_\eta} , \end{aligned} \tag{22}$$

and the definitions

$$\delta = \frac{D\eta}{v} , \quad \gamma = \frac{\mu}{v} , \tag{23}$$

the following implicit equation for the critical control parameter has been derived:⁵

$$vt^{*2} + 2t^*(\mu_0 + 2D\eta) - 4D\eta\tau_\eta(1 - e^{-t^*/\tau_\eta}) - \ln\alpha = 0 . \tag{24}$$

The derivation of the analogous equation using a mean first-passage time approach is very simple. We start by writing the solution of Eq. (22):^{10,11}

$$x(t) = x_0 e^{\mu_0 t + vt^2/2} e^{\sqrt{2D\eta} t \phi} , \tag{25}$$

with

$$\tilde{t} = t + \tau_\eta (e^{-t/\tau_\eta} - 1),$$

and ϕ a Gaussian variable with average zero and standard derivation 1. We find then for the distribution $P(x, t)$ given in (9),

$$P(x, t)dx = \frac{dx}{x\sqrt{4\pi D_\eta \tilde{t}}} \exp \left[-\frac{(\frac{1}{2} \ln \alpha - \mu_0 t - vt^2/2)^2}{4D_\eta \tilde{t}} \right], \quad (26)$$

and we can compute t^* from Eq. (12).

Note that the theory presented above is quite general, and, as such, it is not restricted to the particular choice of $\mu(t)$ we have studied here. It also suggests an alternative method for calculating the moments $\langle x^n(t) \rangle$ than the one used by ZMV. One can simply construct the generating function $\langle e^{ikz} \rangle$ for a given probability density and extract the moments in the normal fashion. The advantage of this method is that higher-order moments are accessible.

D. Combined additive and multiplicative noises

Finally, let us note that if we are interested in deriving the critical control parameter via the $\langle x^2 \rangle$ definition, it is

possible to find an expression even for the system with mixed noises described by the equation

$$\dot{x} = [\mu(t) + \eta(t)]x + \xi(t), \quad (27)$$

where

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t)\eta(s) \rangle = \frac{D_\eta}{\tau_\eta} e^{-|t-s|/\tau_\eta},$$

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(s) \rangle = \frac{D_\xi}{\tau_\xi} e^{-|t-s|/\tau_\xi}, \quad (28)$$

as long as the two Gaussian processes $\eta(t)$ and $\xi(t)$ are uncorrelated and D_η is relatively small (the latter condition being implicit in our assumption that the dynamics is determined by the product of a stochastic variable and a time-dependent coefficient). If we integrate Eq. (27), we obtain

$$x(t) = x_0 \exp \left[\int_0^t \mu(s) ds + \int_0^t \eta(s) ds \right] + \int_0^t \exp \left[\int_s^t \mu(y) dy + \int_s^t \eta(y) dy \right] \xi(s) ds, \quad (29)$$

and for $x^2(t)$ we obtain

$$\begin{aligned} x^2(t) = & x_0^2 \left[\exp \left[\int_0^t \mu(s) ds + \int_0^t \eta(s) ds \right] \right]^2 \\ & + 2x_0 \left[\int_0^t \exp \left[\int_s^t \mu(y) dy + \int_s^t \eta(y) dy \right] \xi(s) ds \right] \exp \left[\int_0^t \mu(s) ds + \int_0^t \eta(s) ds \right] \\ & + \int_0^t \int_0^t \exp \left[\int_s^t \mu(y) dy + \int_s^t \mu(y') dy' + \int_s^t \eta(y) dy + \int_s^t \eta(y') dy' \right] \xi(s) \xi(s') ds ds'. \end{aligned} \quad (30)$$

Given that the two processes $\xi(t)$ and $\eta(t)$ are independent, we can calculate the averages over them independently. Calculating the average over $\eta(t)$, we obtain [assuming also that x_0 and $\eta(t)$ are independent]

$$\begin{aligned} \langle x^2(t) \rangle_\eta = & \left\{ x_0^2 \left[\exp \left[\int_0^t \mu(s) ds + \sqrt{2D_\eta \tilde{t}} \phi \right] \right]^2 \right. \\ & + 2x_0 \int_0^t \exp \left[\int_s^t \mu(y) dy + \sqrt{2D_\eta} [\sqrt{\tilde{t}-\tilde{s}} + \sqrt{\tilde{t}}] \phi + \int_0^t \mu(s) ds \right] \xi(s) ds \\ & \left. + \int_0^t \int_0^t \exp \left[\int_s^t \mu(y) dy + \int_s^t \mu(y') dy' + \sqrt{2D_\eta} (\sqrt{\tilde{t}-\tilde{s}} + \sqrt{\tilde{t}-\tilde{s}'}) \phi \right] \xi(s) \xi(s') ds ds' \right\} e^{-\phi^2/2} \frac{d\phi}{\sqrt{2\pi}}. \end{aligned} \quad (31)$$

Performing the integrations, we find

$$\begin{aligned} \langle x^2(t) \rangle_\eta = & x_0^2 \left[\exp \left[\int_0^t \mu(s) ds + 4D_\eta \tilde{t} \right] \right]^2 + 2x_0 \int_0^t \exp \left[\int_s^t \mu(y) dy + \int_0^t \mu(s) ds + D_\eta (\sqrt{\tilde{t}-\tilde{s}} + \sqrt{\tilde{t}})^2 \right] \xi(s) ds \\ & + \int_0^t \int_0^t \exp \left[\int_s^t \mu(y) dy + \int_s^t \mu(y') dy' + D_\eta (\sqrt{\tilde{t}-\tilde{s}} + \sqrt{\tilde{t}-\tilde{s}'})^2 \right] \xi(s) \xi(s') ds ds'. \end{aligned} \quad (32)$$

This equation can now be averaged over ξ to obtain the value of the critical control parameter. For given $\eta(t)$ and external noise $\xi(t)$, Eq. (32) can be simplified:

$$\langle x^2(t) \rangle_{\eta, \xi} = x_0^2 \left[\exp \left[\int_0^t \mu(s) ds + 4D_\eta t \right] \right]^2 + D_\xi \int_0^t \left[\exp \left[\int_s^t \mu(y) dy + 4D_\eta (t-s) \right] \right]^2 ds. \quad (33)$$

If now we take the usual

$$\mu(t) = \mu_0 + vt, \quad (34)$$

we can immediately write

$$\langle x^2(t) \rangle_\eta = x_0^2 \left[\exp \left[\int_0^t \bar{\mu}(s) ds \right] \right]^2 + D_\xi \int_0^t \left[\exp \left[\int_s^t \bar{\mu}(y) dy \right] \right]^2 ds, \quad (35)$$

where

$$\bar{\mu}(t) = \mu_0 + 2D_\eta + vt. \quad (36)$$

In other words, in the presence of additive and multiplicative white external noises, the critical control parameter should be given by its value for the simpler additive external white noise case, but where μ_0 is now scaled by the multiplicative noise according to Eq. (36).

III. TESTS OF THE THEORY

The theory was tested both by analog experiment¹² and by digital simulations.^{13,14} The basis of these techniques has been described elsewhere,¹²⁻¹⁴ and their application to the particular system currently under study has already been discussed in I. In the present paper, therefore, we simply report the results obtained and compare them with the theoretical predictions.

We consider first the case of external noise. In Figs. 1(a) and 1(b), the scaled bifurcation time z is plotted as a function of the noise reciprocal correlation time parameter c for the two different bifurcation criteria, for various values of the scaled noise intensity b , and of the scaled initial curvature of the potential a . In both parts of the figure, the solid curves (theory) and associated data points refer to the MFPT bifurcation criterion, and the dashed curves (theory) and associated data refer to the moment-average criterion, as indicated in the caption. It may be noted immediately in Fig. 1(a) that, for both criteria, the bifurcation time z tends to its deterministic value in the limit of highly colored noise and hence is only dependent on a . In the white-noise limit, on the other hand, z is dependent only on b . This behavior has been reported previously,^{5,15} but it can now be seen to be the same for both bifurcation criteria. The theoretical predictions (curves) are strongly supported by both the analog experiments and digital simulations. Good agreement between experiment, simulation, and theory is also seen in Fig. 1(b). It may be noted that the constant value taken by z in the highly colored limit depends on the value of a , but not significantly on the bifurcation criterion; in the opposite limit, of white noise, z depends on the bifurcation criterion, but has become almost independent of a . The results obtained for the case of internal noise are shown in Figs. 2(a) and 2(b). Again, the curves represent theory, and the behavior predicted on the basis of the MFPT criterion (solid curves) is different at all values of c from that

predicted from the moment-average criterion (dashed curves). These predictions are fully borne out by the analog and digital data, although the MFPT theoretical curves are both a little bit higher than the measurements. The latter discrepancy is not particularly surprising,

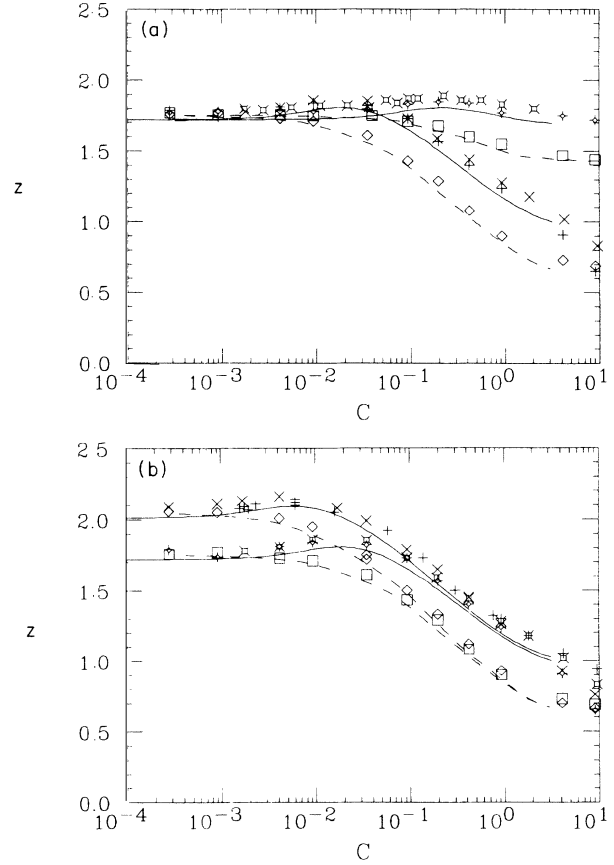


FIG. 1. Theory compared with the results of analog experiments and digital simulations for the case of external noise. In terms of scaled parameters, the bifurcation time z is plotted as a function of the reciprocal correlation time c of the noise. The solid curves are calculated from Eqs. (13) and (15), derived on the basis of the MFPT bifurcation criterion of TSM; the dashed curves represent the result of the ZMV theory. For both parts of the figure, the parameter $\alpha = 1.2$. (a) The scaled initial curvature of the potential, $a = -1.7$. The upper pair of curves in the white-noise limit (right-hand side) correspond to a scaled noise density $b = 0.05$; the lower pair of curves correspond to $b = 0.5$. The data points for the ZMV theory are lozenges, analog experiment with $b = 0.5$; squares, analog simulation with $b = 0.05$. The data points for the MFPT criterion are crosses, analog with $b = 0.5$; pluses, digital with $b = 0.5$; squares with added external lines, analog with $b = 0.05$; lozenges with external lines, digital with $b = 0.05$. (b) Results for a scaled noise intensity $b = 0.5$. The upper pair of curves in the limit of extremely colored noise (left-hand side) correspond to a scaled initial curvature of the potential of $a = -2.0$; for the lower pair of curves, $a = -1.7$. The data points for the ZMV theory are lozenges, analog experiment with $a = -2$; squares, analog simulation with $a = -1.7$. The data points for the MFPT criterion are crosses, analog with $a = -2$; pluses, digital with $a = -2$; squares with added external lines, analog with $a = -1.7$; lozenges with lines, digital with $a = -1.7$.

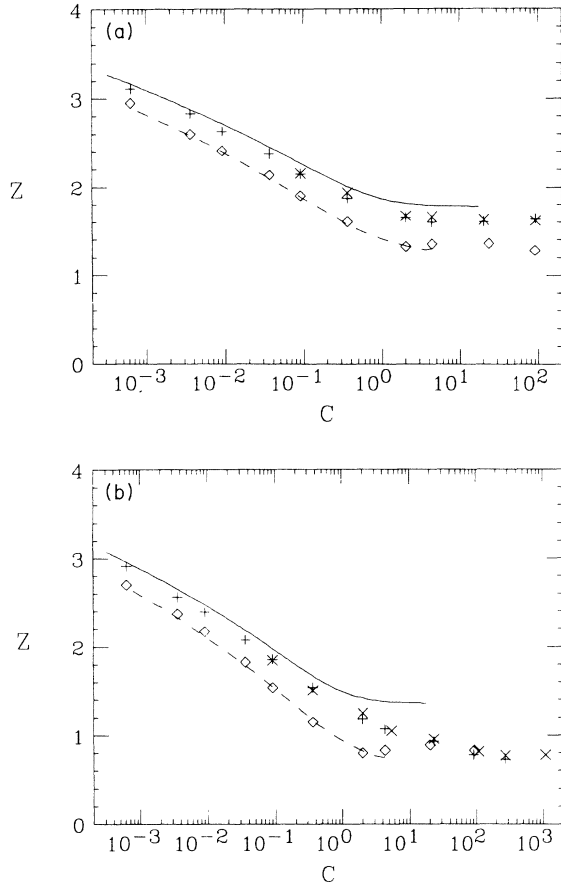


FIG. 2. Theory compared with the results of analog experiments and digital simulations for the case of internal noise. In terms of scaled parameters, the bifurcation time z is plotted as a function of the reciprocal correlation time c of the noise. The solid curves are calculated from Eqs. (13), (19), and (20), derived on the basis of the MFPT bifurcation criterion of TSM; the dashed lines represent the result of the ZMV theory. For both parts of the diagram, the scaled initial curvature of the potential $a = -3.0$. The lozenges are analog experimental measurements based on the ZMV bifurcation criterion. The pluses (analog data) and crosses (digital data) correspond to the MFPT criterion of TSM. (a) With scaled noise intensity $\beta=0.1$; (b) with $\beta=0.35$.

however, because we expect to encounter numerical problems when evaluating (12) in the limit of large c . It should be emphasized that the disagreement is a computation (numerical) one: In the limit of very large c , the theoretical expression derived by taking the limit in (12) before the numerical integration is performed was shown in I to be in excellent agreement with the simulations.

The results obtained for multiplicative noise are shown in Fig. 3 where, again, the solid curves represent the MFPT-based theory and the dashed curves represent the theory based on the moment-average criterion. In each case, z is plotted directly against the noise correlation time τ . The value of z measured in the analog experiment on the basis of the MFPT criterion is almost independent of τ , which is entirely in accord with the theory. In all cases, for both experiment and theory, z becomes in-

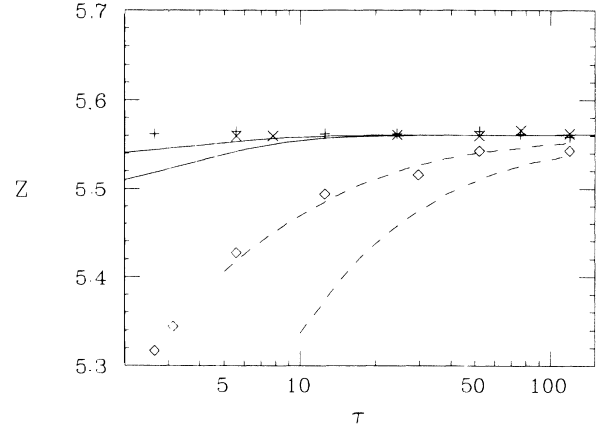


FIG. 3. Theory compared with the results of analog experiments for the case of multiplicative noise. The scaled bifurcation time z is plotted as a function of the correlation time τ of the noise. The solid curves are calculated from Eqs. (12) and (26) derived on the basis of the MFPT bifurcation criterion of TSM: for the upper curve, the scaled noise intensity $\delta=0.1$; for the lower case, $\delta=0.35$. The dashed curves represent the result of the ZMV theory: upper curve, $\delta=0.1$; lower curve, $\delta=0.35$. The pluses and crosses represent the results of analog experiments with $\delta=0.1$ and 0.35 , respectively, using the MFPT criterion. The lozenges represent analog data based on the ZMV criterion with $\delta=0.1$.

dependent of τ in the limit of highly colored noise. The agreement between our extension of the TSM approach and the results of the simulations is clearly excellent. The probability distributions (9) and (26) that we have obtained for the different cases are also solutions of the Fokker-Planck operator associated with the relevant Langevin equation for appropriate initial conditions. The limit of validity of our extension of the TSM approach is essentially given, then, by the treatment of the boundary conditions: In solving the Fokker-Planck operator, we disregard the fact that, to compute a mean first-passage time, $P(x, t)$ should become zero at the boundary x_{th} . In the treatment given here, we do *not* discard contributions to $P(x, t)$ derived from particles that recross the boundary once outside the region of interest. We expect, however, that our approach should be reliable for $\beta < |a| / \ln(z|a|)$, which is always the case for the simulations presented here and also of some relevance to real systems.

IV. CONCLUSION

In this paper, we have performed two main tasks. First, we have used analog experiments to test the applicability of the ZMV theory for colored and multiplicative noise to real physical systems, and have confirmed the results by means of digital simulations. Just as in the case of additive white noise, considered earlier in I, excellent agreement has been obtained between experiment, simulation, and theory.

Second, we have considered the TSM approach to the bifurcation problem, based on an MFPT criterion. We have shown how the TSM approach may be generalized, so that it is applicable to a much larger region of the parameter space, and we have tested the resultant theoretical predictions by means of the experiments and simulations. Excellent agreement has been obtained, demonstrating the veracity of the calculations.

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