

Quantum Noise and Self-Sustained Radiation of \mathcal{PT} -Symmetric Systems

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(Received 6 January 2010; published 8 June 2010)

The observation that \mathcal{PT} -symmetric Hamiltonians can have real-valued energy levels even if they are non-Hermitian has triggered intense activities, with experiments, in particular, focusing on optical systems, where Hermiticity can be broken by absorption and amplification. For classical waves, absorption and amplification are related by time-reversal symmetry. This work shows that microreversibility-breaking quantum noise turns \mathcal{PT} -symmetric systems into self-sustained sources of radiation, which distinguishes them from ordinary, Hermitian quantum systems.

DOI: 10.1103/PhysRevLett.104.233601

PACS numbers: 42.50.Lc, 03.65.Nk

A common factor in quantum systems with a non-Hermitian Hamiltonian is the nonconservation of particle number, either because the system is leaky, or because there is loss or gain in an absorbing or amplifying medium. Ignoring nonlinear effects such as the feedback in a laser, such systems ordinarily do not possess stationary states; instead, they only support decaying quasibound states with complex energy, where the imaginary part $\text{Im}E = -1/2\tau$ (setting $\hbar \equiv 1$) accounts for particle loss with decay rate $1/\tau$ (particle gain is associated to a negative decay rate). A notable exception are non-Hermitian systems with loss and gain combined such that they are invariant under joint parity (\mathcal{P}) and time-reversal (\mathcal{T}) symmetry [1]. If there is no leakage, these \mathcal{PT} -symmetric systems generically possess a set of real-valued energy levels, as well as complex-conjugate pairs of complex energy levels. Systems with entirely real spectrum define a consistent unitary extension of quantum mechanics [2,3]. This observation has led to intense research efforts delivering a new theoretical perspective on systems as varied as quantum field theories and complex crystals [4], while experimental realizations, in particular, focus on optical systems where Hermiticity can be violated by absorption and amplification [5].

For classical waves, amplification and absorption are strictly related by time reversal. The existence of stationary states with real energy can therefore be seen as a consequence of the balance of amplification and absorption in parity-related regions of a \mathcal{PT} -symmetric system. At the heart of absorption and amplification, however, are noisy microscopic quantum processes which effectively break time-reversal symmetry [6]. The objective of the present work is to show that the effects of this quantum noise distinguish \mathcal{PT} -symmetric systems from Hermitian quantum systems, and indeed suggest an alternative interpretation of the physics behind non-Hermitian \mathcal{PT} symmetry. (i) Accounting for quantum noise, \mathcal{PT} -symmetric systems with stationary states are self-sustained sources of radiation. (ii) That the energy of these states is real means that the system is stabilized at the lasing threshold. (iii) When the system is leaky, the emitted radiation breaks parity

symmetry (i.e., the emission pattern is asymmetric). (iv) In the limit of a nonleaking system, the emitted radiation intensity approaches a constant value and provides a direct measure of the nonhermiticity of the system. The internal energy density of radiation then diverges, which entails a practical limitation for the implementation of \mathcal{PT} symmetry in closed systems.

These conclusions are obtained by employing the quantum-optical input-output formalism [7] in its scattering formulation [8–10]. The scattering approach also provides insight into \mathcal{PT} symmetry for classical waves [11,12], which defines the starting point of this Letter.

Scattering approach to non-Hermitian \mathcal{PT} -symmetric systems.—Probing the internal dynamics of an optical system by external radiation naturally leads to the scattering scenario depicted in Fig. 1. The relation $a^{\text{out}} = S a^{\text{in}}$ between incoming and outgoing wave amplitudes is provided by the scattering matrix $S(E) = \begin{pmatrix} r & r' \\ t & t' \end{pmatrix}$, which contains blocks describing reflection (r, r') and transmission (t, t') when probed from the left or right, respectively. Each block consists of an $(N \times N)$ -dimensional matrix, where

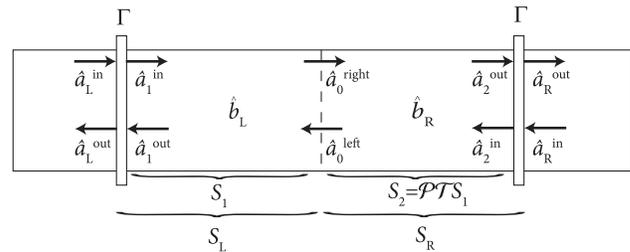


FIG. 1. Illustration of the scattering input-output approach to non-Hermitian \mathcal{PT} -symmetric systems, defining the scattering input-output operators $\hat{a}_{L,R}^{\text{in,out}}$ and internal bosonic noise operators $\hat{b}_{L,R}$ in different parts of the system. The operators $\hat{a}_0^{\text{right,left}}$ and $\hat{a}_{1,2}^{\text{in,out}}$ feature in the intermediate steps of the calculation (for details see the text). Semitransparent mirrors with transmission probability Γ are introduced to study the limit $\Gamma \rightarrow 0$ of a closed (nonleaky) system. In the context of present experiments with optical fibers [5], this sketch relates to the transverse confinement.

N is the number of modes at each entrance. The scattering matrix is generally energy dependent (which arises from the wave-number dependence of constructive and destructive interference), and its poles determine the energies of quasibound states in the system [13].

In general, the scattering matrix fulfills the following two reciprocity relations: the Onsager relation $S(\gamma, -B, E) = S^T(\gamma, B, E)$, and the relation $S(-\gamma, B, E) = [S^\dagger(\gamma, B, E^*)]^{-1}$ of classical microreversibility. Here, γ and B characterize two possible sources of broken time-reversal symmetry: absorption or amplification ($\gamma > 0/\gamma < 0$), which contribute an imaginary symmetric (non-Hermitian) term to the Hamiltonian H , and magneto-optical effects (B), which contribute an imaginary antisymmetric (but still Hermitian) term.

Conventional time-reversal $\mathcal{T}H = H^*$ transforms solutions according to $\mathcal{T}\psi = \psi^*$, which interchanges incoming and outgoing states, and therefore transforms the scattering matrix according to

$$\mathcal{T}S(\gamma, B, E) = [S^*(\gamma, B, E)]^{-1} = S(-\gamma, -B, E^*). \quad (1)$$

Assuming that energy is real, a system has \mathcal{T} symmetry, $\mathcal{T}S = S$ hence $S^* = S^{-1}$, if $S(\gamma, B) = S(-\gamma, -B)$, which requires $\gamma = B = 0$. Parity $\mathcal{P}H(x) = H(-x)$ transforms solutions according to $\mathcal{P}\psi(x) = \psi(-x)$, which exchanges the left and right leads and yields

$$\mathcal{P}S(\gamma, B, E) = \sigma_x S(\gamma, B, E) \sigma_x, \quad (2)$$

where σ_x is a Pauli matrix. The \mathcal{PT} operation on the scattering matrix is therefore given by

$$\mathcal{PT}S(\gamma, B, E) = \sigma_x [S^*(\gamma, B, E)]^{-1} \sigma_x \quad (3)$$

$$= \sigma_x S(-\gamma, -B, E^*) \sigma_x. \quad (4)$$

For Hermitian systems, \mathcal{PT} symmetry implies $S = \sigma_x S^T \sigma_x$ [14]. For non-Hermitian systems, \mathcal{PT} symmetry implies the additional condition $\mathcal{P}\gamma = -\gamma$ [$\gamma(x) = -\gamma(-x)$]; i.e., there is a balance of absorption and amplification in parity-related regions.

Let us now explore from the scattering perspective how real-energy bound states appear in \mathcal{PT} -symmetric systems. As shown in Fig. 1, such systems can be constructed by joining two regions, where the left region, with scattering matrix

$$S_1 = \begin{pmatrix} r_1 & t_1' \\ t_1 & r_1' \end{pmatrix},$$

is \mathcal{PT} symmetric to the right region, $S_2 = \mathcal{PT}S_1$, which using standard block-inversion formulas can be written as

$$S_2 = \begin{pmatrix} \frac{1}{(r_1' - t_1 r_1^{-1} t_1')^*} & (r_1'^{-1} t_1)^* \frac{1}{(t_1' r_1'^{-1} t_1 - r_1)^*} \\ (r_1^{-1} t_1')^* \frac{1}{(t_1 r_1^{-1} t_1' - r_1')^*} & \frac{1}{(r_1 - t_1' r_1'^{-1} t_1)^*} \end{pmatrix}. \quad (5)$$

Bound states can be studied by closing the system off by mirrors with small transmission probability $\Gamma \ll 1$, described by a scattering matrix

$$S_\Gamma = - \begin{pmatrix} \sqrt{1-\Gamma} & i\sqrt{\Gamma} \\ i\sqrt{\Gamma} & \sqrt{1-\Gamma} \end{pmatrix}. \quad (6)$$

The mirrors can be included by wave matching at their inward-facing faces, which amounts to an algebraic elimination of the amplitudes $a_{1,2}^{\text{in,out}}$ in Fig. 1. The scattering matrix of the left half of the system then reads

$$S_L = - \begin{pmatrix} \frac{r_1 + \sqrt{1-\Gamma}}{1 + r_1 \sqrt{1-\Gamma}} & \frac{it_1' \sqrt{\Gamma}}{1 + r_1 \sqrt{1-\Gamma}} \\ \frac{it_1 \sqrt{\Gamma}}{1 + r_1 \sqrt{1-\Gamma}} & \frac{t_1 t_1' \sqrt{1-\Gamma}}{1 + r_1 \sqrt{1-\Gamma}} - r_1' \end{pmatrix}, \quad (7)$$

while the scattering matrix $S_R = \mathcal{PT}S_L$ of the right half again follows from symmetry. These scattering matrices relate amplitudes of ingoing and outgoing modes (defined in Fig. 1) according to

$$\begin{pmatrix} a_{\text{right}}^{\text{out}} \\ a_0^{\text{right}} \end{pmatrix} = S_L \begin{pmatrix} a_0^{\text{in}} \\ a_{\text{left}}^{\text{in}} \end{pmatrix}, \quad \begin{pmatrix} a_0^{\text{left}} \\ a_R^{\text{out}} \end{pmatrix} = S_R \begin{pmatrix} a_0^{\text{right}} \\ a_R^{\text{in}} \end{pmatrix}. \quad (8)$$

The scattering matrix of the composed system is obtained by algebraically eliminating the amplitudes a_0^{left} and a_0^{right} at the interface between both regions. For $\Gamma \rightarrow 0$, these amplitudes become singular when

$$\det \text{Im}(r_L') = \det \left[\text{Im} \left(r_1' - \frac{t_1 t_1'}{1 + r_1} \right) \right] = 0, \quad (9)$$

which (due to the general connection of scattering poles and quasibound states) is the quantization condition of the closed system. Determinantal quantization conditions of this kind have been introduced for general systems in Ref. [13]; they also form the basis of exact numerical techniques as reviewed in [15]. The \mathcal{PT} -specific version (9) of the quantization condition requires that the N real column vectors of $\text{Im}(r_L')$ be linearly dependent, which generically can be achieved by varying a single real parameter such as E (identifying this as a problem of codimension one). Therefore, we recover that closed \mathcal{PT} -symmetric systems typically possess a number of bound states with real energy, even though the Hamiltonian is not Hermitian. The condition (9) also admits complex eigenvalues, which then appear in complex-conjugated pairs.

Quantum noise.—The scattering approach can be extended to include quantum noise by passing from wave amplitudes a^{in} , a^{out} to bosonic annihilation operators \hat{a}^{in} , \hat{a}^{out} , respectively. This defines the scattering variant of the input-output formalism [7–10], which has been used to describe systems that are exclusively absorbing or amplifying. To adapt the approach to \mathcal{PT} -symmetric systems, where both effects are combined, we formally separate the absorbing regions from the amplifying regions, and then join them together similar to the description of classical waves, given above.

For definiteness let us assume that the left half of the system is purely absorbing. For this part, the input-output

scattering relations then take the form

$$\begin{pmatrix} \hat{a}_L^{\text{out}} \\ \hat{a}_0^{\text{right}} \end{pmatrix} = S_L \begin{pmatrix} \hat{a}_L^{\text{in}} \\ \hat{a}_0^{\text{left}} \end{pmatrix} + Q_L \hat{b}_L, \quad (10)$$

which connects the ingoing and outgoing modes to bosonic operators \hat{b}_L and \hat{b}_R representing quantum fluctuations in the left and right part of the medium. These operators can be defined microscopically in the framework of system-and-bath approaches, or phenomenologically as operator-valued Langevin forces [7,8]. The properties of these operators can be determined from the condition that both \hat{a}^{in} and \hat{a}^{out} have to satisfy standard canonical commutation relations. This dictates that the coupling matrix Q_L satisfies the fluctuation-dissipation theorem $Q_L Q_L^\dagger = 1 - S_L S_L^\dagger$ [9]. In the right half of the system, where the medium is amplifying, we have

$$\begin{pmatrix} \hat{a}_0^{\text{left}} \\ \hat{a}_R^{\text{out}} \end{pmatrix} = S_R \begin{pmatrix} \hat{a}_0^{\text{right}} \\ \hat{a}_R^{\text{in}} \end{pmatrix} + Q_R \hat{b}_R^\dagger, \quad (11)$$

where the commutation relations now dictate coupling to creation operators, with $Q_R Q_R^\dagger = S_R S_R^\dagger - 1$. By assumption, the operators \hat{b}_L^\dagger and \hat{b}_R commute with \hat{a}^{in} ; however, according to Eqs. (10) and (11) they do not commute with \hat{a}^{out} , which is a manifestation of broken microreversibility in quantum optics.

We can now describe the full \mathcal{PT} -symmetric system by algebraically eliminating the interface operators \hat{a}_0^{left} and \hat{a}_0^{right} . In the absence of incoming radiation, the intensity emitted to the left and right is then given by $I_L(E) = \frac{1}{2\pi} \times \langle \hat{a}_L^{\text{out}\dagger} \hat{a}_L^{\text{out}} \rangle$, $I_R(E) = \frac{1}{2\pi} \langle \hat{a}_R^{\text{out}\dagger} \hat{a}_R^{\text{out}} \rangle$, which can be evaluated assuming $\langle \hat{b}_L^\dagger \hat{b}_L \rangle = 0$ (ground-state population in the absorbing regions), $\langle \hat{b}_R^\dagger \hat{b}_R \rangle = 0$ (total population inversion in the amplifying regions; these conditions minimize the quantum noise).

Let us first consider the case of a single-mode resonator ($N = 1$) with purely ballistic internal dynamics and absorption in the left region [16], with scattering matrices

$$S_1 = \begin{pmatrix} 0 & t_1 \\ t_1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1/t_1^* \\ 1/t_1^* & 0 \end{pmatrix}, \quad (12)$$

where $|t_1| < 1$. Including the mirrors, the total scattering matrix is

$$S = \begin{pmatrix} \frac{\sqrt{1-\Gamma}(t_1^{*2}-t_1^2)}{t_1^2(1-\Gamma)-t_1^{*2}} & \frac{|t_1|^2\Gamma}{t_1^2(1-\Gamma)-t_1^{*2}} \\ \frac{|t_1|^2\Gamma}{t_1^2(1-\Gamma)-t_1^{*2}} & \frac{\sqrt{1-\Gamma}(t_1^{*2}-t_1^2)}{t_1^2(1-\Gamma)-t_1^{*2}} \end{pmatrix}, \quad (13)$$

and the quantization condition (9) for the closed resonator takes the form $\text{Im}r_1^2 = 0$. Following the quantum-optical procedure described above we find that this resonator emits radiation of intensity

$$I_L(E) = \frac{\Gamma(|t_1|^{-2} - 1)(1 - \Gamma + |t_1|^2)}{2\pi|(t_1/t_1^*)^2 - 1 + \Gamma|^2}, \quad (14)$$

$$I_R(E) = \frac{\Gamma(1 - |t_1|^2)(1 - \Gamma + |t_1|^{-2})}{2\pi|(t_1/t_1^*)^2 - 1 + \Gamma|^2}. \quad (15)$$

Since $|t_1| < 1$ this gives $I_R > I_L$, the difference being

$$\Delta I(E) = I_R(E) - I_L(E) = \frac{\Gamma^2(|t_1|^{-1} - |t_1|)^2}{2\pi|(t_1/t_1^*)^2 - 1 + \Gamma|^2}. \quad (16)$$

Therefore, the emission from the right exit, close to the amplifying region of the medium, is larger than the emission from the left exit, close to the absorbing region of the medium. The overall output intensity to both sides can be written as

$$I(E) = I_L(E) + I_R(E) = \frac{\Gamma(2 - \Gamma)(|t_1|^{-2} - |t_1|^2)}{2\pi|(t_1/t_1^*)^2 - 1 + \Gamma|^2}. \quad (17)$$

Close to quantization in the closed system, [$\Gamma \ll 1$, $E \approx E_0$, where E_0 fulfills the quantization condition $\text{Im}t_1^2(E_0) = 0$], the emission pattern becomes symmetric (as a consequence of \mathcal{PT} invariance) and approaches a Lorentzian of the form

$$I_L(E) = I_R(E) = \frac{\Gamma(|t_0|^{-2} - |t_0|^2)}{2\pi|2i\tau(E - E_0) + \Gamma|^2}. \quad (18)$$

Here $t_0 = t_1(E_0)$, while $\tau = 2 \text{Im}t_1^{-1} dt_1/dE|_{E=E_0} \approx 2L/c$ (with L the length of each region and c the speed of light) is the transmission delay time of propagation between the two mirrors. The full width at half maximum is given by $\Delta E = \Gamma/\tau$. While this width shrinks to zero as the system is closed off, remarkably the total intensity

$$I_{\text{tot}} = \int I(E) dE = \frac{|t_0|^{-2} - |t_0|^2}{2\tau} \quad (19)$$

remains finite, and can be interpreted as a direct measure of the degree of non-Hermiticity of the system (for ballistic transport, Hermiticity implies $|t_0| = 1$.)

In the case of a single-mode resonator with backscattering (where r_1 and r_1' are finite), compact expressions can still be obtained as long as the leakage remains small ($\Gamma \ll 1$), implying according to Eq. (7) that $|r_L + 1|$, $|t_L|$, $|t_L'| \ll 1$. The emission pattern is then still symmetric, with intensity

$$I_L(E) = I_R(E) = \frac{1}{2\pi} \frac{(1 - |r_L'|^2)|t_L'|^2}{|2(\text{Im}r_L') - it_L t_L'|^2}. \quad (20)$$

Linearization around the quantization condition again reveals a Lorentzian line shape, with line width $\Delta E = \text{Re}\{d[(\text{Im}r_L')/t_L t_L'] / dE\}^{-1}$. Accounting for the scaling (7) of scattering coefficients with Γ , the total intensity $I_{\text{tot}} \propto (1 - |r_L'|^2)$ again remains finite as $\Gamma \rightarrow 0$. In the Hermitian case, this limit would imply $|r_L'| = 1$, so that the intensity vanishes. Therefore, the emitted radiation is still a direct measure of the degree of non-Hermiticity.

Following the general formalism described above, the observations for one-dimensional scattering can be ex-

tended to the general case of \mathcal{PT} -symmetric systems with many modes, for which compact expressions are no longer available. The emitted intensity generally remains finite even in the limit of a closed system. Because the expectation values $\langle \hat{a}_0^{\text{left}\dagger} \hat{a}_0^{\text{left}} \rangle \propto \Gamma^{-1}$, $\langle \hat{a}_0^{\text{right}\dagger} \hat{a}_0^{\text{right}} \rangle \propto \Gamma^{-1}$ of the internal operators formally diverge in this limit, this is accompanied by a diverging internal energy density, which can be interpreted as the source of this radiation. Deviations from perfect \mathcal{PT} symmetry can be incorporated into the scattering formalism by taking $S_R \neq \mathcal{PT} S_L$, which results in an asymmetric emission pattern even in the limit of nonleaky mirrors.

In standard laser theory [17], an active optical medium is below threshold (and stable) as long as all fundamental modes are damped, which is characterized by complex energies with a negative imaginary part. With increasing pumping, the loss of the modes within the amplification window is counteracted by gain, and as soon as one of the energies acquires a positive imaginary part the system becomes unstable: the intensity increases exponentially until saturation sets in. Therefore, the radiation features described above are characteristic of a laser stabilized at threshold, the threshold condition being that the energy of the lasing mode is real. The system then is marginally stable, and the internal radiation energy accumulates due to the quantum fluctuations. In practice, this leads to saturation in the amplifying parts of the system and therefore identifies an obstacle for the implementation of strict \mathcal{PT} symmetry in closed optical systems.

Concluding remarks.—In summary, the present Letter demonstrates that accounting for quantum noise, optical realizations of non-Hermitian \mathcal{PT} -symmetric systems emit radiation of an intensity that provides a direct measure of non-Hermiticity. This has both practical as well as fundamental implications. The fundamental consequences arise because non-Hermitian \mathcal{PT} -symmetric systems with an entirely real spectrum define a consistent unitary theory of quantum mechanics [2,3], formalized by the concept of quasi-Hermiticity, which introduces a new scalar product based on a generalized conjugation operation \mathcal{C} that satisfies $\mathcal{C}^2 = 1$, $[\mathcal{C}, H] = [\mathcal{C}, \mathcal{PT}] = 0$. The self-sustained radiation identified here shows that accounting for quantum noise, non-Hermitian \mathcal{PT} -symmetric systems are physically distinct from ordinary Hermitian quantum systems: the canonical commutation relations for the input and output operators are only invariant under unitary transformations, which constraints the possibility to introduce alternative scalar products. This enforces classical distinctions based on the transparency of such systems [12]. From a practical perspective, the self-sustained radiation can be used as an indicator of successfully implemented non-Hermitian \mathcal{PT} symmetry in leaky systems, while the accompanying marginal instability and diverging internal

energy density signifies a practical obstacle for its implementation in the limit of no leakage. Furthermore, these findings identify a hitherto unexplored arena to study quantum fluctuations in active systems close to the lasing threshold. Nonoptical realizations of \mathcal{PT} -symmetric systems offer a wide range of connections to quantum field theories [4], and while the results in the present Letter are not directly transferable in detail, they suggest that quantum noise should provide fundamental insight into such systems, as well.

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- [1] C. M. Bender and S. Boettcher, *Phys. Rev. Lett.* **80**, 5243 (1998).
 - [2] C. M. Bender, D. C. Brody, and H. F. Jones, *Phys. Rev. Lett.* **89**, 270401 (2002); *Am. J. Phys.* **71**, 1095 (2003).
 - [3] F. G. Scholtz, H. B. Geyer, and F. J. W. Hahne, *Ann. Phys. (N.Y.)* **213**, 74 (1992); A. Mostafazadeh, *J. Math. Phys. (N.Y.)* **43**, 205 (2002); **43**, 2814 (2002); **43**, 3944 (2002); *J. Phys. A* **36**, 7081 (2003); arXiv:0810.5643.
 - [4] C. M. Bender, *Rep. Prog. Phys.* **70**, 947 (2007).
 - [5] C. E. Rüter *et al.*, *Nature Phys.* **6**, 192 (2010); Z. H. Musslimani, K. G. Makris, R. El-Ganainy, and D. N. Christodoulides, *Phys. Rev. Lett.* **100**, 030402 (2008); K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, *ibid.* **100**, 103904 (2008); A. Guo *et al.*, *ibid.* **103**, 093902 (2009).
 - [6] R. Loudon, *The Quantum Theory of Light* (Oxford University Press, New York, 2001), 3rd ed.
 - [7] M. J. Collett and C. W. Gardiner, *Phys. Rev. A* **30**, 1386 (1984); C. W. Gardiner and M. J. Collett, *Phys. Rev. A* **31**, 3761 (1985).
 - [8] T. Gruner and D.-G. Welsch, *Phys. Rev. A* **54**, 1661 (1996).
 - [9] C. W. J. Beenakker, *Phys. Rev. Lett.* **81**, 1829 (1998).
 - [10] H. Schomerus, K. Frahm, M. Patra, and C. W. J. Beenakker, *Physica A (Amsterdam)* **278**, 469 (2000).
 - [11] For \mathcal{PT} -symmetric scattering theory in one dimension see F. Cannata, J. P. Dedonder, and A. Ventura, *Ann. Phys. (N.Y.)* **322**, 397 (2007).
 - [12] M. V. Berry, *J. Phys. A* **41**, 244007 (2008); H. F. Jones, *Phys. Rev. D* **76**, 125003 (2007); **78**, 065032 (2008).
 - [13] E. Doron and U. Smilansky, *Phys. Rev. Lett.* **68**, 1255 (1992).
 - [14] H. U. Baranger and P. A. Mello, *Phys. Rev. B* **54**, R14297 (1996); V. A. Gopar, S. Rotter, and H. Schomerus, *Phys. Rev. B* **73**, 165308 (2006); M. Kopp, H. Schomerus, and S. Rotter, *Phys. Rev. B* **78**, 075312 (2008); R. S. Whitney, H. Schomerus, and M. Kopp, *Phys. Rev. E* **80**, 056209 (2009); **80**, 056210 (2009).
 - [15] A. Bäcker, in *The Mathematical Aspects of Quantum Maps*, edited by M. D. Esposito and S. Graffi, Lecture Notes in Physics Vol. 618 (Springer, Berlin, 2003), p. 91.
 - [16] M. Znojil, *Phys. Lett. A* **285**, 7 (2001).
 - [17] A. E. Siegmann, *Lasers* (University Science Books, Mill Valley, CA, 1986).