Segre products of cluster algebras

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23rd April 2024

Abstract

We show that under suitable assumptions the Segre product of two graded cluster algebras has a natural cluster algebra structure.

1 Introduction

The map $\sigma: \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{n+m+nm}$ of projective spaces defined by

$$\sigma((x_0: \ldots : x_n), (y_0: \ldots : y_m)) = (x_0y_0: x_1: y_0: \ldots : x_ny_m)$$

is known as the Segre embedding—it is injective and its image is a subvariety of \mathbb{P}^{n+m+nm} . We may then define the Segre product of two projective varieties $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ as the image of $X \times Y$ with respect to the Segre embedding. We denote the Segre product by $X \overline{\otimes} Y \stackrel{\text{def}}{=} \sigma(X \times Y)$.

In what follows, rather than the geometric setting described above, we will be interested in the dual notion of the Segre product of graded algebras. Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ and $B = \bigoplus_{i \in \mathbb{N}} B_i$ be N-graded K-algebras. Then their Segre product, $A \otimes B$ is the N-graded algebra

$$A\overline{\otimes}B \stackrel{\text{\tiny def}}{=} \bigoplus_{i \in \mathbb{N}} A_i \otimes_{\mathbb{K}} B_i \tag{1}$$

with the usual tensor product algebra multiplication. Letting X and Y be projective varieties with homogeneous coordinate rings A and B respectively, the Segre product $A \otimes B$ is the homogeneous coordinate ring of $X \otimes Y$.

Cluster algebras are a class of combinatorially rich algebras arising in a number of algebraic and geometric contexts (see [FWZ21] and references

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therein). The additional data of a cluster structure leads to the existence of canonical bases, closely related to the canonical bases arising in Lie theory. Important examples of cluster algebras of this type include coordinate algebras of projective varieties and their various types of cells, e.g. Grassmannians ([Sco06]) and Schubert cells ([GLS11]), positroid varieties and positroid cells ([GL19]), etc.

In all known examples when the cluster algebra is the coordinate algebra of a projective variety, we have a compatible grading on the cluster algebra, with all cluster variables being homogeneous. Such cluster algebras are naturally called *graded cluster algebras* and the general theory of these is set out in work of the first author ([Gra15]).

In this note, inspired by [Pre23, Remark 4.14], we define a cluster algebra structure on the Segre product of graded cluster algebras. This generalises the particular case arising in [Pre23] in the study of cluster algebra structures on positroid varieties and in doing so, we are able to clarify the required input data to be able to form a Segre product.

We show that from the point of view of cluster algebras, forming the Segre product is given by a gluing operation on suitable frozen variables. We also record some simple observations on the preservation or otherwise of cluster-algebraic properties under taking Segre products.

As we will see, the standard Segre construction imposes significant restrictions on both the graded cluster algebras and the choice of clusters at which one can glue. The latter is perhaps surprising since most cluster algebra constructions are agnostic as to choices of initial cluster.

Acknowledgements

The authors acknowledge financial support from the Engineering and Physical Sciences Research Council (studentship ref. 2436773, https://gtr.ukri.org/projects?ref=studentship-2436773).

2 Segre Products of Graded Cluster Algebras

It was shown in [GL19] that coordinate rings of positroid varieties in the Grassmannian have cluster algebra structures. In [Pre23], the Segre product of two such cluster algebras is shown to have a cluster structure. In what follows, we aim to generalise this construction to the case of graded skew-symmetric cluster algebras.

We start by establishing some notation; readers unfamiliar with graded cluster algebras may wish to refer to [Gra15] for further details and examples.

First, let $\mathcal{A}_i = (\underline{x}_i, \underline{ex}_i, B_i, G_i)$ be (skew-symmetric) graded cluster algebras, for $i \in \{1, 2\}$, such that

- $\underline{x}_1 = \{x_1, \ldots, x_{n_1}\}$ and $\underline{x}_2 = \{y_1, \ldots, y_{n_2}\}$ are the respective initial clusters;
- $\underline{ex}_i \subsetneq \{1, \ldots, n_i\}$ is the set of indices corresponding to mutable cluster variables;
- every frozen variable (i.e. those elements with index in $\{1, \ldots, n_i\} \setminus \underline{ex}_i$) is invertible;
- B_i is an exchange matrix with skew-symmetric principal part;
- $G_i \in \mathbb{Z}^{n_i}$ is a grading vector, i.e. a vector such that $B_i^T G_i = 0$.

Throughout, we will work over a field \mathbb{K} , so that our cluster algebras are \mathbb{K} -algebras and we take all tensor products to be over \mathbb{K} . As we will see, the underlying field plays essentially no role in our construction.

Let $s_i \in \{1, \ldots, n_i\} \setminus \underline{ex}_i$ be an index corresponding to a frozen cluster variable. We denote by $B_i^{s_i}$ the s_i th row of B_i and by $\widehat{B}_i^{s_i}$ the exchange matrix obtained from B_i with the s_i th row removed.

Remark 2.1. In the above we require at least one frozen cluster variable in each cluster algebra—this will be important when defining a cluster structure on their Segre product since this will involve 'gluing' at frozen variables.

We have also asked that every frozen vertex is invertible, which is a common but not universal assumption in cluster theory. In fact, an examination of our construction shows that this assumption can be weakened to only asking that the glued frozen variables are invertible, which may be a more appropriate assumption for geometric applications.

We wish to define a cluster algebra structure on the Segre product $\mathcal{A}_1 \overline{\otimes} \mathcal{A}_2$. Following the approach of [Pre23], we aim to construct a new cluster algebra from \mathcal{A}_1 and \mathcal{A}_2 by gluing at frozen variables of the same degree, which we will show coincides with the Segre product under suitable further conditions.

2.1 A gluing construction

Fix $s_1 \in \{1, \ldots, n_1\} \setminus \underline{ex}_1$ and $s_2 \in \{1, \ldots, n_2\} \setminus \underline{ex}_2$ such that $(G_1)_{s_1} = (G_2)_{s_2}$. We will identify the frozen variables x_{s_1} and y_{s_2} , denoting a new proxy variable replacing both of these by x_s .

The initial data for our new cluster algebra is as follows. For the initial cluster, we take

$$\underline{x} = (\underline{x}_1 \setminus \{x_{s_1}\}) \cup (\underline{x}_2 \setminus \{y_{s_2}\}) \cup \{x_s\}.$$

The index set corresponding to mutable variables is given by $\underline{ex} = \underline{ex}_1 \cup \underline{ex}_2$, and for the initial exchange matrix, we take

$$\underline{B} = \begin{bmatrix} \widehat{B}_1^{s_1} & 0\\ 0 & \widehat{B}_2^{s_2}\\ B_1^{s_1} & B_2^{s_2} \end{bmatrix}.$$

Finally, for the initial grading vector we take

$$\underline{G} = ((\widehat{G}_1^{s_1})^T \ (\widehat{G}_2^{s_2})^T \ G_1^s)^T$$

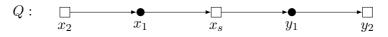
where $\widehat{G}_i^{s_i}$ is the grading vector G_i with the s_i th entry removed and $G_1^s \stackrel{\text{def}}{=} G_{s_1}^1 = G_{s_2}^2$. We can now define a cluster algebra $\mathcal{A}_1 \Box \mathcal{A}_2 = \mathcal{A}(\underline{x}, \underline{\text{ex}}, \underline{B}, \underline{G})$ from this initial data.

The process of gluing at frozen variables with matching degree is illustrated in the example below. Here and elsewhere, 1 denotes the vector $(1, \ldots, 1)^T$.

Example 2.2. Let $A_1 = (\underline{x}_1 = \{x_1, x_2, x_3\}, \underline{ex}_1 = \{1\}, Q_1, G_1 = 1)$ and $A_2 = (\underline{x}_2 = \{y_1, y_2, y_3\}, \underline{ex}_2 = \{1\}, Q_2, G_1 = 1)$ be cluster algebras with exchange quivers as follows:



The quiver obtained by 'gluing' at the frozen variables x_3 and y_3 is shown below—we denote the new variable by x_s .



The cluster algebra $\mathcal{A}_1 \Box \mathcal{A}_2$ is then given by the initial data

 $(\underline{x} = \{x_1, x_2, y_1, y_2, x_s\}, \underline{ex} = \{1, 3\}, Q, G = 1.$

We will show in Theorem 2.9 that this gives a cluster structure on the Segre product $\mathcal{A}_1 \overline{\otimes} \mathcal{A}_2$.

We record some straightforward observations about the cluster algebra $\mathcal{A}_1 \Box \mathcal{A}_2$.

Lemma 2.3. Let \mathcal{A}_1 and \mathcal{A}_2 be graded cluster algebras. Fix $s_1 \in \{1, \ldots, n_1\} \setminus \underline{ex}_1$ and $s_2 \in \{1, \ldots, n_2\} \setminus \underline{ex}_2$ such that $(G_1)_{s_1} = (G_2)_{s_2}$. Then the cluster algebras $\mathcal{A}_1 \Box \mathcal{A}_2$ and $\mathcal{A}_2 \Box \mathcal{A}_1$ are isomorphic as cluster algebras.

Proof. This is clear from comparing the initial data for $\mathcal{A}_1 \Box \mathcal{A}_2$ and $\mathcal{A}_2 \Box \mathcal{A}_1$ and in particular noting that the two initial clusters are equal up to permutation of the entries.

Lemma 2.4. Let A_1 and A_2 be graded cluster algebras. Fix $s_1 \in \{1, \ldots, n_1\} \setminus \underline{ex}_1$ and $s_2 \in \{1, \ldots, n_2\} \setminus \underline{ex}_2$ such that $(G_1)_{s_1} = (G_2)_{s_2}$. Then

- (i) $\mathcal{A}_1 \Box \mathcal{A}_2$ is of finite type if and only if \mathcal{A}_1 and \mathcal{A}_2 are;
- (ii) writing κ(A) for the number of cluster variables of a cluster algebra
 A, we have κ(A₁□A₂) = κ(A₁) + κ(A₂) 1 when these numbers are all finite; and
- (iii) writing $K(\mathcal{A})$ for the number of clusters of \mathcal{A} , we have $K(\mathcal{A}_1 \Box \mathcal{A}_2) = K(\mathcal{A}_1)K(\mathcal{A}_2)$ when these numbers are all finite.

Proof. This follows from observing that our gluing process does not introduce any new arrows between mutable vertices. Since mutation is a local phenomenon and concentrated on mutable vertices, it is straightforward to see that mutating at vertices indexed by \underline{ex}_1 is independent of mutating at vertices indexed by \underline{ex}_2 and the variables obtained are exactly as if the gluing had not been carried out.

There is an overall reduction of one in the number of cluster variables because we have glued two previously distinct frozen variables; note that this highlights the difference between this construction and the usual "disconnected" product of cluster algebras (where one simply takes the union of clusters and direct sum of exchange matrices).

Remark 2.5. One might hope that this construction extends straightforwardly to graded quantum cluster algebras (cf. [GL14]). However, computation in small examples shows that this is not the case.

For if one tries the naïve approach in which initial quantum cluster variables from \mathcal{A}_1 commute with those from \mathcal{A}_2 , one rapidly finds situations in which after performing a mutation, the new variable does not quasi-commute with the rest of its cluster. For it to do so requires the compatibility condition between the exchange and quasi-commutation matrices for the glued data and this imposes a collection of "cross-term" requirements between B_1 and L_2 (respectively, B_2 and L_1) in respect of the glued frozen variables.

2.2 Relationship with the Segre product

The cluster algebra $\mathcal{A}_1 \Box \mathcal{A}_2$ defined above does not have any immediately obvious relationship with the Segre product $\mathcal{A}_1 \overline{\otimes} \mathcal{A}_2$. Our main goal in what follows is to show that there indeed is one and furthermore, we will establish under what conditions these algebras are actually isomorphic.

We first identify a candidate isomorphism, analogous to the map δ^{src} defined in [Pre23].

Proposition 2.6. Let $\mathcal{A}_i = (\underline{x}_i, \underline{ex}_i, B_i, G_i)$, i = 1, 2 be graded cluster algebras and fix $s_1 \in \{1, \ldots, n_1\} \setminus \underline{ex}_1$ and $s_2 \in \{1, \ldots, n_2\} \setminus \underline{ex}_2$.

Then the map $\varphi : \mathcal{A}_1 \Box \mathcal{A}_2 \to \mathcal{A}_1 \otimes \mathcal{A}_2$ given on initial cluster variables by

$$\varphi(x_j) = x_j \otimes y_{s_2}^{\deg x_j} \quad \text{for } j \in \{1, \dots, n_1\},$$

$$\varphi(y_j) = x_{s_1}^{\deg y_j} \otimes y_j \quad \text{for } j \in \{1, \dots, n_2\} \text{ and }$$

$$\varphi(x_s) = x_{s_1} \otimes y_{s_2}$$

is an injective algebra homomorphism, with the property that the above formulæ hold for any cluster of $\mathcal{A}_1 \Box \mathcal{A}_2$.

Proof. Let φ denote the algebra homomorphism between fields of rational functions

$$\varphi \colon \mathbb{K}(\underline{x}) \to \mathbb{K}(\underline{x}_1 \otimes y_{s_2}^{\deg x_\bullet}, x_{s_1}^{\deg y_\bullet} \otimes \underline{x}_2, x_{s_1} \otimes y_{s_2})$$

obtained from the above specification on generators of the domain, where $\underline{x}_1 \otimes y_{s_1}^{\deg x_{\bullet}} = \{x \otimes y_{s_2}^{\deg x} \mid x \in \underline{x}_1\}$ and similarly for $x_{s_1} \otimes \underline{x}_2$.

We claim that the map φ has a natural extension to all cluster variables via cluster mutation. To prove this, we proceed by induction on the number of mutation steps from the initial cluster. We compute φ for a one-step mutation from the initial cluster as follows.

We first consider the case in which $x_k \in \underline{x}_1$, $k \in \underline{ex}_1$. To reduce the proliferation of subscripts, we will write B_{jk}^i for the (j,k)-entry of B_i and G_j^i for the *j*th entry of G_i . Note that $G_j^1 = \deg x_j$ and $G_j^2 = \deg y_j$. We also set $[n]_+ = \max\{n, 0\}$ and $[n]_- = \max\{-n, 0\}$.

We have

$$\begin{split} \varphi(\mu_k(x_k)) &= \varphi\bigg(\frac{1}{x_k} \Bigg[x_{s_1}^{[B_{n_1+n_2-1,k}]_+} \bigg(\prod_{B_{jk}>0} x_j^{B_{jk}} \bigg) \bigg(\prod_{B_{n_1+j-1,k}>0} y_j^{B_{n_1+j-1,k}} \bigg) \\ &+ x_{s_1}^{[B_{n_1+n_2-1,k}]_-} \bigg(\prod_{B_{jk}<0} x_j^{-B_{jk}} \bigg) \bigg(\prod_{B_{n_1+j-1,k}<0} y_j^{-B_{n_1+j-1,k}} \bigg) \Bigg] \bigg) \\ &= \varphi\bigg(\frac{1}{x_k} \Bigg[x_{s_1}^{[B_{n_1+n_2-1,k}]_+} \bigg(\prod_{B_{jk}>0} x_j^{B_{jk}} \bigg) \\ &+ x_{s_1}^{[B_{n_1+n_2-1,k}]_-} \bigg(\prod_{B_{jk}<0} x_j^{-B_{jk}} \bigg) \Bigg] \bigg) \\ &= \varphi\bigg(\frac{1}{x_k} \Bigg[\prod_{B_{jk}^{1}>0} x_j^{B_{jk}^{1}} + \prod_{B_{jk}^{1}<0} x_j^{-B_{jk}^{1}} \bigg] \bigg) \end{split}$$

$$\begin{split} &= \frac{1}{x_k \otimes y_{s_2}^{G_k^1}} \Biggl[\prod_{B_{jk}^1 > 0} \left(x_j^{B_{jk}^1} \otimes y_{s_2}^{G_j^1 B_{jk}^1} \right) + \prod_{B_{jk}^1 < 0} \left(x_j^{-B_{jk}^1} \otimes y_{s_2}^{-G_j^1 B_{jk}^1} \right) \Biggr] \\ &= \frac{1}{x_k \otimes y_{s_2}^{G_k^1}} \Biggl[\prod_{B_{jk}^1 > 0} \left(x_j^{B_{jk}^1} \right) \otimes y_{s_2}^d + \prod_{B_{jk}^1 < 0} \left(x_j^{-B_{jk}^1} \right) \otimes y_{s_2}^d \Biggr] \\ &= \frac{1}{x_k} \Biggl(\prod_{B_{jk}^1 > 0} x_j^{B_{jk}^1} + \prod_{B_{jk}^1 < 0} x_j^{-B_{jk}^1} \Biggr) \otimes y_{s_2}^{d-\deg x_k} \\ &= \mu_k(x_k) \otimes y_{s_2}^{d-\deg x_k} \\ &= \mu_k(x_k) \otimes y_{s_2}^{\deg \mu_k(x_k)} \end{split}$$

where

$$d = \sum_{B_{jk}^1 > 0} B_{jk}^1 G_j^1 = \sum_{B_{jk}^1 < 0} -B_{jk}^1 G_j^1$$

noting that the two are equal since $B_1^T G_1 = 0$ and $\deg \mu_k(x_k) = d - \deg x_k$. An analogous argument shows that $\varphi(\mu_k(y_k)) = x_{s_1}^{\deg \mu_k(y_k)} \otimes \mu_k(y_k)$ for

 $y_k \in \underline{y}$. Now let $z = (z_1, \dots, z_{m-1})$ be a cluster m mutation steps away from

Now, let $\underline{z} = (z_1, \ldots, z_{n_1+n_2-1})$ be a cluster m mutation steps away from the initial cluster \underline{x} , i.e. $\underline{z} = \mu_{\underline{p}}(\underline{x})$ for some mutation path \underline{p} of length m, and assume that

$$\varphi(z_j) = z_j \otimes y_{s_2}^{\deg z_j}$$

for $z_j \in \mathcal{A}_1$. Denote by B' the corresponding exchange matrix. We claim that, for $z_k \in \mathcal{A}_1$,

$$\varphi(\mu_k(z_k)) = \mu_k(z_k) \otimes y_{s_2}^{\deg \mu_k(z_k)}.$$

Indeed, the same calculation as above with \underline{z} and B' shows that this is the case and similarly $\varphi(\mu_k(z_k)) = x_{s_1}^{\deg \mu_k(z_k)} \otimes \mu_k(z_k)$, when $z_k \in \mathcal{A}_2$.

The above allows us to define $\varphi \colon \mathcal{A}_1 \Box \mathcal{A}_2 \to \mathcal{A}_1 \otimes \mathcal{A}_2$ on the generating set of cluster variables as above. We have seen that this respects the defining (exchange) relations and can therefore be extended to an algebra homomorphism. It is clearly injective on the generating set of the domain and therefore injective.

Note that

$$\begin{aligned} \varphi(x_j) &= x_j \otimes y_{s_2}^{\deg x_j} \in (\mathcal{A}_1)_{\deg x_j} \otimes (\mathcal{A}_2)_{\deg x_j \deg y_{s_2}} \\ \varphi(y_j) &= x_{s_1}^{\deg y_j} \otimes y_j \in (\mathcal{A}_1)_{\deg y_j \deg x_{s_1}} \otimes (\mathcal{A}_2)_{\deg y_j} \\ \varphi(x_s) &= x_{s_1} \otimes y_{s_2} \in (\mathcal{A}_1)_{\deg x_{s_1}} \otimes (\mathcal{A}_2)_{\deg y_{s_2}} \end{aligned}$$

and so we do not land in the Segre product without extra constraints. However, we also see that the obstacle is the degree of the gluing frozen variables: fixing these to be 1, we immediately obtain that the map is a graded homomorphism to the Segre product.

Proposition 2.7. Let $A_i = (\underline{x}_i, \underline{ex}_i, B_i, G_i)$, i = 1, 2 be graded cluster algebras such that there exist $s_1 \in \{1, \ldots, n_1\} \setminus \underline{ex}_1$ and $s_2 \in \{1, \ldots, n_2\} \setminus \underline{ex}_2$ of degree 1.

Then the map $\varphi : \mathcal{A}_1 \Box \mathcal{A}_2 \to \mathcal{A}_1 \overline{\otimes} \mathcal{A}_2$ given on initial cluster variables by

$$\varphi(x_j) = x_j \otimes y_{s_2}^{\deg x_j} \quad for \ j \in \{1, \dots, n_1\},$$

$$\varphi(y_j) = x_{s_1}^{\deg y_j} \otimes y_j \quad for \ j \in \{1, \dots, n_2\} \text{ and }$$

$$\varphi(x_s) = x_{s_1} \otimes y_{s_2}$$

is an injective graded algebra homomorphism, with the property that the above formulæ hold for any cluster of $A_1 \Box A_2$.

It remains to show that φ is an isomorphism, for then the cluster structure defined on $\mathcal{A}_1 \Box \mathcal{A}_2$ indeed induces a cluster structure on the Segre product $\mathcal{A}_1 \overline{\otimes} \mathcal{A}_2$. This again requires a further assumption.

Definition 2.8. Let $\mathcal{A} = (\underline{x}, \underline{ex}, B, G)$ be a graded cluster algebra. We say that a cluster \underline{y} of \mathcal{A} is homogeneous of degree d if all cluster variables in y have the same degree d.

Theorem 2.9. Let $\mathcal{A}_i = (\underline{x}_i, \underline{ex}_i, B_i, G_i = 1)$, i = 1, 2 be graded cluster algebras such that \underline{x}_1 and \underline{x}_2 are homogeneous of degree 1 and fix $s_1 \in \{1, \ldots, n_1\} \setminus \underline{ex}_1$ and $s_2 \in \{1, \ldots, n_2\} \setminus \underline{ex}_2$.

Then the map $\varphi: \mathcal{A}_1 \Box \mathcal{A}_2 \to \mathcal{A}_1 \overline{\otimes} \mathcal{A}_2$ given on initial cluster variables by

$$\varphi(x_j) = x_j \otimes y_{s_2} \quad for \ j \in \{1, \dots, n_1\},$$

$$\varphi(y_j) = x_{s_1} \otimes y_j \quad for \ j \in \{1, \dots, n_2\} \text{ and }$$

$$\varphi(x_s) = x_{s_1} \otimes y_{s_2}$$

is a graded algebra isomorphism, with the property that the above formulæ hold for any cluster of $\mathcal{A}_1 \Box \mathcal{A}_2$.

Thus the construction above endows $\mathcal{A}_1 \overline{\otimes} \mathcal{A}_2$ with the structure of a cluster algebra.

Proof. It remains to check surjectivity. Note that a generating set for $\mathcal{A}_1 \overline{\otimes} \mathcal{A}_2$ is given by taking the elementary tensors with components in generating sets for \mathcal{A}_1 and \mathcal{A}_2 , i.e.

$$\{z_1 \otimes z_2 | z_1 \in (\mathcal{A}_1)_d, z_2 \in (\mathcal{A}_2)_d \text{ cluster variables}, d \in \mathbb{Z}\}$$

Now

$$z_1 \otimes z_2 = (z_1 \otimes y_{s_2})(x_{s_1} \otimes z_2)(x_{s_1}^{-d} \otimes y_{s_2}^{-d}) = \varphi(z_1)\varphi(z_2)\varphi(x_s)^{-d}.$$

Hence, Im φ contains a generating set for $\mathcal{A}_1 \overline{\otimes} \mathcal{A}_2$, and so φ is surjective. The claim follows. **Remark 2.10.** Notice that in proving surjectivity, we required $\varphi(x_s) = x_{s_1} \otimes y_{s_2}$, and hence x_{s_1} and y_{s_2} themselves, to be invertible, but no other frozen variables needed to be invertible for the proof to hold.

The following is now immediate from Lemmas 2.3 and 2.4.

Corollary 2.11. Let $\mathcal{A}_i = (\underline{x}_i, \underline{ex}_i, B_i, G_i = 1)$, i = 1, 2 be graded cluster algebras such that \underline{x}_1 and \underline{x}_2 are homogeneous of degree 1 and fix $s_1 \in \{1, \ldots, n_1\} \setminus \underline{ex}_1$ and $s_2 \in \{1, \ldots, n_2\} \setminus \underline{ex}_2$.

Then

- (i) the cluster algebras A₁ ⊗A₂ and A₂ ⊗A₁ are isomorphic as cluster algebras;
- (ii) $\mathcal{A}_1 \overline{\otimes} \mathcal{A}_2$ is of finite type if and only if \mathcal{A}_1 and \mathcal{A}_2 are;
- (iii) writing $\kappa(\mathcal{A})$ for the number of cluster variables of a cluster algebra \mathcal{A} , we have $\kappa(\mathcal{A}_1 \otimes \mathcal{A}_2) = \kappa(\mathcal{A}_1) + \kappa(\mathcal{A}_2) 1$ when these numbers are all finite; and
- (iv) writing $K(\mathcal{A})$ for the number of clusters of \mathcal{A} , we have $K(\mathcal{A}_1 \otimes \mathcal{A}_2) = K(\mathcal{A}_1)K(\mathcal{A}_2)$ when these numbers are all finite.

Remark 2.12. In the above theorem, we require the input clusters to be homogeneous of degree one. This was necessary to ensure that the image of φ generates the Segre product as defined. We note that this condition is very restrictive and it would be desirable for it to be weakened. However, without it, it does not seem feasible to describe the image in as simple a fashion as for the standard Segre product.

We also observe that, as a result, the construction of $\mathcal{A}_1 \Box \mathcal{A}_2$ and hence the cluster structure induced on $\mathcal{A}_1 \overline{\otimes} \mathcal{A}_2$ is strongly "rooted", i.e. dependent on a choice of initial clusters with specific properties. Even in a graded cluster algebra that has a homogeneous cluster, only with very specific exchange matrices we will find that other clusters are also homogeneous.

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