# Deformed cluster maps of type $A_{2 N}$ 

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#### Abstract

We extend recent work of the third author and Kouloukas by constructing deformations of integrable cluster maps corresponding to the Dynkin types $A_{2 N}$, lifting these to higher-dimensional maps possessing the Laurent property and demonstrating integrality of the deformations for $N \leq 3$. This provides the first infinite class of examples (in arbitrarily high rank) of such maps and gives information on the associated discrete integrable systems. Key to our approach is a "local expansion" operation on quivers which allows us to construct and study mutations in type $A_{2 N}$ from those in type $A_{2(N-1)}$.


## 1 Introduction

Cluster algebras are a class of commutative algebras, constructed as subalgebras of rational function fields, which were introduced by Fomin and Zelevinsky in [FZ02]. These algebras are built differently from many other commutative algebras as cluster algebras are not presented with generators and relations from the beginning. Instead, we start with initial data, given by two objects,

- initial cluster variables, $n$ distinguished generators $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)$ and
- an exchange quiver $Q$, a finite directed graph with $n$ nodes which does not contain loops or oriented 2-cycles.

The pair of objects $(\mathbf{x}, Q)$ is called an initial seed. Then we apply a special iterative process called mutation to produce more cluster variables and exchange quivers. Continued application of the mutation process results in constructing the algebra, which is called the associated cluster algebra, as the subalgebra of $\mathbb{Q}\left(x_{1}, \ldots, x_{n}\right)$ generated by all the cluster variables.

Fomin and Zelevinsky further extended the notion of mutation to define the alternative version known as a $Y$-seed pattern. This introduces coefficient variables. Similar to cluster mutation, the coefficient variables have their own dynamics described by the mutation.

[^0]Along with the Laurent phenomenon, which says that every cluster variable can be expressed as a Laurent polynomial in the initial (or indeed any) seed (rather than just being a rational function), the other main result they obtained is the classification of cluster algebras of finite type. A cluster algebra is said to be of finite type if it has only finitely many cluster variables. Such cluster algebras are classified by quivers that are orientations of a Dynkin diagram of a simple Lie algebra.

These results raised the importance of the question of studying the recurrence relation which is given by mutations in Y-seeds. Such a relation is equivalent to the difference equation known as a $Y$-system. Y-systems were discovered by Zamolodchikov in Zam91 and he showed that the solutions of the difference equation are also solutions of Bethe ansatz equation associated with conformal theories of ADE scattering diagrams. Furthermore, it was observed that the solutions appeared to be periodic with a particular period; this was known as Zamolodchikov's periodicity conjecture. In [FZ03], this was investigated via the cluster algebra setting and Fomin and Zelevinsky showed a specific product of mutations exhibited Zamolodchikov periodicity.

This periodicity phenomenon allows the construction of a periodic map, the periodic cluster map introduced by the third author and Fordy in [FH14]. This map is composed of the mutations such that the cluster returns to the initial cluster and the associated quiver $Q$ is mutation periodic [FM11] with respect to the mutations, i.e. $\mu_{i_{r}} \mu_{i_{r-1}} \cdots \mu_{i_{1}} Q=\rho(Q)$, where $\rho$ is a permutation of the nodes. With this notion of a cluster map, one can consider the Liouville-Arnold definition of integrability for maps Mae87] (see further detail in [FH14]) and hence study them as discrete integrable systems.

Recently, the third author and Kouloukas ([HK23]) introduced deformation theory of coefficientfree cluster mutation, which preserves the symplectic form that is compatible with mutation. They showed that the composition of deformed mutations can be regarded as a symplectic map with the condition that it is a cluster map. They presented several examples, including deformed integrable cluster maps associated with Dynkin types $A_{2}, A_{3}$ and $A_{4}$.

In this paper, we consider the deformation of an integrable cluster map corresponding to the general even-dimensional case of Dynkin type $A_{2 N}$.

In Section 2.1, we give some brief background on cluster algebras, mutation periodicity and cluster maps. We follow this in Section 2.2 by recalling the associated definitions to consider these as discrete integrable systems and in Section 2.3, we give a short introduction to the heuristic method known as singularity analysis for difference equations. Section 2.4 introduces the deformation theory of the third author and Kouloukas and Section 2.5 shows how singularity analysis may be used to Laurentify the deformed map, that is, lift this to an undeformed cluster map on a higher-dimensional space.

In Section 3, we begin the analysis of cluster maps associated to type $A_{2 N}$ in general. We give the associated Poisson structure, construct appropriate first integrals and hence show that the periodic cluster map associated to type $A_{2 N}$ is integrable (Theorem 3.3).

In Section 3.2 we examine the base case for our "inductive" approach, namely type $A_{6}$. Using this and a comparison with type $A_{4}$, we see that the Laurentification of the former is obtained from that of the latter by insertion of a particular quiver, in a form of local expansion: these are shown in Figures 1 and 2 (occurring later as Figures 6 and 7 ).

Writing this in terms of exchange matrices, we can construct an associated family of quivers by repeated local expansion. The particular structure of this expansion allows us to show that


Figure 1: Extension from $Q_{A_{4}}$ to $Q_{A_{6}}$

(a) Subquiver in $Q_{A_{4}}$

(b) Subquiver in $Q_{A_{6}}$

Figure 2: Local expansion of the subquiver in $Q_{A_{4}}$
this does indeed give the Laurentification of the type $A_{2 N}$ deformed cluster map (Section 3.4).

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## 2 Preliminaries

### 2.1 Cluster algebras

In this section, we recall the definition of two types of mutation, quiver mutation and cluster mutation, and introduce an example to see the construction of cluster algebras.

Let $Q=(V, E)$ be a quiver with $n$ nodes, $V=\{1,2, \ldots, n\}$ and directed edges $E$. We assume that $Q$ does not possess any loops or oriented 2 -cycles. However, multiple edges between nodes are allowed and when there are many edges between two vertices we write $\xrightarrow{i}$ as a shorthand for $i$ parallel arrows.

Definition 2.1. Let $Q$ be a quiver. Quiver mutation at node $k$, to obtain the new quiver $\mu_{k}(Q)$, is performed by following the steps below:

1. For each full subquiver $i \xrightarrow{p} k \xrightarrow{q} j$, insert a (multiple) edge $i \xrightarrow{p q} j$
2. Reverse all arrows which are connected to $k, j \xrightarrow{q} k \xrightarrow{p} i$
3. Remove any 2-cycles which are formed by inserting arrows.

Example 2.2 (Quiver mutation at node 2).


As we assumed that the quiver has no 2-cycles and no loops, then we can identify a $N \times N$ skew-symmetric matrix which corresponds to the quiver.

Definition 2.3. Let $Q$ be a quiver with $n$ vertices and no 2-cycles and no loops. Then one can encode this quiver in the $n \times n$ skew-symmetric integer matrix $B=B(Q)$ by setting the matrix entry $b_{i j}$ to be the number of arrows $i$ to $j$. Then we refer to this matrix $B$ as an exchange matrix.

As the quiver $Q$ can be represented by the exchange matrix $B=B(Q)$, one can formulate the quiver mutation in terms of entries of $B$, giving $\mu_{k}(B)$. We refer to this formula as matrix mutation.

Definition 2.4. Let $B$ be an exchange matrix and let $B^{\prime}=\mu_{k}(B)$ be the new exchange matrix obtained by applying mutation to the exchange matrix $B$ in direction $k$. The entries of $B^{\prime}, b_{i j}^{\prime}$, are given by

$$
b_{i j}^{\prime}= \begin{cases}b_{i j} & \text { if } i=k \text { or } j=k  \tag{1}\\ b_{i j}+\frac{1}{2}\left(\left|b_{i k}\right| b_{k j}+b_{i k}\left|b_{k j}\right|\right) & \text { otherwise }\end{cases}
$$

Alongside quiver (correspondingly, matrix) mutation, cluster variables transform under cluster mutation.

Definition 2.5. Let $\mathcal{F}$ be the field of rational functions in $n$ independent variables $x_{1}, \ldots, x_{n}$ over $\mathbb{C}$ and set $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right) \in \mathcal{F}^{n}$. The cluster mutation of $\mathbf{x}$ in direction $k$ is $\mu_{k}(\mathbf{x})=$ $\left(x_{1}, \ldots, x_{k-1}, x_{k}^{\prime}, \ldots x_{n}\right)$ where $x_{k}^{\prime}$ is the element defined by the expression,

$$
\mu_{k}\left(x_{i}\right)=x_{i}^{\prime}= \begin{cases}x_{i} & \text { if } i \neq k  \tag{2}\\ \frac{1}{x_{k}}\left(\prod_{\substack{j=1 \\ b_{j k}>0}}^{n} x^{b_{j k}}+\prod_{\substack{j=1 \\ b_{j k}<0}}^{n} x^{-b_{j k}}\right) & \text { if } i=k\end{cases}
$$

This expression is known as a (coefficient free) exchange relation.

Note that we could work over other base fields than $\mathbb{C}$ but we will restrict to this choice in order to employ geometric methods later.

Given an initial seed ( $\mathbf{x}, B$ ) of size $n$, one can apply the mutations in $N$ possible directions. This results in $n$ new seeds. In succession, the mutations can be applied to each such seed in $n$
possible directions again, and so on. It is important to note that consecutive mutations in the same direction do not yield anything new. This is due to the mutation being involutive: $\mu_{i}^{2}=\mathrm{id}$. Then, in this way, we may label the vertices of a rooted $n$-valent tree by clusters. Note that in general mutations on different vertices do not commute (in the sense that $\mu_{i} \circ \mu_{j} \neq \mu_{j} \circ \mu_{i}$ ) unless the vertices are "far apart" (i.e. are at least distance 2 apart in the quiver). Then one can produce the collection of cluster variables which are induced by iterated cluster mutations in all directions; the set of all cluster variables obtained in this way generates the so-called cluster algebra.

Definition 2.6 (Cluster algebra). The cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is a $\mathbb{C}=$ subalgebra of the field $\mathcal{F}$ whose generating set is the set of all cluster variables produced by all possible sequences of mutations applied to the initial seed $(\mathbf{x}, B)$.

If $\mathbf{x}$ is of size $n$ (so $B$ is an $n \times n$ matrix) we say $\mathbf{x}, \mathcal{B}$ ) is of rank $n$.
It is possible to generalize the notion of a cluster algebra by introducing coefficients. In order to define this, let us consider the general class of exchange matrices as follows.

Definition 2.7. An $n \times n$ integer matrix $B$ is called a skew-symmetrizable exchange matrix if there exists an integer diagonal matrix $D$, which satisfies $(D B)^{T}=-D B$. We refer such a matrix $D$ as a skew-symmetrizer.

Note that $D$ is the identity matrix if $B$ is a skew-symmetric matrix. In (2), we emphasized the coefficient-free exchange relation because the notion of cluster algebras can be extended by introducing frozen variables in clusters which do not mutate. We call $\tilde{\mathbf{x}}=$ $\left(x_{1}, x_{2}, \ldots, x_{n}, x_{n+1}, \ldots, x_{n+m}\right)$ an extended cluster, formed by mutable cluster variables $x_{1}, \ldots, x_{n}$ and the frozen variables $x_{n+1}, \ldots, x_{n+m}$. If the $(n+m) \times n$ matrix $\tilde{B}$ has upper $n \times n$ submatrix a skew-symmetrizable matrix, then we call $\tilde{B}$ an extended exchange matrix. Thus we obtain a cluster algebra with initial seed ( $\tilde{\mathbf{x}}, \tilde{B}$ ) generated by cluster variables induced by the sequence of mutations $\mu_{k}(\tilde{\mathbf{x}}, \tilde{B})=\left(\tilde{\mathbf{x}}^{\prime}, \tilde{B}^{\prime}\right)$ where $\tilde{\mathbf{x}}^{\prime}=\left(x_{1}, \ldots, x_{k}^{\prime}, \ldots, x_{n}, x_{n+1}, \ldots, x_{n+m}\right)$ and $\tilde{B}^{\prime}$ are given by the cluster mutation,

$$
\begin{equation*}
x_{k}^{\prime} x_{k}=\alpha_{k} \prod_{\substack{j=1 \\ b_{j k}>0}}^{n} x^{b_{j k}}+\beta_{k} \prod_{\substack{j=1 \\ b_{j k}<0}}^{n} x^{-b_{j k}} \tag{3}
\end{equation*}
$$

where the coefficients are

$$
\begin{equation*}
\alpha_{k}=\prod_{\substack{j=1 \\ b_{j+n, k}>0}}^{n} x^{b_{j+n, k}}, \quad \beta_{k}=\prod_{\substack{j=1 \\ b_{j+n, k}<0}}^{n} x^{-b_{j+n, k}} \tag{4}
\end{equation*}
$$

and the matrix mutation (1). Note that the formulæ for $\tilde{\mathbf{x}}^{\prime}$ and $\tilde{B}^{\prime}$ are the same as previously given except the range of indices is now from 1 to $n+m$. We refer to these more general cluster algebras as cluster algebras of geometric type.

Cluster algebras possess several interesting structural features. One of the most significant features of cluster algebras is that cluster variables, obtained from the sequence of mutations, are expressed as Laurent polynomials in initial cluster variables. This is known as the Laurent phenomenon and is stated as follows.

Theorem 2.8 (Laurent phenomenon). Every cluster variable generated by the cluster mutations is in the Laurent polynomial ring in its initial cluster variables.

This property of cluster mutations is key for our later results. We will also consider another important feature which may be present (but need not always be, unlike the Laurent phenomenon), namely periodicty of (i) quiver/matrix mutations, which was introduced by Fordy and Marsh [FM11], and (ii) cluster mutations, which was shown by Fomin and Zelevensky in [FZ07.

Definition 2.9 (Mutation periodic). Let $Q$ be a quiver with $n$ vertices. Then the quiver is mutation periodic with period $m$ if there exists a sequence of quiver mutations which is equivalent to the cyclic permutation of the labels of the quiver/matrix $Q$ i.e.

$$
\mu_{i_{m}} \mu_{i_{m-1}} \cdots \mu_{i_{2}} \mu_{i_{1}}(Q)=\rho^{m}(Q)
$$

for $n \geq m$, where $\rho=(n, 1,2, \ldots, n-1)$ is the cyclic permutation. We say a skew-symmetric exchange matrix is mutation periodic if its associated quiver is.

Example 2.10 (Type $A_{2}$ ). The quiver associated with type $A_{2}$ is drawn as

$$
1 \longrightarrow 2
$$

The mutation $\mu_{1}$ on the quiver is given by just reversing the arrow. Permuting the labels via the transposition $(1,2)$ will therefore return the quiver to its original state.

In a more general setting, the periodicity of the exchange matrix was defined by Nakanishi Nak11.

In the example above, the quiver satisfies the relation $\mu_{1}(Q)=\rho(Q)$ for $\rho=(1,2)$. Suppose we define the map $\varphi=\rho^{-1} \mu_{1}$; then the map preserves the structure of the quiver. This induces a birational map that we refer to as a cluster map.

Definition 2.11 (Cluster map). Let $(\mathbf{x}, Q)$ be an initial seed with an initial cluster $\mathbf{x}$ and mutation periodic quiver $Q$. Then a birational map $\varphi: \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$ such that

$$
\varphi=\rho \mu_{i_{m}} \mu_{i_{m-1}} \cdots \mu_{i_{2}} \mu_{i_{1}}
$$

for some $i_{j}$ and $\rho$ satisfying $\varphi(\mathrm{x}, Q)=(\varphi(\mathrm{x}), Q)$ is called a cluster map.

Notice that a single mutation is not identified as a single birational map. This is because the exponent of each cluster variable in the exchange relation changes along the mutations. Furthermore, each mutation cannot be specified simply by the integers labelling a sequence of mutations; it is specified by discrete steps in an $n$-valent tree. However periodicity enables the sequence of mutations, that return us to the initial exchange matrix, to be identified by the birational mapping.

### 2.2 Discrete integrable systems from cluster algebras

A discrete dynamical system can be described by the points induced by the finite iteration of a mapping, i.e. a finite degree of freedom system in a discrete mapping. For example, the orbits given by iteration of the cluster map $\varphi: \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$ can be considered as a discrete dynamical system. Therefore the notion of integrability (Mae87] HLK20] [FH14]) in discrete systems can
be applied to the cluster map by introducing a suitable Poisson bracket that is compatible with the cluster algebra.

In this section, we will see how the cluster map can be both a Poisson and a symplectic map, which leads us to the notion of integrability for cluster algebras and maps. We begin with some basic facts regarding the Poisson brackets defined on manifolds.

Let $M$ be a smooth manifold and let $f, g, h \in C^{\infty}(M)$ be smooth functions defined on $M$. A skew-symmetric bilinear map $\{\cdot, \cdot\}: C^{\infty}(M) \times C^{\infty}(M) \rightarrow C^{\infty}(M)$ is called a Poisson bracket if it satisfies the properties

1. Leibiniz rule: $\{f g, h\}=f\{g, h\}+\{f, h\} g$
2. Jacobi identity: $\{f,\{g, h\}\}+\{g,\{h, f\}\}+\{h,\{f, g\}\}=0$

In the local coordinates $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)$, the explicit form of the Poisson bracket between smooth functions $f$ and $g$ is written as

$$
\begin{equation*}
\{f, g\}=\sum \mathbf{P}_{i j}(\mathbf{x}) \frac{\partial f}{\partial x_{i}} \frac{\partial g}{\partial x_{j}} \tag{5}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{P}_{i j}(\mathbf{x})=\left\{x_{i}, x_{j}\right\} \tag{6}
\end{equation*}
$$

is known as the Poisson tensor. Furthermore due to the Poisson bracket satisfying the Leibniz rule and taking the form (5), we can use vector fields to express the Poisson bracket as follows:

$$
\begin{equation*}
X_{H}(f)=\{f, H\} \tag{7}
\end{equation*}
$$

Such a vector field is called a Hamiltonian vector field.
Let us consider a cluster algebra of $\operatorname{rank} n, \mathcal{A}(\mathbf{x}, B)$, where the $B$ is an $n \times n$ exchange matrix and $D$ is an $n \times n$ diagonal skew-symmetrizer for $B$, so that $(D B)^{T}=-(D B)$. In the case of cluster algebras, we are interested in Poisson brackets taking a particularly nice form on clusters as

$$
\begin{equation*}
\left\{x_{i}, x_{j}\right\}=P_{i j} x_{i} x_{j} \tag{8}
\end{equation*}
$$

where $P=\left(P_{i j}\right)$ is a $n \times n$ skew-symmetric matrix, which is referred as the associated Poisson matrix. The terminology for a bracket of this form is known as a log-canonical Poisson bracket, as coined by Gekhtman, Shapiro and Vainshtein in [GSV10.

Definition 2.12. For a cluster algebra $\mathcal{A}(\mathbf{x}, B)$, a Poisson bracket $\{\cdot, \cdot\}$ on $\mathcal{F}$ is said to be mutation compatible if the bracket restricted to any cluster is log-canonical.

Imposing the condition of mutation compatibility, one can obtain the following result (further details can be found in [GSV10], [IN11]).

Theorem 2.13. Assume that $B$ is skew-symmetrizable with skew-symmtrizer $D$ and that $B$ is of full rank. Then the Poisson matrix

$$
\begin{equation*}
P=\lambda D B^{-1}, \quad \lambda \in \mathbb{Q} \tag{9}
\end{equation*}
$$

on the initial cluster $\mathbf{x}$ extends to a mutation compatible Possion bracket. In addition to this, the product $P B$ is mutation invariant.

The requirement of full rank for this result is the reason we consider type $A_{2 N}$ here.
The fact that the Poisson tensor $P$ is directly proportional to the inverse of the exchange matrix $B$ enables us to identify the cluster map as a Poisson map, in the following sense.
Definition 2.14 (Poisson map). Let $M$ be an $n$-dimensional smooth manifold equipped with the Poisson bracket $\{\cdot, \cdot\}$ and let $\mathbf{x}=\left(x_{1}, x_{2}, \ldots, x_{2 n}\right)$ be local coordinates of $M$. Then a map $\varphi: M \rightarrow M$ is a Poisson map if it preserves the Poisson bracket, that is,

$$
\begin{equation*}
\varphi^{*}\left\{x_{i}, x_{j}\right\}=\left\{\varphi^{*} x_{i}, \varphi^{*} x_{j}\right\} \tag{10}
\end{equation*}
$$

In the case of cluster algebras, one can show that the cluster mutation $\mu_{k}$ is Poisson if and only if the corresponding Poisson matrix is invariant under $\mu_{k}$. But for a single mutation, this cannot happen. However, due to the periodicity, a cluster map is a Poisson map, which is associated with a log-canonical Poisson bracket.

It is well-known that Poisson structures are closely related to symplectic structures. For instance, a Poisson manifold equipped with a degenerate Poisson tensor can be foliated into symplectic submanifolds and furthermore, all symplectic manifolds are Poisson manifolds. For the cluster algebra case, it was shown in [GSV10] that there exists a symplectic form associated to each cluster in a cluster algebra, which is written in the log-canonical form

$$
\begin{equation*}
\omega=\sum_{i<j} \frac{b_{i j}}{x_{i} x_{j}} \mathrm{~d} x_{i} \wedge \mathrm{~d} x_{j}=\sum_{i<j} b_{i j} \mathrm{~d} \log x_{i} \wedge \mathrm{~d} \log x_{j} \tag{11}
\end{equation*}
$$

which is mutation compatible, i.e. the cluster mutation $\mu_{k}$ of $(\mathbf{x}, B)$ to $\left(\mathrm{x}^{\prime}, B^{\prime}\right)$ yields $\omega^{\prime}=$ $\sum_{i<j} b_{i j}^{\prime} \mathrm{d} \log x_{i}^{\prime} \wedge \mathrm{d} \log x_{j}^{\prime}$. Note that if the exchange matrix is degenerate then the bilinear form (11) is a pre-symplectic form; when it is non-degenerate, we have an honest symplectic form.

Similar to the Poisson case, it was shown in GSV10 that a cluster map preserves the (pre)symplectic form as $\varphi(B)=B]$, so $\varphi^{*} \omega=\omega$, so cluster maps are symplectic maps. Therefore one can apply to them the definition of a Liouville integrable map ([Ves91, Mae87]) as follows.

Definition 2.15 ((Integrable map)). Let $\psi: \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$ be a Poisson map with respect to a bracket $\{$,$\} such that the Poisson tensor has constant rank 2 r$. Then $\psi$ is said to be integrable if

1. there exist $n-2 r$ independent Casimir functions $\mathcal{C}_{k}$, i.e. $\varphi^{*}\left(\mathcal{C}_{k}\right)=\mathcal{C}_{k}$, satisfying $\left\{\mathcal{C}_{k}, f(\mathbf{x})\right\}=$ 0 for all functions $f(\mathbf{x})$ and
2. there exist $r$ independent first integrals $h_{j}, j=1, \ldots, r$ with $\left.\varphi^{*}\left(h_{j}\right)=h_{j}\right)$ such that $\left\{h_{i}, h_{j}\right\}=0$

Then in order to prove our results on integrablility, when the associated Poisson tensor is non-degenerate, we will show that the cluster map is a symplectic map and an integrable map by finding sufficiently many first integrals in involution with respect to the log-canonical Poisson bracket.

### 2.3 Singularity analysis of difference equations

In this section, we will recall a the heuristic approach, called the singularity confinement test, and then consider a particular example to demonstrate the process of the test.

The singularity confinement test was proposed by Grammaticos, Ramani and Papageorgion in GRP91 in order to assess the integrability of a discrete dynamical system. The motivation for such criterion came from the local singularity analysis of the solutions of ODEs, called the Painlevé test, which was used to detect the ordinary differential equations with the Painelevé property (solutions of the ordinary differential equation possessing removable singularities).

In ARS78, Ablowitz, Ramani and Segur made a conjecture that there exists a connection between non-linear integrable PDEs and non-linear ODEs with the Painlevé property such that any ODE which emerged from the reduction of an integrable PDE is of Painlevé type. Then, passing the Painlevé test gives a necessary condition for the integrability of the system. Thus Grammaticos, Ramani and Papageorgion in GRP91 adapt these notions for discrete systems to identify the discrete analogue of Painlevé equations by performing the singularity confinement test. Let us consider the following examples to see the procedure.

Example 2.16 (Lyness recurrence). Let us consider the autonomous difference equation known as the Lyness recurrence,

$$
x_{n+2} x_{n}=a x_{n+1}+b
$$

where $a$ and $b$ are non-zero parameters. It is clear that one of the potential singularities of the equation is $x_{n}=0$. By setting suitable initial data, the iteration of the recurrence reaches the singularity. Let us assume that the step $n_{0}$ iteration gives $x_{n_{0}}=0$ and $x_{n_{0}+1}=u$ (finite regular value). Then the further iterations give

$$
\begin{aligned}
& x_{n_{0}+2}=\frac{a x_{n_{0}+1}+b}{x_{n_{0}}}=\infty \\
& x_{n_{0}+3}=\infty+\frac{b}{u}=\infty \\
& x_{n_{0}+4}=\frac{\infty}{\infty}+\frac{b}{\infty}
\end{aligned}
$$

As one can see $\frac{\infty}{\infty}$ is an ambiguity term (a true singularity, with loss of information). To resolve $i t$, we study the neighbourhood of the singularity by introducing the small quantity $\epsilon$. Thus let $x_{n_{0}}=\epsilon$; then

$$
\begin{aligned}
& x_{n_{0}}=\epsilon \\
& x_{n_{0}+1}=u \\
& x_{n_{0}+2}=(a u+b) \epsilon^{-1} \\
& x_{n_{0}+3}=\frac{a(a u+b)}{u} \epsilon^{-1}+\frac{b}{u} \\
& x_{n_{0}+4}=\frac{a^{2}}{u}+O(\epsilon) \\
& x_{n_{0}+5}=\left(\frac{a^{3}+b u}{a(a u+b)}\right) \epsilon+O\left(\epsilon^{2}\right) \\
& x_{n_{0}+6}=\frac{b u}{a^{2}}+O(\epsilon)
\end{aligned}
$$

Notice that $x_{n_{0}+4}$ is no longer an ambiguity term and now it is well defined. As $\epsilon \rightarrow 0$, the sequence given by the iteration is

$$
\left(\cdots, \epsilon, u, \epsilon^{-1}, \epsilon^{-1}, a^{2} / u, \epsilon, b\right.
$$

$$
(\cdots, 0, R, \infty, \infty, R, 0, R, \cdots)
$$

The sequence above is referred to as the associated singularity pattern. The iteration enters 0 and then it passes through poles $x_{i}=\infty$ and zeroes $x_{j}=0$, after which the next iteration depends on the initial value $u$. We call this singularity is confined.

Note that the difference equation above can be represented by a two-dimensional complex birational map of the form

$$
\psi:\binom{x_{1, n}}{x_{2, n}} \rightarrow\binom{\frac{b+a x_{2, n}}{x_{1, n}}}{x_{1, n}}
$$

In LG04, Lafortune and Goriely defined the singularity confinement property for discrete mappings.
Definition 2.17. For the class of $N$-dimensional birational maps of the form

$$
\psi:\left(\begin{array}{c}
x_{1, n} \\
x_{2, n} \\
\vdots \\
x_{N, n}
\end{array}\right) \rightarrow\left(\begin{array}{c}
x_{1, n+1} \\
x_{2, n+1} \\
\vdots \\
x_{N, n+1}
\end{array}\right)
$$

a singularity of the map is defined as a point $\mathbf{y}=\left(y_{1}, y_{2}, \ldots, y_{N}\right)$ where the right-hand side of the map is undefined. This singularity is said to be confined if there exists a positive integer $M$ such that $\lim _{\mathbf{x} \rightarrow \mathbf{y}} \psi^{M}(\mathbf{x})=\psi_{0}$ exists.

Now we consider an example of particular relevance to us; we will shortly see that it arises from a deformed $A_{2}$ cluster map.

Example 2.18. Consider

$$
\begin{equation*}
\psi:\binom{x_{1}}{x_{2}} \rightarrow\binom{\frac{1+a_{1} x_{2}}{x_{1}}}{\frac{x_{1}+a_{2}\left(1+a_{1} x_{2}\right)}{x_{1} x_{2}}} \tag{12}
\end{equation*}
$$

This birational map has of two singularities $x_{1}=0$ and $x_{2}=0$. We begin singularity analysis with the singularity $x_{1}=0$. Once again we study the singularity by introducing the small quantity $\epsilon \ll 0$. By setting the initial data $x_{1}=u, x_{2}=\frac{1+\epsilon}{a_{1}}$, the next iteration of the map reaches $\left(\epsilon, \frac{\left(1+\epsilon a_{2}\right) a_{1}}{\epsilon u-1}\right)$, where it is at a singularity:

$$
\begin{aligned}
& \binom{-\left(a_{1}-1\right)\left(a_{1}+1\right) \epsilon^{-1}-a_{1}^{2}\left(u+a_{2}\right)}{\frac{a_{2}\left(a_{1}-1\right)\left(a_{1}+1\right)}{a_{1}} \epsilon^{-1}+\frac{u a_{2}+a_{2}^{2}-1}{a_{1}}} \rightarrow\binom{-a_{2}+\epsilon\left(u+a_{2}\right) a_{2}+O\left(\epsilon^{2}\right)}{\epsilon \frac{-\left(a_{2}-1\right)\left(a_{2}+1\right) a_{2}}{a_{2}\left(a_{1}-1\right)\left(a_{1}+1\right)}+O\left(\epsilon^{2}\right)} \\
& \rightarrow\binom{-\frac{1}{a_{2}}+O(\epsilon)}{\frac{-a_{2}^{2}+u\left(a_{1}^{2}-1\right) a_{2}+a_{1}^{2}}{a_{1}\left(a_{2}^{2}-1\right)}+O(\epsilon)}
\end{aligned}
$$

As $\epsilon \rightarrow 0$, the sequence becomes

$$
\begin{equation*}
\binom{\infty}{\infty} \rightarrow\binom{-a_{2}}{0^{1}} \rightarrow\binom{-\frac{1}{a_{2}}}{\frac{-a_{2}^{2}+u\left(a_{1}^{2}-1\right) a_{2}+a_{1}^{2}}{a_{1}\left(a_{2}^{2}-1\right)}} \tag{13}
\end{equation*}
$$

Therefore there exists a limit $\lim _{\mathbf{x} \rightarrow \mathbf{y}} \psi^{M}(\mathbf{x}) \neq 0$, which is non-vanishing. Therefore the singularity is confined.

For the case of the singularity $x_{2}=0$, we take the same procedure with the initial iterates $x_{1}=a_{2}\left(1+a_{1} u\right) /(\epsilon u-1)$ and then $\epsilon \rightarrow 0$ give us the following sequence of iterations,

$$
\begin{equation*}
\binom{-a_{2}}{\infty} \rightarrow\binom{\infty}{\left(a_{1}-a_{1} a_{2}^{2}\right) /\left(a_{2}^{2}-1\right)} \rightarrow\binom{0}{-1 / a_{1}} \rightarrow\binom{R}{R} \tag{14}
\end{equation*}
$$

where $R$ is some non-zero regular value. Therefore both singularities are confined.

Example 2.18 showed that this particular map possesses the confinement property. In addition to this, one can show that the map (12) is Liouville integrable, as there exists a first integral that is invariant under the map, given by

$$
\begin{equation*}
x_{1}+\frac{1}{x_{1}}+\frac{a_{1}^{2}}{x_{1}}+\frac{a_{1}}{a_{1} x_{2}}+\frac{a_{1} a_{2}}{x_{2}}+\frac{a_{1}}{x_{1} x_{2}}+\frac{a_{1} x_{1}}{x_{2}}+\frac{a_{1} x_{2}}{a_{2}}+\frac{a_{1} x_{2}}{x_{1}} \tag{15}
\end{equation*}
$$

From these examples, it may appear that the confinement property leads to finding an integrable map, but in fact, this is not the correct statement. This was shown by Hietarinta and Viallet in [HV98], who provided a counter-example, passing the confinement property test but nonintegrable. Therefore the confinement property is a necessary condition for integrability but not sufficient.

There is an alternative procedure to find a singularity confinement pattern by considering birational maps over the finite field that is reduction modulo prime $p$ of $\mathbb{Q}$. In [KMTT12], the authors introduced an arithmetic version of the singularity confinement test such that the iteration of the map $x_{n+1}=\varphi\left(x_{n}\right)$ is indeterminate if the $p$-adic norm yields $\left\|x_{n+1}\right\|_{p}>1$. In other words, the $\epsilon^{-1}$ in the singularity confinement test correlates to the prime $p$ if $\left\|x_{n+1}\right\|_{p}>1$. Let us again consider the map $\psi$ in example 2.18 to see the process.

Example 2.19. Setting initial clusters $\left(x_{1}, x_{2}\right)=(1,1)$ and the paramters to be $a_{1}=2$ and $a_{2}=3$, we consider the orbit given by iteration of the map $\psi$ as shown in the table below,

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $x_{1, n}$ | 1 | 3 | 7 | $\frac{3^{3}}{5 \cdot 7}$ | $\frac{5 \cdot 103}{3^{2} \cdot 11}$ | $\frac{3 \cdot 11 \cdot 401}{29 \cdot 103}$ | $\frac{11 \cdot 29 \cdot 419}{3 \cdot 137 \cdot 401}$ |
| $x_{2, n}$ | 1 | $2 \cdot 5$ | $\frac{11}{5}$ | $\frac{2^{2} \cdot 29}{7 \cdot 11}$ | $\frac{7 \cdot 137}{3 \cdot 29}$ | $\frac{2 \cdot 3 \cdot 3049}{103 \cdot 137}$ | $\frac{17 \cdot 43^{2} \cdot 103}{401 \cdot 3049}$ |

where each iteration is factored into primes.
We can see that for $p=7,103,401$ etc., the $p$-adic norm for $x_{1, n}$ and $x_{2, n}$ exhibits the patterns

$$
\begin{align*}
& \left|x_{1, n}\right|_{p}: 1, p^{-1}, p, 1,1 \\
& \left|x_{2, n}\right|_{p}: 1,1, p, p^{-1}, 1 \tag{16}
\end{align*}
$$

As for the prime $p=5,11,29,137$ etc., the pattern is

$$
\begin{align*}
& \left|x_{1, n}\right|_{p}: 1,1,1, p, p^{-1}, 1 \\
& \left|x_{2, n}\right|_{p}: 1, p^{-1}, p, 1,1,1 \tag{17}
\end{align*}
$$

If we take $x_{i, n}$ modulo $p$ then the sequences (16) and (17) are analogous to the following singularity pattern:

$$
\begin{gather*}
\binom{\epsilon}{R} \rightarrow\binom{\epsilon^{-1}}{\epsilon^{-1}} \rightarrow\binom{R}{\epsilon} \rightarrow\binom{R}{R}  \tag{18a}\\
\binom{R}{\epsilon} \rightarrow\binom{R}{\epsilon^{-1}} \rightarrow\binom{\epsilon^{-1}}{R} \rightarrow\binom{\epsilon}{R} \rightarrow\binom{R}{R} \tag{18b}
\end{gather*}
$$

which is identical to the patterns shown in Example 2.18.

### 2.4 Deformation of cluster mutations

In this section, we will briefly review the deformation theory which preserves the presymplectic form, introduced by the third author and Kouloukas [HK23]. Let us consider the exchange relation (2) and express it in the following form:

$$
x_{j}^{\prime}= \begin{cases}x_{k}^{-1} f_{k}\left(M_{k}^{+}, M_{k}^{-}\right), & \text {for } j=k  \tag{19}\\ x_{j}, & \text { for } j \neq k\end{cases}
$$

where $f_{k}: \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ is a differentiable function and

$$
\begin{equation*}
M_{k}^{+}=\prod_{i=1}^{N} x_{i}^{\left[b_{i k}\right]_{+}}, \quad M_{k}^{-}=\prod_{i=1}^{N} x_{i}^{\left[-b_{i k}\right]+} \tag{20}
\end{equation*}
$$

Note that if $f_{k}\left(M_{k}^{+}, M_{k}^{-}\right)=M_{k}^{+}+M_{k}^{-}$, then the mutation is the ordinary coefficient-free cluster mutation (2). In HK23], the third author and Kouloukas introduced such functions $f$ in order to extend the definition of cluster mutation $\mu_{k}$, while still wishing to maintain the property of preserving the pre-symplectic form $\omega$. The first key lemma and theorem in this setting are the following.

Lemma 2.20 ([HK23]). If $\left(B^{\prime}, \mathbf{x}^{\prime}\right)=\mu_{k}(B, \mathbf{x})$ is defined as in (19), then the symplectic form $\omega$ is preserved, i.e.

$$
\begin{equation*}
\sum_{i<j} \frac{b_{i j}^{\prime}}{x_{i}^{\prime} x_{j}^{\prime}} \mathrm{d} x_{i}^{\prime} \wedge \mathrm{d} x_{j}^{\prime}=\sum_{i<j} \frac{b_{i j}}{x_{i} x_{j}} \mathrm{~d} x_{i} \wedge \mathrm{~d} x_{j} \tag{21}
\end{equation*}
$$

if and only if

$$
\begin{equation*}
f_{k}\left(M_{k}^{+}, M_{k}^{-}\right)=M_{k}^{+} g_{k}\left(\frac{M_{k}^{-}}{M_{k}^{+}}\right) \tag{22}
\end{equation*}
$$

for some differentiable function $g_{k}: \mathcal{F} \rightarrow \mathcal{F}$

Theorem 2.21 ([HK23]). Let $\mu_{i_{l}}, \ldots, \mu_{i_{2}} \mu_{i_{1}}$ be a sequence of generalised mutations of the form (19) such that

$$
\mu_{i_{l}} \cdots \mu_{i_{2}} \mu_{i_{1}}(B, \mathbf{x})=(B, \tilde{\mathbf{x}})
$$

with $f_{i_{j}}$ being of the form (22). Then $\varphi: \mathbf{x} \rightarrow \tilde{\mathbf{x}}$ is such that $\varphi^{*} \omega=\omega$, for $\omega$ log-canonical as per (11).

Thus exchange matrices that are periodic under a particular sequence of mutations (or more generally, are periodic up to a permutation) give rise to parametric cluster maps that preserve the pre-symplectic form (19). In other words, by adjusting the function $f$ in (22), one can generalise the symplectic cluster map. Furthermore, if the map is integrable then one can find a family of deformed integrable maps. Let us consider several examples of the deformed integrable maps.

Example 2.22 (Dynkin type $A_{2}$ ). The type $A_{2}$ quiver, as in Example 2.10, corresponds to the exchange matrix given by

$$
B_{A_{2}}=\left(\begin{array}{cc}
0 & -1  \tag{23}\\
1 & 0
\end{array}\right)
$$

The matrix is invariant under the action of the sequence of mutations $\mu_{2} \mu_{1}$, i.e. $\mu_{2} \mu_{1}\left(B_{A_{2}}\right)=$ $B_{A_{2}}$. This composition of mutations enables us define a cluster map $\varphi_{A_{2}}=\mu_{2} \mu_{1}$ such that it preserves the symplectic form

$$
\begin{equation*}
\omega_{A_{2}}=\mathrm{d} \log x_{1} \wedge \mathrm{~d} \log x_{2} \tag{24}
\end{equation*}
$$

As the map satisfies the condition of Theorem 2.20, one can define a new symplectic map $\tilde{\varphi}$, which is constructed by the composition of mutations $\tilde{\mu}_{k}$ for $1 \leq k \leq 2$, where $\tilde{\mu}_{k}$ are deformed mutations in the direction $k$,

$$
\begin{equation*}
\tilde{\mu}_{k}\left(x_{k}\right)=x_{k}^{-1} M_{k}^{+} g_{k}\left(\frac{M_{k}^{-}}{M_{k}^{+}}\right) \tag{25}
\end{equation*}
$$

with $M_{k}^{ \pm}$defined in (20).
By setting the function $g_{k}(x)=a_{k}+x$, the mutated variables $\tilde{\mu}_{k}\left(x_{k}\right)$ can be written as

$$
\begin{equation*}
\tilde{\mu}_{k}\left(x_{k}\right)=x_{k}^{-1}\left(M_{k}^{-}+a_{k} M_{k}^{+}\right) \tag{26}
\end{equation*}
$$

The deformed cluster map $\tilde{\varphi}_{A_{2}}$, which preserves the symplectic form $\omega_{A_{2}}$, (24), is then $\psi$ as in Example 2.18. As mentioned there, the map $\psi$ is integrable as one can find the first integral (15), which is invariant under the map. Thus deformed map $\tilde{\varphi}_{A_{2}}$ is integrable.

Example 2.23 (Dynkin type $A_{4}$ ). The quiver with linear orientation of Dynkin type $A_{4}$ can be drawn as

$$
1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4
$$

Let $B_{A_{4}}$ be exchange matrix associated with type $A_{4}$ quiver, which is given by the skew-symmetric matrix

$$
B_{A_{4}}=\left(\begin{array}{cccc}
0 & -1 & 0 & 0  \tag{27}\\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0
\end{array}\right)
$$

This matrix has period 4 with respect to a sequence of cluster mutations:

$$
\mu_{4} \mu_{3} \mu_{2} \mu_{1}\left(B_{A_{4}}\right)=B_{A_{4}}
$$

We define the cluster map $\varphi_{A_{4}}$ with the composition of mutations above, $\varphi_{A_{4}}=\mu_{4} \mu_{3} \mu_{2} \mu_{1}$. Once again, we fix the function $f_{k}$ in the cluster mutation as (26) and then apply the scaling action on the cluster, i.e. $x_{i} \rightarrow c_{i} x_{i}$, to adjust the parameters into $b_{i}=1$ and $a_{i}=1$ for $i \in\{2,3\}$. This yields the following parametric map

$$
\tilde{\varphi}_{A_{4}}:\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \rightarrow\left(x_{1} 1, x_{1} 2, x_{3}^{\prime}, x_{4}^{\prime}\right)
$$

where the mutated variables $x_{i}^{\prime}$ are given by the following relations:

$$
\begin{array}{ll}
\mu_{1}: & x_{1} x_{1}^{\prime}=b_{1}+a_{1} x_{2} \\
\mu_{2}: & x_{2} x_{2}^{\prime}=1+x_{1}^{\prime} x_{3}  \tag{28}\\
\mu_{3}: & x_{3} x_{3}^{\prime}=1+x_{2}^{\prime} x_{4} \\
\mu_{4}: & x_{4} x_{4}^{\prime}=b_{4}+a_{4} x_{3}^{\prime}
\end{array}
$$

One can check that the original cluster map $\varphi_{A_{4}}$ is periodic with period $6\left(\varphi_{A_{4}}^{6}(\mathbf{x})=\mathbf{x}\right)$. In our case, the periodicity takes an important role in constructing first integrals which are invariant under the type $A_{4}$ cluster map. This property allows us to define the symmetric functions,

$$
\begin{equation*}
I_{1}=\sum_{j=0}^{6} L_{j}, \quad I_{2}=\prod_{j=0}^{6} L_{j} \tag{29}
\end{equation*}
$$

where $L_{i}=\left(\varphi^{*}\right)^{i}\left(x_{1}\right)$. The symmetric functions are first integrals associated with $\varphi_{A_{4}}$ as they satisfy $\varphi_{A_{4}}^{*}\left(I_{i}\right)=I_{i}$. In addition to this, they are in involution with respect to the Poisson bracket, i.e.

$$
\begin{equation*}
\left\{x_{i}, x_{j}\right\}=P_{i j} x_{i} x_{j} \tag{30}
\end{equation*}
$$

where

$$
P=\left(\begin{array}{cccc}
0 & 1 & 0 & 1  \tag{31}\\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & -1 & 0
\end{array}\right)
$$

With these properties holding, we see that the cluster map $\varphi_{A_{4}}$ is Liouville integrable. In the same fashion, one can show that the deformed $\operatorname{map} \tilde{\varphi}_{A_{4}}$ possesses integrability under certain conditions. The most natural candidates for the first integrals are (29), as the Poisson structure remains as per (31). However, they are not preserved under $\tilde{\varphi}_{A_{4}}$.

To resolve this problem, we consider expanded first integrals, which are expressed into a sum of monomials and modify them by inserting arbitrary coefficients into each term:

$$
\begin{equation*}
\tilde{I}_{1}=\sum_{i} \alpha_{i} J_{i}, \quad \tilde{I}_{2}=\sum_{j} \beta_{j} K_{j} \tag{32}
\end{equation*}
$$

where $J_{i}$ and $K_{j}$ are monomials arising from the first integrals $I_{1}$ and $I_{2}$ respectively, and $\alpha_{i}$, $\beta_{j}$ are arbitrary coefficients. Then by imposing the condition $\tilde{\varphi}_{A_{4}}^{*}\left(\tilde{I}_{i}\right)=\tilde{I}_{i}$, one can constrain the coefficients $\alpha_{i}, \beta_{j}$ and find the necessary and sufficient conditions for integrability with $\tilde{I}_{i}$
being first integrals. Thus if we fix the parameters $b_{1}=1=b_{4}$, then $\tilde{I}_{1}$ and $\tilde{I}_{2}$ are first integrals,

$$
\begin{align*}
\tilde{I}_{1}= & \frac{1}{x_{1} x_{2} x_{3} x_{4}}\left(a_{1} a_{4} x_{1} x_{2}+a_{1} a_{4}^{2} x_{1} x_{2} x_{3}+a_{1} x_{1} x_{2} x_{3}+a_{1} a_{4} x_{1} x_{2} x_{3}^{2}+a_{1} a_{4} x_{1} x_{4}+a_{1} a_{4} x_{1} x_{2}^{2} x_{4}\right. \\
& +a_{1} a_{4} x_{3} x_{4}+a_{1} a_{4} x_{1}^{2} x_{3} x_{4}+a_{4} x_{2} x_{3} x_{4}+a_{1}^{2} a_{4} x_{2} x_{3} x_{4}+a_{4} x_{1}^{2} x_{2} x_{3} x_{4}+a_{1} a_{4} x_{2}^{2} x_{3} x_{4} \\
& \left.+a_{1} a_{4} x_{1} x_{3}^{2} x_{4}+a_{1} a_{4} x_{1} x_{2} x_{4}^{2}+a_{1} x_{1} x_{2} x_{3} x_{4}^{2}\right) \\
& \tilde{I}_{2}=\frac{\left(a_{1}+x_{2}\right)\left(x_{1}+x_{3}\right)\left(x_{1}+x_{3}\right)\left(a_{4}+x_{3}\right)\left(x_{1} x_{2}+a_{4} x_{1} x_{2} x_{3}+x_{1} x_{4}+x_{3} x_{4}+a_{1} x_{2} x_{3} x_{4}\right)}{x_{1} x_{2}^{2} x_{3}^{2} x_{4}} \tag{33}
\end{align*}
$$

This allows us to conclude that the deformed $\operatorname{map} \tilde{\varphi}_{A_{4}}$ is Liouville integrable.

The examples above show that $\tilde{\varphi}_{A_{2}}$ and $\tilde{\varphi}_{A_{4}}$ are integrable symplectic maps. However, as a result of applying the deformation, the map no longer generates cluster variables, belonging to a Laurent polynomial ring. Thus in general, the deformed map is not a cluster map. To restore the property, we require a process called Laurentification, which will be introduced in the next section.

### 2.5 Laurentification

In this section, we will introduce a specific projectivization that lifts our deformed mapwhich is not given by a cluster algebra structure, hence we do not have Laurent polynomial expressions for the iterates - to a higher dimensional one which is and does. This lifting is called Laurentification and was introduced and studied by the third author and collaborators ([Hon07),HKQ18]). Thus this procedure helps us to resolve the problem that emerges from the deformation theory.

To be a little more concrete, recall that one of the key features of a cluster algebra is the Laurent phenomenon, where every variable induced by cluster mutation can be expressed as a Laurent polynomial in the initial cluster variables. This implies that a cluster map, which is composed of certain mutations, has the Laurent property, in the following sense.

Definition 2.24 (Laurent property). Let $\mathbf{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in \mathcal{F}^{n}$ be an initial cluster and let $\psi: \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$ be an associated cluster map. Then $\psi$ is said to have the Laurent property if for all $n$, the $n$th iterates of $\psi$ are given by Laurent polynomials in the Laurent polynomial ring $\mathbb{C}\left[x_{1}^{ \pm}, x_{2}^{ \pm}, \ldots, x_{n}^{ \pm}\right]$

The deformed map $\tilde{\varphi}_{A_{2}}$ in Example 2.22 is not a cluster map. This is because the iteration of the map, beginning from the initial cluster $\left(x_{1}, x_{2}\right)$ yields the new cluster

$$
\begin{equation*}
\left(\tilde{\varphi}_{A_{2}}\right)^{2}: \mathbf{x} \rightarrow\binom{\frac{a_{1} a_{2}+a_{1} x_{1}+a_{1}^{2} a_{2} x_{2}+x_{1} x_{2}}{x_{2}\left(1+a_{1} x_{2}\right)}}{\frac{x_{1}\left(a_{1} a_{2}^{2}+a_{1} a_{2} x_{1}+x_{2}+a_{1}^{2} a_{2}^{2} x_{2}+a_{2} x_{1} x_{2}+a_{1} x_{2}^{2}\right)}{\left(1+a_{1} x_{2}\right)\left(a_{2}+x_{1}+a_{1} a_{2} x_{2}\right)}} \tag{34}
\end{equation*}
$$

which consists of rational expressions whose denominator is no longer monomial as the parameters prevent the cancellation with the numerator. Thus deformation of the cluster map destroyed the Laurent property of the undeformed counterpart to this map.

To restore the property one must try to lift the map to a higher dimensional space, where the Laurent property is restored.

Definition 2.25 (Laurentification). Let $\varphi: \mathbb{C}^{M} \rightarrow \mathbb{C}^{M}$ be a birational map. A birational map $\psi: \mathbb{C}^{N} \rightarrow \mathbb{C}^{N}$ (for some $N \geq M$ ) is said to be a Laurentification of $\varphi$ if $\psi$ lifts $\varphi$ and $\psi$ has the Laurent property.

Note that there are several methods which fit the description of Laurentification such as recursive factorization, which was introduced by Hamad and Kamp in HvdK16. They showed that certain QRT maps, which do not generate the elements of the Laurent polynomial ring, can be transformed into Somos-4 and Somos-5 recurrence relations with periodic coefficients.

Here we take an approach, that is, defining the rational map $\pi$ as dependent variable transformations whose structures are identified by the singularity confinement patterns induced by a deformed integrable map. This method was applied to several examples in HKQ18, [HK23]. To see the significance of this approach, let us consider the Laurentification of the deformed maps $\tilde{\varphi}_{A_{2}}$ and $\tilde{\varphi}_{A_{4}}$

Example 2.26 (Laurentification of $\tilde{\varphi}_{A_{2}}$ ). The singularity confinement pattern given by $\tilde{\varphi}_{A_{4}}$ (shown in Example 2.18) defines a rational map,

$$
\pi:\left(x_{1}, x_{2}\right) \rightarrow\left(\tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3}\right)
$$

which is equivalent to the dependent variable transformation

$$
\begin{equation*}
x_{1, n}=\frac{\sigma_{n} \tau_{n+1}}{\sigma_{n+1} \tau_{n}}, \quad x_{2, n}=\frac{\sigma_{n+3} \tau_{n-1}}{\sigma_{n+2} \tau_{n}} \tag{35}
\end{equation*}
$$

where $\tau$ and $\sigma$ represent (18a) and (18b) respectively. Substituting (35) gives the deformed map on the space of tau-functions, $\tilde{\psi}=\psi \circ \pi$, which is equivalent to the following system of equations

$$
\begin{align*}
\tau_{n+2} \sigma_{n} & =\sigma_{n+2} \tau_{n}+a_{1} \sigma_{n+3} \tau_{n-1}  \tag{36a}\\
\sigma_{n+4} \tau_{n-1} & =\sigma_{n+2} \tau_{n+1}+a_{2} \sigma_{n+1} \tau_{n+2} \tag{36b}
\end{align*}
$$

If the relations above are to be regarded as exchange relations, then there must be initial data formed by initial clusters and exchange matrix. The initial clusters can be extracted from (36a) and (36b), as follows. Let us denote

$$
\begin{equation*}
\left(\tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3}\right)=\left(\tilde{x}_{1}, \tilde{x}_{1}, \ldots, \tilde{x}_{7}\right) \tag{37}
\end{equation*}
$$

The symplectic form $\omega_{A_{2}}$ on the space of tau-functions can be written as

$$
\begin{equation*}
\tilde{\omega}_{A_{2}}=\pi^{*} \omega_{A_{2}}=\sum_{i<j} \tilde{b}_{i j} \mathrm{~d} \log \tilde{x}_{i} \wedge \mathrm{~d} \log \tilde{x}_{j} \tag{38}
\end{equation*}
$$

where the $\tilde{b}_{i j}$ are entries of a new exchange matrix,

$$
\tilde{B}_{A_{2}}=\left(\begin{array}{ccccccc}
0 & 1 & -1 & -1 & 1 & 0 & 0  \tag{39}\\
-1 & 0 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & 0 & 0 & 0 & -1 & 1 \\
1 & -1 & 0 & 0 & 0 & -1 & 1 \\
-1 & 1 & 0 & 0 & 0 & 1 & -1 \\
0 & -1 & 1 & 1 & -1 & 0 & 0 \\
0 & 1 & -1 & -1 & 1 & 0 & 0
\end{array}\right)
$$

Let us consider the extended initial cluster $\left(\tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3}, a_{1}, a_{2}\right)$ and extended exchange matrix,

$$
\hat{B}_{A_{2}}=\left(\begin{array}{ccccccc}
0 & 1 & -1 & -1 & 1 & 0 & 0  \tag{40}\\
-1 & 0 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & 0 & 0 & 0 & -1 & 1 \\
1 & -1 & 0 & 0 & 0 & -1 & 1 \\
-1 & 1 & 0 & 0 & 0 & 1 & -1 \\
0 & -1 & 1 & 1 & -1 & 0 & 0 \\
0 & 1 & -1 & -1 & 1 & 0 & 0 \\
0 & 1 & 1 & -1 & -1 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 1 & 1
\end{array}\right)
$$

The cluster mutation in direction 4 acting on the initial clusters (37) gives a new cluster:

$$
\begin{equation*}
\mu_{4}:\left(\tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3}, a_{1}, a_{2}\right) \rightarrow\left(\tau_{-1}, \tau_{0}, \tau_{1}, \tau_{2}, \sigma_{1}, \sigma_{2}, \sigma_{3}, a_{1}, a_{2}\right) \tag{41}
\end{equation*}
$$

, where the fourth component of the tuple sees $\sigma_{0}$ replaced by the new variable $\tau_{2}$. Such a mutation induces the following exchange relation

$$
\begin{equation*}
\tau_{2} \sigma_{0}=\tau_{-1} \sigma_{3}+a_{1} \tau_{0} \sigma_{2} \tag{42}
\end{equation*}
$$

Further applying the mutation in the direction 1,

$$
\begin{equation*}
\mu_{1}:\left(\tau_{-1}, \tau_{0}, \tau_{1}, \tau_{2}, \sigma_{1}, \sigma_{2}, \sigma_{3}, a_{1}, a_{2}\right) \rightarrow\left(\sigma_{4}, \tau_{0}, \tau_{1}, \tau_{2}, \sigma_{1}, \sigma_{2}, \sigma_{3}, a_{1}, a_{2}\right) \tag{43}
\end{equation*}
$$

gives another form of exchange relation

$$
\begin{equation*}
\sigma_{4} \tau_{-1}=\sigma_{2} \tau_{1}+a_{2} \sigma_{1} \tau_{2} \tag{44}
\end{equation*}
$$

Notice that equations (42) and (43) are (36a) and (36b) with $n=0$ respectively. Applying the mutations in a similar way consecutively onto the initial seed ( $\tilde{\mathbf{x}}, \hat{B}_{A_{2}}$ ), we see that

$$
\begin{equation*}
\mu_{37} \mu_{26} \mu_{15} \mu_{74} \mu_{63} \mu_{52} \mu_{41}\left(\tilde{\mathbf{x}}, \hat{B}_{A_{2}}\right)=\left(\tilde{\mathbf{x}}^{\prime}, \hat{B}_{A_{2}}\right), \quad \text { where } \mu_{i j}=\mu_{i} \mu_{j}, \tag{45}
\end{equation*}
$$

generates a new set of cluster variables $\tilde{\mathbf{x}}^{\prime}=\left(\tau_{6}, \tau_{7}, \tau_{8}, \sigma_{7}, \sigma_{8}, \sigma_{9}, \sigma_{10}, a_{1}, a_{2}\right)$, which are in the form of (42) and (43), and moreover the exchange matrix $\hat{B}_{A_{2}}$ is invariant under the sequence of mutations. Recall that the exponents of monomials in exchange relations depend on the entries of the exchange matrix. Thus the mutations (45) acting on the new seed ( $\tilde{\mathbf{x}}^{\prime}, \hat{B}_{A_{2}}$ ) gives cluster variables which are expressed by (42) and (43). This implies that by the Laurent phenomenon, variables induced by the iteration of the deformed map $\tilde{\varphi}_{A_{2}}$ belong to the Laurent polynomial ring in the initial tau-variables.

Example 2.27 (Laurentification of $\tilde{\varphi}_{A_{4}}$ ). In order to define the associated rational map $\pi$, we need to determine the singularity structure for the deformed map $\tilde{\varphi}_{A_{4}}$. Performing the empirical p-adic method, one can find four types of singularity patterns as follows

$$
\begin{aligned}
&(1): \ldots \rightarrow(\epsilon, R, R, R) \rightarrow\left(\epsilon^{-1}, \epsilon^{-1}, \epsilon^{-1}, \epsilon^{-1}\right) \rightarrow(R, R, R, \epsilon) \rightarrow \ldots \\
&(2): \ldots \rightarrow(R, R, R, \epsilon) \rightarrow\left(R, R, R, \epsilon^{-1}\right) \rightarrow\left(R, R, \epsilon^{-1}, R\right) \rightarrow\left(R, \epsilon^{-1}, R, R\right) \\
& \rightarrow\left(\epsilon^{-1}, R, R, R\right) \rightarrow(\epsilon, R, R, R) \rightarrow \ldots \\
&(3): \ldots \rightarrow(R, \epsilon, R, R) \rightarrow \ldots \\
&(4): \ldots \rightarrow(R, R, \epsilon, R) \rightarrow \ldots
\end{aligned}
$$

where the $\epsilon$ in the pattern (3) and (4) corresponds to the primes which can be seen only in $x_{3, n}$ and $x_{4, n}$ respectively. Then we define the rational map $\pi$,

$$
\pi: \mathbf{x}_{0}=\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \rightarrow \tilde{\mathbf{x}}_{0}=\left(q_{0}, \tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4}, \sigma_{5}, p_{0}\right)
$$

which is equivalent to the following dependent variable transformation:

$$
\begin{equation*}
x_{1, n}=\frac{\sigma_{n} \tau_{n+1}}{\sigma_{n+1} \tau_{n}} \quad x_{2, n}=\frac{p_{n}}{\sigma_{n+2} \tau_{n}} \quad x_{3, n}=\frac{q_{n}}{\sigma_{n+3} \tau_{n}} \quad x_{4, n}=\frac{\sigma_{n+5} \tau_{n-1}}{\sigma_{n+4} \tau_{n}} \tag{46}
\end{equation*}
$$

where $\sigma_{n}, \tau_{n}, p_{n}, q_{n}$ correspond to the singularities in (1), (2), (3), (4) respectively. We define $a$ new initial seed $\left(\hat{\mathbf{x}}_{0}, \hat{B}_{A_{4}}\right)$, where $\hat{\mathbf{x}}_{0}$ is the extended cluster obtained by adding frozen variables $a_{1}$ and $a_{4}$,

$$
\hat{\mathbf{x}}_{0}=\left(q_{0}, \tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4}, \sigma_{5}, p_{0}, a_{1}, a_{4}\right)
$$

and $\hat{B}_{A_{4}}$ is the deformed exchange matrix, which is depicted by the quiver in Figure 3. Then


Figure 3: Type $A_{4}$ deformed quiver
the deformed map $\tilde{\varphi}_{A_{4}}$ is Laurentified to the cluster map

$$
\psi_{A_{4}}=\tilde{\varphi}_{A_{4}} \pi=\hat{\rho}_{A_{4}}^{-1} \mu_{2} \mu_{1} \mu_{11} \mu_{5}, \quad \text { for } \hat{\rho}_{A_{4}}=(2,3,4,5,6,7,8,9,10)
$$

on ( $\hat{\mathbf{x}}_{0}, \hat{B}_{A_{4}}$ ), which generates the cluster variables expressed by the following recurrence relations:

$$
\begin{align*}
\tau_{n+2} \sigma_{n} & =\sigma_{n+2} \tau_{n}+a_{1} p_{n} \\
p_{n+1} p_{n} & =\sigma_{n+3} \sigma_{n+2} \tau_{n} \tau_{n+1}+q_{n} \sigma_{n+1} \tau_{n+2}  \tag{47}\\
q_{n+1} q_{n} & =\sigma_{n+4} \sigma_{n+3} \tau_{n} \tau_{n+1}+p_{n+1} \sigma_{n+5} \tau_{n-1} \\
\sigma_{n+6} \tau_{n-1} & =\sigma_{n+4} \tau_{n+1}+a_{1} q_{n+1}
\end{align*}
$$

This example follows the working in [HK23].

## 3 The cluster map of type $A_{2 N}$

The examples above showed that the singularity confinement patterns of type $A_{2}$ and $A_{4}$ deformed integrable maps allow us to define rational maps $\pi$ such that $\psi=\tilde{\varphi} \pi$ is a cluster map on the space of tau functions.

Therefore it is natural to ask whether Laurentification can be successfully applied to the defomed integrable maps associated with general type $A$. We are able to answer this positively for cluster algebras of type $A_{2 N}$.

### 3.1 Initial analysis

The exchange matrix associated to a linear orientation of a Dynkin diagram of type $A_{2 N}$ can be expressed as

$$
B=\left(\begin{array}{ccccccc}
0 & 1 & 0 & 0 & 0 & \cdots & 0  \tag{48}\\
-1 & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & -1 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & 0 & 1 & 0 \\
0 & \cdots & 0 & 0 & -1 & 0 & 1 \\
0 & \cdots & 0 & 0 & 0 & -1 & 0
\end{array}\right)
$$

This exchange matrix is mutation periodic with periodicity $2 N$ under the particular sequence of mutations $\mu_{2 N} \mu_{2 N-1} \cdots \mu_{2} \mu_{1}$. Let $\varphi_{A_{2 N}}$ be the associated birational map; then $\varphi_{A_{2 N}}(B)=B$. On the cluster, $\varphi_{A_{2 N}}: \mathbf{x} \rightarrow \mathbf{x}^{\prime}$ gives the following exchange relations

$$
\begin{align*}
& x_{1}^{\prime} x_{1}=1+x_{2} \\
& x_{2}^{\prime} x_{2}=1+x_{1}^{\prime} x_{3} \\
& x_{3}^{\prime} x_{3}=1+x_{2}^{\prime} x_{4} \\
& \vdots  \tag{49}\\
& x_{2 N-1}^{\prime} x_{2 N-1}=1+x_{2 N-2}^{\prime} x_{2 N} \\
& x_{2 N}^{\prime} x_{2 N}=1+x_{2 N-1}^{\prime}
\end{align*}
$$

The matrix (48) is mutation equivalent to a matrix which represents a bipartite graph. Therefore due to Zamolodchikov periodicity, the map $\varphi$ is periodic with period $2 N+3$, i.e. $\left(\varphi_{A_{2 N}}^{*}\right)^{2 N+3}(\mathbf{x})=$ $\mathbf{x}$. The sequence of cluster variables, which are generated by $\left(\varphi_{A_{2 N}}^{*}\right)^{i}\left(x_{1}\right)=L_{i}$, are given as follows,

$$
\begin{align*}
& L_{0}=x_{1}, \quad L_{1}=\frac{1+x_{2}}{x_{1}}, \quad L_{2}=\frac{x_{1}+x_{3}}{x_{2}}, \cdots, \quad L_{2 N-1}=\frac{x_{2 N-2}+x_{2 N}}{x_{2 N-1}}, \quad L_{2 N}=\frac{1+x_{2 N-1}}{x_{2 N}}, \\
& L_{2 N+1}=x_{2 N}, \\
& L_{2 N+2}=\frac{\left.\prod_{i=1}^{2 N-1} x_{i}+\prod_{i=1}^{2 N-2} x_{i}+\left(\left(\prod_{i=1}^{2 N-3} x_{i}\right)+\left(\prod_{i=1}^{2 N-4} x_{i}+\cdots\left(1+x_{2}\right) x_{3}\right) \cdots\right) x_{2 N-1}\right) x_{2 N}}{\prod_{i=1}^{2 N} x_{i}} \tag{50}
\end{align*}
$$

Let $P$ be the standard Poisson structure for type $A_{2 N}$, given by

$$
P_{i j}=\left(B^{-1}\right)_{i j}=\sum_{l=0}^{N-1} \delta_{i, i+2 N-2 l-j}-\sum_{l=0}^{N-1} \delta_{j+2 N-2 l-i, j}
$$

Then the associated Poisson bracket is

$$
\begin{equation*}
\left\{x_{i}, x_{j}\right\}=P_{i j} x_{i} x_{j} \tag{51}
\end{equation*}
$$

which simplifies to give us the following log-canonical Poisson bracket relations:

$$
\begin{equation*}
\left\{x_{2 r-1}, x_{2 s}\right\}=x_{2 r-1} x_{2 s} \tag{52}
\end{equation*}
$$

Using the above, these relations give us the following Poisson bracket relations on the space of functions $L_{i}$ :

$$
\begin{array}{rlrl}
\left\{L_{0}, L_{1}\right\} & =L_{0} L_{1}-1 \\
\left\{L_{0}, L_{2 j}\right\} & =-L_{0} L_{2 j} & \text { for } 1 \leq j \leq N  \tag{53}\\
\left\{L_{0}, L_{2 j+1}\right\} & =L_{0} L_{2 j+1} \quad & \text { for } 1 \leq j \leq N
\end{array}
$$

To find further relations, we can use one of the properties of the cluster map, namely, preservation of the Poisson bracket. For example, let us consider the Poisson bracket $\left\{L_{0}, L_{1}\right\}$. By the relation above, $L_{0}$ and $L_{1}$ satisfy

$$
\left\{L_{0}, L_{1}\right\} \circ \varphi_{A_{2 N}}=\left\{\varphi_{A_{2 N}}^{*}\left(L_{0}\right), \varphi_{A_{2 N}}^{*}\left(L_{1}\right)\right\}
$$

The left-hand side of the equation can be written as

$$
\left\{L_{0}, L_{1}\right\} \circ \varphi=L_{1} L_{2}-1
$$

Since $\varphi_{A_{2 N}}^{*}$ will shift the index $i$ of $L_{i}$ by 1 , the right hand side is $\left\{L_{1}, L_{2}\right\}$. Thus altogether, we obtain the bracket relation

$$
\left\{L_{1}, L_{2}\right\}=L_{1} L_{2}-1
$$

Arguing in this way, the Poisson brackets between the $L_{i}$ are given by

$$
\begin{aligned}
\left\{L_{i}, L_{i+1}\right\} & =L_{i} L_{i+1}-1 & & \text { for } i \geq 0 \\
\left\{L_{i}, L_{j}\right\} & =(-1)^{i+j+1} L_{i} L_{j} & & \text { for } i+1<j
\end{aligned}
$$

Combining the Poisson relations above, we can represent the Poisson bracket as the sum of the two homogenous terms, $P=\mathbf{P}^{(2)}+\mathbf{P}^{(0)}$. In fact, the two terms give rise individually to Poisson brackets, $\{\cdot, \cdot\}_{2}$ and $\{\cdot, \cdot\}_{0}$ as the Jacobi identities are homogeneous.

Lemma 3.1. For $i=0, \ldots, 2 N+2$, the set of functions $L_{i}$ (50) generate a Poisson subalgebra with the brackets

$$
\begin{align*}
\left\{L_{i}, L_{j}\right\} & =\left\{L_{i}, L_{j}\right\}_{2}+\left\{L_{i}, L_{j}\right\}_{0} \\
& =\mathbf{P}_{i j}^{(2)}+\mathbf{P}_{i j}^{(0)} \tag{54}
\end{align*}
$$

The corresponding Poisson tensors $\mathbf{P}^{(2)}$ and $\mathbf{P}^{(0)}$ are given by

$$
\mathbf{P}_{i k}^{(2)}=C_{i k}^{(2)} L_{i} L_{k}, \quad \mathbf{P}_{i k}^{(0)}=C_{i k}^{(0)}
$$

where the skew-symmetric matrices $C^{(2)}$ and $C^{(0)}$ are Toeplitz matrices with their top rows given by $C_{1 k}^{(2)}=(0,1,-1, \ldots, 1,-1)$ and $C_{1 k}^{(0)}=(0,-1,0, \ldots, 0,1)$ respectively.

Proof. Immediate from the above relations.

Thus the Poisson bracket on the space of functions generated by the $L_{i}$ can be split into two distinguished Poisson brackets. This is the condition for the system to be bi-Hamiltonian, as defined by Magri in Mag78. This property leads to the existence of Poisson-commuting first integrals which can be constructed by the so-called Magri-Lenard scheme, that is, recursion relations

$$
\begin{equation*}
\mathbf{P}^{(0)} \mathrm{d} I_{n}=\mathbf{P}^{(2)} \mathrm{d} I_{n+1} \tag{55}
\end{equation*}
$$

for the first integrals $I_{n}$.

Note that both $C^{(2)}$ and $C^{(0)}$ are both odd-dimensional skew-symmetric matrices, which implies they are singular. This tells us that $\{\cdot, \cdot\}_{0}$ and $\{\cdot, \cdot\}_{2}$ are both degenerate and thus there exist Casimir functions for each Poisson bracket. Starting from the Casimir function $I_{1}$ for $\mathbf{P}^{(0)}$, the relation (55) yields the sequence of relations

$$
\begin{align*}
\mathbf{P}_{0} \boldsymbol{\nabla} I_{1} & =0 \\
\mathbf{P}_{0} \boldsymbol{\nabla} I_{k} & =\mathbf{P}_{2} \boldsymbol{\nabla} I_{k-1} \quad \text { for } 2 \leq k \leq M  \tag{56}\\
\mathbf{P}_{2} \boldsymbol{\nabla} I_{M} & =0
\end{align*}
$$

which ends at the Casimir function $I_{M}$ of $\mathbf{P}_{2}$. From the structure of the Poisson tensors $\mathbf{P}_{0}$ and $\mathbf{P}_{2}$, the Casimir functions $I_{1}$ and $I_{M}$ can be written as

$$
\begin{equation*}
I_{1}=\sum_{j} L_{j}, \quad I_{M}=\prod_{j} L_{j} \tag{57}
\end{equation*}
$$

By substituting these Casimir functions into the relation (56), we can find the other first integrals $I_{k}$ (of degree $2 k+1$ ) associated with the cluster map $\varphi_{A_{2 N}}$.

As mentioned above, by general theory, if the relations (56) hold, then the associated first integrals are in involution with respect to the Poisson brackets $\{\cdot, \cdot\}_{0}$ and $\{\cdot, \cdot\}_{2}$, which we record in following.

Lemma 3.2. Let $I_{i}$ be the first integrals which are obtained from the sequence of relations (56). Then

$$
\begin{equation*}
\left\{I_{i}, I_{j}\right\}_{2}=0=\left\{I_{i}, I_{j}\right\}_{0} \tag{58}
\end{equation*}
$$

for any $i, j$.

Proof. Starting from $\left\{I_{i}, I_{j}\right\}_{0}$ we find that

$$
\left\{I_{i}, I_{j}\right\}_{0}=\left(\boldsymbol{\nabla} I_{i}\right) \mathbf{P}^{(0)}\left(\boldsymbol{\nabla} I_{j}\right)=\left(\boldsymbol{\nabla} I_{i}\right) \mathbf{P}^{(2)}\left(\boldsymbol{\nabla} I_{j-1}\right)=\left\{I_{i}, I_{j-1}\right\}_{2}
$$

Thus if we apply the relations subsequently then

$$
\begin{equation*}
\left\{I_{i}, I_{j}\right\}_{0}=\left\{I_{i}, I_{j-1}\right\}_{2}=\left\{I_{i+1}, I_{j-1}\right\}_{0} \tag{59}
\end{equation*}
$$

By repeating the process, we reach

$$
\left\{I_{i}, I_{j}\right\}_{0}=\left\{I_{k}, I_{k}\right\}_{l}=0 \quad k \in\{1,2, \ldots, M\}
$$

One can show that $\left\{I_{i}, I_{j}\right\}_{2}=0$ by taking the same steps and hence first integrals $I_{m}$ are Poisson-commuting with respect to $\{\cdot, \cdot\}_{0}$ and $\{\cdot, \cdot\}_{2}$.

This immediately yields the integrability of the cluster map $\varphi_{A_{2 N}}$.
Theorem 3.3. The periodic cluster map $\varphi_{A_{2 N}}$ associated with type $A_{2 N}$ is integrable.

It is important to remark that the same method above has been used in [For11]. In this paper, Fordy studied the period 1 cluster map associated with affine type $A_{2 N}^{(1)}$ and found the existence of functions periodic under the map. It turns out that the Poisson structures of those functions take a form which is similar to (54). Thus by the Magri-Lenard scheme, the integrability of the map was shown. Soon after, in [FH14], Fordy and the third author gave explicit formulæ for Poisson-commuting first integrals corresponding to the map:

Theorem 3.4 ([FH14]). In the case of affine type $A_{2 N}^{(1)}$, there exists periodic functions $J_{n+p}=J_{n}$ and Casimir function $\mathcal{K}$ with respect to the Poisson bracket in the form (54)

$$
\left(P^{(2)}+P^{(0)}\right) \boldsymbol{\nabla} \mathcal{K}=0
$$

such that there exist expressions

$$
\begin{equation*}
\mathcal{K}_{n}^{(p+3)}=\mathcal{R}^{(p)} \mathcal{K}^{(p+1)} \quad \text { with } \mathcal{K}^{(2)}=J_{n} \quad \mathcal{K}^{(3)}=J_{n} J_{n+1}-2 \tag{60}
\end{equation*}
$$

where $p=2 N-1$ represents is the period of the function $J_{n}$, so $J_{n}=J_{n+p}$, and

$$
\begin{equation*}
\mathcal{R}^{(p)}=-1+J_{n+p} J_{n+p+1}-J_{n+p} \frac{\partial}{\partial J_{n}}-J_{n+p+1} \frac{\partial}{\partial J_{n+p-1}}+J_{n+p} J_{n+p+1} \frac{\partial^{2}}{\partial J_{n} \partial J_{n+p-1}} \tag{61}
\end{equation*}
$$

Moreover, the Poisson matrices are compatible and therefore one can define a bi-Hamiltonian ladder such that $\mathcal{K}$ can be written in the following form:

$$
\begin{equation*}
\mathcal{K}=\sum_{j=1}(-1)^{j} h_{j} \tag{62}
\end{equation*}
$$

with $h_{j}$ is a homogeneous polynomial of degree $2 j-1$. The $h_{j}$ are first integrals which are in involution with respect to the Poisson bracket.

Now we understand that the (undeformed) maps $\varphi_{A_{2 N}}$ of even type $A$ are integrable cluster maps. Next we consider the deformation of the corresponding cluster map. As our method will be inductive, in the next section, we will consider first in detail how the $A_{6}$ case may be related that of $A_{4}$, which we have seen earlier.

### 3.2 The periodic type $A_{6}$ cluster map

In this section, we consider the deformation of the periodic cluster map of type $A_{6}$. The exchange matrix of (linearly oriented) type $A_{6}$ is given by

$$
B_{A_{6}}=\left(\begin{array}{cccccc}
0 & -1 & 0 & 0 & 0 & 0  \tag{63}\\
1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right)
$$

The corresponding matrix possesses period 6 with respect to a sequence of cluster mutations:

$$
\mu_{6} \mu_{5} \mu_{4} \mu_{3} \mu_{2} \mu_{1}(B)=B
$$

Given the initial cluster $\mathbf{x}_{0}=\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}\right)$, let us denote by $\varphi_{A_{6}}$ the composition of mutations above i.e. $\varphi_{A 6}=\mu_{6} \mu_{5} \mu_{4} \mu_{3} \mu_{2} \mu_{1}$. Once more, with $f_{k}(x)=b_{k} M_{k}^{-}+a_{k} M_{k}^{+}$in (19), we define the modified mutations $\tilde{\mu}_{k}\left(x_{k}\right)=x_{k}^{-1}\left(b_{k} M_{k}^{-}+a_{k} M_{k}^{+}\right)$for $k=1, \ldots, 6$, which yields deformed map $\tilde{\varphi}_{A_{6}}=\tilde{\mu}_{6} \tilde{\mu}_{5} \tilde{\mu}_{4} \tilde{\mu}_{3} \tilde{\mu}_{2} \tilde{\mu}_{1}$ equivalent to the following exchange relations,

$$
\begin{align*}
\mu_{1}: & x_{1} x_{1}^{\prime}=b_{1}+a_{1} x_{2} \\
\mu_{i}: & x_{i} x_{i}^{\prime}=1+x_{i-1}^{\prime} x_{i+1} \quad(2 \leq i \leq 4) \\
\mu_{5}: & x_{5} x_{5}^{\prime}=b_{5}+a_{5} x_{4}^{\prime} x_{6}  \tag{64}\\
\mu_{6}: & x_{6} x_{6}^{\prime}=b_{6}+a_{6} x_{5}^{\prime}
\end{align*}
$$

where the parameters $b_{i}$ and $a_{i}$ for $i=2,3,4$ are re-scaledad to 1 by a scaling action on the cluster variables $x_{i} \rightarrow c_{i} x_{i}$.

For integrability of the deformed map, we know from Section 3.1, that the first integrals of the original cluster map $\varphi_{A_{6}}$ can be found by the relation (56). By solving the equations, these first integrals are given by the following:

$$
\begin{align*}
& I_{1}=\sum_{j=0}^{8} L_{j}, \\
& I_{2}=\prod_{j=0}^{8} L_{j},  \tag{65}\\
& I_{3}=\sum_{j=0}^{8} L_{j} L_{j+1}\left(L_{j+2}+L_{j+4}+L_{j+6}\right)+\sum_{j=0}^{2} L_{j} L_{j+3} L_{j+6}
\end{align*}
$$

where $L_{i}=\left(\varphi^{*}\right)^{i}\left(x_{1}\right)$. Recall, as in the Example 2.23, the deformed first integrals arise from expanded first integrals whose coefficients in each term are determined by imposing the invariance property $\tilde{\varphi}^{*}(K)=K$.

Adopting the same approach, by taking linear combinations of the monomials that appear in $I_{1}, I_{2}, I_{3}$ and using computer algebra to analyse the action of the deformed cluster map, we obtain the following three independent rational functions:

$$
\left.\begin{array}{l}
\tilde{I}_{1}=\frac{1}{a_{5}^{3} a_{6} x_{1} x_{2} x_{3} x_{4} x_{5} x_{6}}\left(\begin{array}{c}
a_{1} a_{5}^{2} a_{6} x_{1} x_{2} x_{3} x_{4}+a_{1} x_{1} x_{2} x_{3} x_{4} x_{5}+a_{1} a_{5}^{4} a_{6}^{2} x_{1} x_{2} x_{3} x_{4} x_{5} \\
+a_{1} a_{5}^{2} a_{6} x_{1} x_{2} x_{3} x_{4} x_{5}^{2}+a_{1} a_{5}^{3} a_{6} x_{1} x_{2} x_{3} x_{6}+a_{1} a_{5}^{3} a_{6} x_{1} x_{2} x_{3} x_{4}^{2} x_{6} \\
+a_{1} a_{5}^{3} a_{6} x_{1} x_{2} x_{5} x_{6}+a_{1} a_{5}^{3} a_{6} x_{1} x_{2} x_{3}^{2} x_{5} x_{6}+a_{1} a_{5}^{3} a_{6} x_{1} x_{4} x_{5} x_{6} \\
+a_{1} a_{5}^{3} a_{6} x_{1} x_{2}^{2} x_{4} x_{5} x_{6}+a_{1} a_{5}^{3} a_{6} x_{3} x_{4} x_{5} x_{6}+a_{1} a_{5}^{3} a_{6} x_{1}^{2} x_{3} x_{4} x_{5} x_{6} \\
+a_{5}^{3} a_{6} x_{2} x_{3} x_{4} x_{5} x_{6}+a_{1}^{2} a_{5}^{3} a_{6} x_{2} x_{3} x_{4} x_{5} x_{6}+a_{5}^{3} a_{6} x_{1}^{2} x_{2} x_{3} x_{4} x_{5} x_{6} \\
+a_{1} a_{5}^{3} a_{6} x_{2}^{2} x_{3} x_{4} x_{5} x_{6}+a_{1} a_{5}^{3} a_{6} x_{1} x_{3}^{2} x_{4} x_{5} x_{6}+a_{1} a_{5}^{3} a_{6} x_{1} x_{2} x_{4}^{2} x_{5} x_{6} \\
+a_{1} a_{5}^{3} a_{6} x_{1} x_{2} x_{3} x_{5}^{2} x_{6}+a_{1} a_{5}^{4} a_{6} x_{1} x_{2} x_{3} x_{4} x_{6}^{2}+a_{1} a_{5}^{2} x_{1} x_{2} x_{3} x_{4} x_{5} x_{6}^{2}
\end{array}\right) \\
\tilde{I}_{2}=\left(a_{1}+x_{2}\right)\left(\frac{x_{1}+x_{3}}{x_{2}}\right)\left(\frac{x_{2}+x_{4}}{x_{3}}\right)\left(\frac{x_{3}+x_{5}}{x_{4}}\right)\left(\frac{x_{4}+a_{5} x_{6}}{x_{5}}\right)\left(\frac{x_{5}+a_{5}^{2} a_{6}}{a_{5}}\right)
\end{array}\right) \quad \begin{aligned}
& \left(\begin{array}{l}
a_{5}^{2} a_{6} x_{1} x_{2} x_{3} x_{4} x_{5}+a_{1} a_{5} x_{2} x_{3} x_{4} x_{5} x_{6}+a_{5} x_{1} x_{2} x_{3} x_{6} \\
+a_{5} x_{1} x_{2} x_{5} x_{6}+a_{5} x_{1} x_{4} x_{5} x_{6}+a_{5} x_{3} x_{4} x_{5} x_{6}+x_{1} x_{2} x_{3} x_{4} \\
a_{5} x_{1} x_{2} x_{3} x_{4} x_{5} x_{6}
\end{array}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
& P=x_{2} x_{3}^{2} x_{6} a_{6}^{3}\left(a_{1}\left(x_{2}+x_{4}\right) x_{3}+x_{1}\left(a_{1}+x_{2}\right) x_{4}+a_{1} x_{2} x_{5}\right) x_{5} x_{4} x_{1} a_{5}^{7}+x_{3}\left(\left(a_{1}\left(x_{2}+x_{4}\right) x_{3}^{2}\right.\right. \\
& \left.+\left(\left(x_{2} x_{1}+a_{1}\left(x_{1}+x_{5}\right)\right) x_{4}+x_{2}^{2} x_{5}\right) x_{3}+x_{1} x_{5}\left(x_{2}+x_{4}\right)\left(a_{1}+x_{2}\right)\right) x_{2} x_{5} x_{4}^{2} x_{1} a_{6} \\
& +x_{6}^{2}\left(\left(x_{5}\left(a_{1}+x_{2}\right)\left(a_{1} x_{2}+1\right) x_{4}+a_{1} x_{2} x_{1}\right)\left(x_{2}+x_{4}\right) x_{3}^{2}+\left(\left(\left(a_{1}^{2}+2\right) x_{2}+2 a_{1}\right) x_{5} x_{1} x_{4}^{2}\right.\right. \\
& \left.+x_{2}\left(a_{1} x_{5}\left(x_{1}+x_{5}\right) x_{2}^{2}+\left(\left(a_{1}^{2}+1\right) x_{5}^{2}+x_{1}^{2}\right) x_{2}+a_{1}\left(x_{1}+x_{5}\right)^{2}\right) x_{4}+2 a_{1} x_{1} x_{2}^{2} x_{5}\right) x_{3} \\
& \left.\left.+\left(\left(a_{1} x_{2}^{2} x_{5}+a_{1} x_{1}+x_{2} x_{1}\right) x_{4}+a_{1} x_{2} x_{5}\right) x_{5} x_{1}\left(x_{2}+x_{4}\right)\right)\right) a_{6}^{2} a_{5}^{6} \\
& +x_{6} a_{6}^{2} x_{4}\left(\left(x_{2}+x_{4}\right)\left(x_{5}\left(a_{1}+x_{2}\right)\left(a_{1} x_{2}+1\right) x_{4}+\left(x_{2} x_{5}^{2}+2 a_{1}\right) x_{2} x_{1}\right) x_{3}^{3}\right. \\
& +\left(\left(a_{1} x_{2}^{2} x_{5}+\left(a_{1}^{2} x_{1}+x_{5} a_{1}^{2}+2 x_{1}+x_{5}\right) x_{2}+\left(2 x_{1}+x_{5}\right) a_{1}\right) x_{5} x_{4}^{2}+x_{2}\left(a_{1} x_{2}^{2} x_{5}+2 x_{2} x_{1}\right.\right. \\
& \left.\left.+a_{1}\left(x_{5}^{3}+2 x_{1}+3 x_{5}\right)\right) x_{1} x_{4}+x_{2}^{2}\left(\left(x_{1} x_{5}+x_{5}^{2}+1\right) x_{2}+a_{1}\left(x_{1} x_{5}+2\right)\right) x_{5} x_{1}\right) x_{3}^{2} \\
& +\left(\left(\left(x_{5} a_{1}^{2}+x_{1}+2 x_{5}\right) x_{2}+a_{1}\left(x_{1}+2 x_{5}\right)\right) x_{4}\right. \\
& \left.\left.+x_{2}\left(\left(x_{1} x_{5}^{2}+x_{1}+x_{5}\right) x_{2}+a_{1} x_{1}\left(x_{5}^{2}+1\right)\right)\right) x_{5} x_{1}\left(x_{2}+x_{4}\right) x_{3}+x_{1}^{2} x_{5}^{2}\left(x_{2}+x_{4}\right)^{2}\left(a_{1}+x_{2}\right)\right) a_{5}^{5} \\
& +x_{3} a_{6}\left(x _ { 2 } \left(a_{1}\left(x_{5}^{2}+1\right)\left(x_{2}+x_{4}\right) x_{3}^{2}+\left(\left(x_{5}^{2}+1\right)\left(x_{2} x_{1}+a_{1}\left(x_{1}+x_{5}\right)\right) x_{4}\right.\right.\right. \\
& \left.\left.+\left(x_{5}^{3}+x_{5}\right) x_{2}^{2}+a_{1}^{2} x_{2} x_{5}^{2}\right) x_{3}+x_{1} x_{5}\left(x_{5}^{2}+1\right)\left(x_{2}+x_{4}\right)\left(a_{1}+x_{2}\right)\right) x_{4}^{2} x_{1} a_{6} \\
& +x_{6}^{2} x_{5}\left(\left(x_{5}\left(a_{1}+x_{2}\right)\left(a_{1} x_{2}+1\right) x_{4}+a_{1} x_{2} x_{1}\right)\left(x_{2}+x_{4}\right) x_{3}^{2}\right. \\
& +\left(x_{1}\left(x_{2}^{2} a_{1}^{2}+\left(a_{1}^{2}+2\right) x_{5} x_{2}+2 a_{1} x_{5}\right) x_{4}^{2}\right. \\
& \left.+x_{2}\left(a_{1} x_{5}\left(x_{1}+x_{5}\right) x_{2}^{2}+\left(\left(a_{1}^{2}+1\right) x_{5}^{2}+x_{1}^{2}\right) x_{2}+a_{1}\left(x_{1}+x_{5}\right)^{2}\right) x_{4}+2 a_{1} x_{1} x_{2}^{2} x_{5}\right) x_{3} \\
& \left.\left.+\left(\left(a_{1} x_{2}^{2} x_{5}+a_{1} x_{1}+x_{2} x_{1}\right) x_{4}+a_{1} x_{2} x_{5}\right) x_{5} x_{1}\left(x_{2}+x_{4}\right)\right)\right) a_{5}^{4}+\left(a_{1} x_{1}\left(x_{2}+x_{4}\right)\left(a_{1} x_{5}+2\right) x_{3}^{2}\right. \\
& +\left(a_{1} x_{1} x_{2}\left(a_{1}+x_{5}\right) x_{4}^{2}+\left(a_{1} x_{5}\left(a_{1}+x_{5}\right) x_{2}+x_{5}^{2}+a_{1}\left(x_{1}^{2}+1\right) x_{5}+2 x_{1}^{2}\right)\left(a_{1}+x_{2}\right) x_{4}\right. \\
& \left.+a_{1} x_{1} x_{2}\left(a_{1} x_{5}^{2}+a_{1}+3 x_{5}\right)\right) x_{3} \\
& \left.+a_{1} x_{1} x_{5}\left(x_{2}+x_{4}\right)\left(x_{2} x_{4}+1\right)\left(a_{1}+x_{5}\right)\right) x_{2} x_{3} x_{6} a_{6} x_{5} x_{4} a_{5}^{3}+x_{2} x_{3}\left(\left(a_{1}\left(x_{2}+x_{4}\right) x_{3}^{2}\right.\right. \\
& \left.+\left(\left(x_{2} x_{1}+a_{1}\left(x_{1}+x_{5}\right)\right) x_{4}+x_{2}\left(x_{2} x_{5}+a_{1}^{2}\left(x_{5}^{2}+1\right)\right)\right) x_{3}+x_{1} x_{5}\left(x_{2}+x_{4}\right)\left(a_{1}+x_{2}\right)\right) a_{6} \\
& \left.+a_{1}^{2} x_{2} x_{3} x_{5} x_{6}^{2}\right) x_{5} x_{4}^{2} x_{1} a_{5}^{2}+a_{1}^{2} x_{2}^{2} x_{4}^{2} x_{5}^{2} x_{3}^{2} x_{1}
\end{aligned}
$$

Then computer-aided calculation enables us to verify the following.
Theorem 3.5. The conditions $b_{1}=1 \underset{\tilde{I}_{5}}{=}$ and $b_{6} a_{5}^{2}=1$ on the parameters are necessary and sufficient conditions for $\tilde{I}_{1}, \tilde{I}_{2}$ and $\tilde{I}_{3}$ to be first integrals that are preserved by the type $A_{6}$ deformed map, i.e. $\tilde{\varphi}_{A_{6}}\left(\tilde{I}_{i}\right)=\tilde{I}_{i}$, and are in involution with respect to the Poisson bracket. Hence $\tilde{\varphi}_{A_{6}}$ is a Liouville integrable map whenever these conditions on the parameters hold.

The deformed map $\tilde{\varphi}_{A_{6}}$ is a Liouville integrable symplectic map, however, it is no longer a cluster map as the generated variables stop being Laurent polynomial after some iterations. Therefore once again, we look to lift the deformed map to a higher dimensional space by Laurentification. Following the same process as in the previous examples, we study the singularity structures of the deformed map $\tilde{\varphi}_{A_{6}}$ by observing the $p$-adic properties of iterates defined over $\mathbb{Q}$. Then we observe the following singularity patterns:

$$
\begin{equation*}
\ldots \rightarrow(\epsilon, R, R, R, R, R) \rightarrow\left(\epsilon^{-1}, \epsilon^{-1}, \epsilon^{-1}, \epsilon^{-1}, \epsilon^{-1}, \epsilon^{-1}\right) \rightarrow(R, R, R, R, R, \epsilon) \tag{1}
\end{equation*}
$$

(2) : $\quad \ldots \rightarrow(R, R, R, R, R, \epsilon) \rightarrow\left(R, R, R, R, R, \epsilon^{-1}\right) \rightarrow\left(R, R, R, R, \epsilon^{-1}, R\right)$
$\rightarrow\left(R, R, R, \epsilon^{-1}, R, R\right) \rightarrow\left(R, R, \epsilon^{-1}, R, R, R\right) \rightarrow\left(R, \epsilon^{-1}, R, R, R, R\right)$
$\rightarrow\left(\epsilon^{-1}, R, R, R, R, R\right) \rightarrow(\epsilon, R, R, R, R, R) \rightarrow \ldots$

$$
\begin{equation*}
\ldots \rightarrow(R, \epsilon, R, R, R, R) \rightarrow \ldots \tag{3}
\end{equation*}
$$

(4) : $\ldots \rightarrow(R, R, \epsilon, R, R, R) \rightarrow \ldots$
(5) : $\ldots \rightarrow(R, R, R, \epsilon, R, R) \rightarrow \ldots$
(6) : $\ldots \rightarrow(R, R, R, R, \epsilon, R) \rightarrow \ldots$

By introducing the tau-functions $\tau_{n}, \sigma_{n}, p_{n}, r_{n}, q_{n}, w_{n}$, one can construct the rational map $\pi_{A_{6}}$ in the same way as (35) in Example 2.22 .

Definition 3.6. Given the conditions $b_{5}=1=b_{1}$ and $b_{6} a_{5}^{2}=1$, the singularity confinement patterns in (67) enable us to define a rational map $\pi_{A_{6}}$, which is identified as the dependent variable transformation

$$
\begin{align*}
x_{1, n}=\frac{\sigma_{n} \tau_{n+1}}{\sigma_{n+1} \tau_{n}} \quad x_{2, n} & =\frac{p_{n}}{\sigma_{n+2} \tau_{n}} \quad x_{3, n}=\frac{r_{n}}{\sigma_{n+3} \tau_{n}} \quad x_{4, n}=\frac{q_{n}}{\sigma_{n+4} \tau_{n}}  \tag{68}\\
x_{5, n} & =\frac{w_{n}}{\sigma_{n+5} \tau_{n}} \quad x_{6, n}=\frac{\sigma_{n+7} \tau_{n-1}}{\sigma_{n+6} \tau_{n}}
\end{align*}
$$

where $\tau$ and $\sigma$ represent patterns (1) and (2) respectively. The variables $p, r, q, w$ correspond to the patterns (3)-(6)

If we substitute these directly into the components (28) of $\tilde{\varphi}_{A_{6}}$ with the conditions $b_{6} a_{5}^{2}=1$, $b_{i}=1=a_{j}$ for $i=1, \ldots, 5$ and $j=2,3,4$ one obtains the following system of equations:

$$
\begin{align*}
\tau_{n+2} \sigma_{n} & =\sigma_{n+2} \tau_{n}+a_{1} p_{n} \\
p_{n+1} p_{n} & =\sigma_{n+3} \sigma_{n+2} \tau_{n} \tau_{n+1}+r_{n} \sigma_{n+1} \tau_{n+2} \\
r_{n+1} r_{n} & =\sigma_{n+4} \sigma_{n+3} \tau_{n} \tau_{n+1}+q_{n} p_{n+1}  \tag{69}\\
q_{n+1} q_{n} & =\sigma_{n+5} \sigma_{n+4} \tau_{n} \tau_{n+1}+w_{n} r_{n+1} \\
w_{n+1} w_{n} & =\sigma_{n+6} \sigma_{n+5} \tau_{n} \tau_{n+1}+a_{5} \sigma_{n+7} q_{n+1} \tau_{n-1} \\
\sigma_{n+8} \tau_{n-1} & =b_{6} \sigma_{n+6} \tau_{n+1}+a_{6} w_{n+1}
\end{align*}
$$

Once again, we begin by presenting the initial data as

$$
\begin{aligned}
& \left(\tilde{x}_{1}, \tilde{x}_{2}, \tilde{x}_{3}, \tilde{x}_{4}, \tilde{x}_{5}, \tilde{x}_{6}, \tilde{x}_{7}, \tilde{x}_{8}, \tilde{x}_{9}, \tilde{x}_{10}, \tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{13}, \tilde{x}_{14}, \tilde{x}_{15}\right) \\
& \quad=\left(q_{0}, w_{0}, \tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4}, \sigma_{5}, \sigma_{6}, \sigma_{7}, p_{0}, r_{0}\right)
\end{aligned}
$$

Then the corresponding exchange matrix can be found by reading off the coefficients of the new pre-symplectic form, $\pi_{A_{6}}^{*} \omega_{A_{6}}$. This matrix is given by

$$
\tilde{B}_{A_{6}}=\left(\begin{array}{ccccccccccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1  \tag{70}\\
. & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\
. & . & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
. & . & . & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
. & . & . & . & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
. & . & . & . & . & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
. & . & . & . & . & . & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
. & . & . & . & . & . & . & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
. & . & . & . & . & . & . & . & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
. & . & . & . & . & . & . & . & . & 0 & 1 & 0 & 0 & 0 & 1 \\
. & . & . & . & . & . & . & . & . & . & 0 & 1 & -1 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 0 & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}\right)
$$

If this case is similar to that of type $A_{2}$ and $A_{4}$, then one would expect to be able to find an extended exchange matrix which contains entries corresponding to frozen variables $a_{1}, a_{5}$
and $a_{6}$. Then it should be invariant under a certain sequence of mutation, generating cluster variables expressed by the relations (69). However, the first few iterations of the recurrence (69) give rise to the variable whose denominator is given by

$$
a_{5}^{2} \sigma_{0} \tau_{-1} p_{0} r_{0} q_{0} w_{0}
$$

which shows that the denominator contains the frozen variable.
In cluster mutation, the frozen variables are only apparent in the numerator of the fraction in a Laurent expression. This indicates with the condition $b_{6} a_{5}^{2}=1$, we cannot generate cluster variables corresponding to the recurrence (69). Thus to achieve our goal, one is required to put further constraints on the parameters. The simplest choice is to fix $b_{6}=1=a_{5}$, which satisfies $b_{6} a_{5}^{2}=1$. By adjusting the parameters, this leads to the following theorem.

Theorem 3.7. The sequence mutations in a cluster algebra defined by (70) with two frozen variables $a_{1}, a_{6}$ generates the sequences of tau functions $\left(\sigma_{n}\right),\left(p_{n}\right),\left(r_{n}\right),\left(w_{n}\right),\left(q_{n}\right),\left(\tau_{n}\right)$ satisfying

$$
\begin{align*}
\tau_{n+2} \sigma_{n} & =\sigma_{n+2} \tau_{n}+a_{1} p_{n} \\
p_{n+1} p_{n} & =\sigma_{n+3} \sigma_{n+2} \tau_{n} \tau_{n+1}+r_{n} \sigma_{n+1} \tau_{n+2} \\
r_{n+1} r_{n} & =\sigma_{n+4} \sigma_{n+3} \tau_{n} \tau_{n+1}+q_{n} p_{n+1}  \tag{71}\\
q_{n+1} q_{n} & =\sigma_{n+5} \sigma_{n+4} \tau_{n} \tau_{n+1}+w_{n} r_{n+1} \\
w_{n+1} w_{n} & =\sigma_{n+6} \sigma_{n+5} \tau_{n} \tau_{n+1}+\sigma_{n+7} q_{n+1} \tau_{n-1} \\
\sigma_{n+8} \tau_{n-1} & =\sigma_{n+6} \tau_{n+1}+a_{6} w_{n+1}
\end{align*}
$$

which are elements of the Laurent polynomial ring

$$
\mathbb{Z}_{>0}\left[a_{1}, a_{6}, \sigma_{0}^{ \pm}, \sigma_{1}^{ \pm}, \sigma_{2}^{ \pm}, \sigma_{3}^{ \pm}, \sigma_{4}^{ \pm}, \sigma_{5}^{ \pm}, \sigma_{6}^{ \pm}, \sigma_{7}^{ \pm}, \tau_{-1}^{ \pm}, \tau_{0}^{ \pm}, \tau_{1}^{ \pm}, p_{0}^{ \pm}, r_{0}^{ \pm}, w_{0}^{ \pm}, q_{0}^{ \pm}\right]
$$

Proof. Let us extend the initial data by inserting the frozen variables:

$$
\begin{aligned}
& \left(\tilde{x}_{1}, \tilde{x}_{2}, \tilde{x}_{3}, \tilde{x}_{4}, \tilde{x}_{5}, \tilde{x}_{6}, \tilde{x}_{7}, \tilde{x}_{8}, \tilde{x}_{9}, \tilde{x}_{10}, \tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{13}, \tilde{x}_{14}, \tilde{x}_{15}, \tilde{x}_{16}, \tilde{x}_{17}\right) \\
& =\left(q_{0}, w_{0}, \tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4}, \sigma_{5}, \sigma_{6}, \sigma_{7}, p_{0}, r_{0}, a_{1}, a_{6}\right)
\end{aligned}
$$

We add two new rows, whose entries correspond to the frozen variables, to the exchange matrix (70) to define the extended exchange matrix

$$
\hat{B}_{A_{6}}=\left[\begin{array}{ccccccccccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1  \tag{72}\\
. & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\
. & . & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
. & . & . & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
. & . & . & . & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
. & . & . & . & . & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
. & . & . & . & . & . & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
. & . & . & . & . & . & . & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
. & . & . & . & . & . & . & . & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
. & . & . & . & . & . & . & . & . & 0 & 1 & 0 & 0 & 0 & 1 \\
. & . & . & . & . & . & . & . & . & . & 0 & 1 & -1 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 0 & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 \\
0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{array}\right]
$$



Figure 4: Quiver corresponding to $\hat{B}_{A_{6}}$
which can be depicted by the quiver in Figure 4.
We then apply the mutation sequence $\mu_{3} \mu_{2} \mu_{1} \mu_{15} \mu_{14} \mu_{6}$ to this quiver. If we arrange the nodes and edges of the mutated quiver as per Figure 5, then one can see that this is identical to the initial quiver except that specific labels are shifted by 1 . Thus the block mutation $\mu_{3} \mu_{2} \mu_{1} \mu_{15} \mu_{14} \mu_{6}$ is equivalent to permuting the labels of the nodes in $Q_{6}$ and hence

$$
\begin{equation*}
\mu_{3} \mu_{2} \mu_{1} \mu_{15} \mu_{14} \mu_{6}\left(Q_{6}\right)=\rho_{6} Q_{6} \tag{73}
\end{equation*}
$$

where $\rho_{6}$ is the permutation

$$
\rho_{6}=\left(\begin{array}{ccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17  \tag{74}\\
1 & 2 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 3 & 14 & 15 & 16 & 17
\end{array}\right)
$$

The labels of nodes $6,14,15,1,2$ and 3 are replaced by $7,14,15,1,2$ and 4 respectively. Thus if we apply the mutations in the same order as previously, once again the structure of the quiver remains the same except the labels of the nodes are shifted. Thus if we take the inverse of the permutation on each side of (73), then one has

$$
\begin{equation*}
\psi_{A_{6}}:=\rho_{6}^{-1} \mu_{3} \mu_{2} \mu_{1} \mu_{15} \mu_{14} \mu_{6}\left(Q_{A_{6}}\right)=Q_{A_{6}} \tag{75}
\end{equation*}
$$

and it is clear that the composition of mutations on the left-hand side is a cluster map. The first iteration of the map gives rise to a new seed, which contains cluster variables that are expressed by (71) with $n=0$ :

$$
\begin{align*}
\psi_{A_{6}} & :\left(q_{0}, w_{0}, \tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4}, \sigma_{5}, \sigma_{6}, \sigma_{7}, p_{0}, r_{0}, a_{1}, a_{6}\right)  \tag{76}\\
& \rightarrow\left(q_{1}, w_{1}, \tau_{0}, \tau_{1}, \tau_{2}, \sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4}, \sigma_{5}, \sigma_{6}, \sigma_{7}, \sigma_{8}, p_{1}, r_{1}, a_{1}, a_{6}\right)
\end{align*}
$$

Notice that the subscript of the variables $q, w, \tau, \sigma$ is shifted by 1 . Therefore successive applying the map $\psi_{A_{6}}$ will induce a series of seeds that consist of the cluster variables

$$
\begin{equation*}
\left(q_{n}, w_{n}, \tau_{n-1}, \tau_{n}, \tau_{n+1}, \sigma_{n}, \sigma_{n+1}, \sigma_{n+2}, \sigma_{n+3}, \sigma_{n+4}, \sigma_{n+5}, \sigma_{n+6}, \sigma_{n+7}, p_{n}, r_{n}, a_{1}, a_{6}\right) \tag{77}
\end{equation*}
$$

for $n \in \mathbb{N}$, satisfying (71). Therefore every tau function can be generated by applying $\psi_{A_{6}}$ repeatedly. Hence $\tau_{n}, \sigma_{n}, p_{n}, w_{n}, q_{n}, r_{n}$ are in the Laurent polynomial ring by the Laurent phenomenon.


Figure 5: Mutated quiver $Q_{A_{6}}^{\prime}=\mu_{3} \mu_{2} \mu_{1} \mu_{15} \mu_{14} \mu_{6}\left(Q_{A_{6}}\right)$. It has the same structure as Figure 4 with permuted labellings.

### 3.3 Local expansion

To investigate generalizing the integrable cluster map to arbitrary even rank, we begin by exploring the relation between deformed quivers/exchange matrices of type $A_{4}$ and type $A_{6}$. In [HK23], the variable transformation (46) was substituted into the pre-symplectic form and then one constructs the new exchange matrix with initial cluster ( $q_{0}, \tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4}, \sigma_{5}, p_{0}, a_{1}, a_{4}$ ), which is associated with type $A_{4}$.

Comparison between $Q_{A_{4}}$ and $Q_{A_{6}}$ indicates that $Q_{A_{6}}$ can be obtained from $Q_{A_{4}}$ by local expansion, as illustrated in Figure 6, that is by removing edges between the four-cycle formed by the nodes $1,7,8$ and 11 in $Q_{A_{4}}$ and including new nodes and edges in the quiver as per Figure 7.


Figure 6: Extension from $Q_{A_{4}}$ to $Q_{A_{6}}$

Recall that each node in the deformed quiver corresponds to a tau function, e.g. for $Q_{A_{4}}$, the sequence of nodes $(1,2,3,4,5,6,7,8,9,10,11,12,13)$ corresponds to the sequence of functions $\left(q_{0}, \tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4}, \sigma_{5}, p_{0}, a_{1}, a_{4}\right)$. The figures show that $Q_{6}$ can be built from $Q_{4}$ by carrying out the local expansion on the four-cycle subquiver with nodes corresponding to the functions $\sigma_{3}, \sigma_{4}, q_{0}$ and $p_{0}$. We relabel $\sigma_{3}, \sigma_{4}$ and $\sigma_{5}$ as $\sigma_{5}, \sigma_{6}, \sigma_{7}$ respectively and denote new variables by $\sigma_{3}, \sigma_{4}$. Then inserting the new four-cycle quiver on the nodes $\sigma_{3}, \sigma_{4}, p_{1}$ and $q_{1}$, the new edges will give us the deformed quiver $Q_{\left(A_{6}\right)}$.

We will show that this pattern continues: one can recursively apply the same local expansion

(a) Subquiver in $Q_{A_{4}}$

(b) Subquiver in $Q_{A_{6}}$

Figure 7: Local expansion of the subquiver in $Q_{A_{4}}$
by a four-cycle quiver to obtain the deformed quiver $Q_{A_{2 N}}$ with nodes corresponding to

$$
\begin{gathered}
\left(p_{1}^{N-2} \ldots, p_{1}^{1}, p_{1}^{0}, \tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3}, \ldots, \sigma_{2 N+1}, p_{2}^{0}, p_{2}^{1}, \ldots, p_{2}^{N-2}, a_{1}, a_{2 N}\right) \\
\quad=(1,2,3, \ldots, 4 N+3,4 N+4,4 N+5)
\end{gathered}
$$

What does this expansion tell us? The local expansion above gives insight into the structure of the tau functions in the $x_{i}$ variables. Let us compare the tau functions in type $A_{4}$ and type $A_{6}$ cases. In the setting $\left(\tilde{x}_{1}, \tilde{x}_{2}, \ldots, \tilde{x}_{11}\right)=\left(q_{0}, \tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \ldots, \sigma_{5}, p_{0}\right)$, the variables $x_{i, n}$, induced by the deformed map associated to type $A_{4}$, are defined as

$$
x_{1, n}=\frac{\tilde{x}_{5} \tilde{x}_{4}}{\tilde{x}_{6} \tilde{x}_{3}}, \quad x_{2, n}=\frac{\tilde{x}_{11}}{\tilde{x}_{7} \tilde{x}_{3}}, \quad x_{3, n}=\frac{\tilde{x}_{1}}{\tilde{x}_{8} \tilde{x}_{3}}, \quad x_{4, n}=\frac{\tilde{x}_{10} \tilde{x}_{2}}{\tilde{x}_{9} \tilde{x}_{3}}
$$

The local expansion above $\left(q_{0}, \tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \ldots, \sigma_{5}, p_{0}\right) \rightarrow\left(q_{1}, q_{0}, \tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \ldots, \sigma_{7}, p_{0}, p_{1}\right)$ is equivalent to shifting the subscript of the variables $\tilde{x}_{i} \rightarrow \tilde{x}_{i+2}$ for $i=1,2, \ldots, 7,8$ and $\tilde{x}_{j} \rightarrow \tilde{x}_{j+3}$ for $i=8,9,10,11$ and imposing the new variables

$$
x_{3}=\frac{\tilde{x}_{15}}{\tilde{x}_{9} \tilde{x}_{4}}, \quad x_{4}=\frac{\tilde{x}_{1}}{\tilde{x}_{10} \tilde{x}_{4}}
$$

Then one obtains the variable transformation in (35) whose tau functions are denoted as $\left(\tilde{x}_{1}, \tilde{x}_{2}, \ldots, \tilde{x}_{15}\right)=\left(q_{1}, q_{0}, \tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \ldots, \sigma_{7}, p_{0}, p_{1}\right)$. The recursive local expansion constructs the following $x_{i}$ variables associated to the type $A_{2 N}$ deformed map,

$$
\begin{align*}
& x_{1}=\frac{\tilde{x}_{N+3} \tilde{x}_{N+2}}{\tilde{x}_{N+4} \tilde{x}_{N+1}}, \quad x_{2}=\frac{\tilde{x}_{3 N+5}}{\tilde{x}_{N+5} \tilde{x}_{N+1}}, \quad x_{3}=\frac{\tilde{x}_{3 N+6}}{\tilde{x}_{N+6} \tilde{x}_{N+1}}, \ldots, \\
& x_{N}=\frac{\tilde{x}_{4 N+3}}{\tilde{x}_{2 N+3} \tilde{x}_{N+1}}, x_{N+1}=\frac{\tilde{x}_{1}}{\tilde{x}_{2 N+4} \tilde{x}_{N+1}}, \quad x_{N+2}=\frac{\tilde{x}_{2}}{\tilde{x}_{2 N+5} \tilde{x}_{N+1}}, \ldots,  \tag{78}\\
& x_{2 N-1}=\frac{\tilde{x}_{N-1}}{\tilde{x}_{3 N+2} \tilde{x}_{N+1}}, \quad x_{2 N}=\frac{\tilde{x}_{3 N+4} \tilde{x}_{N}}{\tilde{x}_{3 N+3} \tilde{x}_{N+1}}
\end{align*}
$$

where
$\left(\tilde{x}_{1}, \tilde{x}_{2}, \ldots, \tilde{x}_{4 N+3}\right)=\left(q_{N-2} \ldots, q_{1}, q_{0}, \tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3}, \ldots, \sigma_{2 N+1}, p_{0}, p_{1}, \ldots, p_{N-2}, a_{1}, a_{2 N}\right)$.

The symplectic form associated to $A_{2 N}$ is defined by

$$
\omega=\sum_{i<j} b_{i j} \mathrm{~d} \log x_{i} \wedge \mathrm{~d} \log x_{j}
$$

where $b_{i j}$ are entries of the exchange matrix $B_{A_{2 N}}$ (48), defined by the following

$$
\left(B_{A_{2 N}}\right)_{i j}= \begin{cases}1 & \text { if } j=i+1 \\ 0 & \text { otherwise }\end{cases}
$$

Now,

$$
\begin{aligned}
\pi^{*} \omega & =\tilde{\omega} \\
& =\mathrm{d} \log \left(\frac{\tilde{x}_{N+3} \tilde{x}_{N+2}}{\tilde{x}_{N+4} \tilde{x}_{N+1}}\right) \wedge \mathrm{d} \log \left(\frac{\tilde{x}_{3 N+5}}{\tilde{x}_{N+5} \tilde{x}_{N+1}}\right)+\mathrm{d} \log \left(\frac{\tilde{x}_{3 N+5}}{\tilde{x}_{N+5} \tilde{x}_{N+1}}\right) \wedge \log \left(\frac{\tilde{x}_{3 N+6}}{\tilde{x}_{N+6} \tilde{x}_{N+1}}\right) \\
& +\sum_{l=6}^{N+2} \mathrm{~d} \log \left(\frac{\tilde{x}_{3 N+l}}{\tilde{x}_{N+l} \tilde{x}_{N+1}}\right) \wedge \mathrm{d} \log \left(\frac{\tilde{x}_{3 N+(l+1)}}{\tilde{x}_{N+(l+1)} \tilde{x}_{N+1}}\right)+\mathrm{d} \log \left(\frac{\tilde{x}_{4 N+3}}{\tilde{x}_{2 N+3} \tilde{x}_{N+1}}\right) \wedge \mathrm{d} \log \left(\frac{\tilde{x}_{1}}{\tilde{x}_{2 N+4} \tilde{x}_{N+1}}\right) \\
& +\sum_{m} \mathrm{~d} \log \left(\frac{\tilde{x}_{m}}{\tilde{x}_{2 N+3+m} \tilde{x}_{N+1}}\right) \wedge \mathrm{d} \log \left(\frac{\tilde{x}_{m+1}}{\tilde{x}_{2 N+3+(m+1)} \tilde{x}_{N+1}}\right) \\
& +\mathrm{d} \log \left(\frac{\tilde{x}_{N-1}}{\tilde{x}_{3 N+2} \tilde{x}_{N+1}}\right) \wedge \mathrm{d} \log \left(\frac{\tilde{x}_{3 N+4} \tilde{x}_{N}}{\tilde{x}_{3 N+3} \tilde{x}_{N+1}}\right)
\end{aligned}
$$

To simplify the calculation, let us define $\alpha_{i}=\mathrm{d} \log \tilde{x}_{i}$ and $f_{j}=\alpha_{3 N+j}-\alpha_{N+j}$ and $g_{k}=$ $\alpha_{k}-\alpha_{k+2 N+3}$. Then the pre-symplectic form can be re-written as

$$
\begin{aligned}
\tilde{\omega} & =\left(\alpha_{N+3}+\alpha_{N+2}-\alpha_{N+4}\right) \wedge\left(f_{5}-\alpha_{N+1}\right)+f_{5} \wedge f_{6}+f_{6} \wedge \alpha_{N+1} \\
& +\left(\sum_{l=6}^{N+2} f_{l} \wedge f_{l+1}-f_{l} \wedge \alpha_{N+1}-\alpha_{N+1} \wedge f_{l+1}\right)+f_{4 N+3} \wedge g_{1}-\alpha_{N+1} \wedge g_{1} \\
& +\left(\sum_{m=1}^{N-2} g_{m} \wedge g_{m+1}-g_{m} \wedge \alpha_{N+1}-\alpha_{N+1} \wedge g_{m+1}\right) \\
& +g_{N-1} \wedge\left(\alpha_{3 N+4}+g_{N}\right)-\alpha_{N+1} \wedge\left(\alpha_{3 N+4}+g_{N}\right)-g_{N-1} \wedge \alpha_{N+1}
\end{aligned}
$$

Combining and cancelling, we obtain

$$
\begin{aligned}
\tilde{\omega} & =\left(\alpha_{N+3}+\alpha_{N+2}-\alpha_{N+4}\right) \wedge\left(f_{5}-\alpha_{N+1}\right) \\
& +\left(\sum_{l=5}^{N+2} f_{l} \wedge f_{l+1}\right)+f_{4 N+3} \wedge g_{1}+\left(\sum_{m=1}^{N-2} g_{m} \wedge g_{m+1}\right) \\
& +g_{N-1} \wedge\left(\alpha_{3 N+4}+g_{N}\right)-\alpha_{N+1} \wedge\left(\alpha_{3 N+4}+g_{N}\right)-g_{N-1} \wedge \alpha_{N+1}
\end{aligned}
$$

Therefore $\tilde{\omega}$ is expressed as

$$
\sum_{r<s} \tilde{b}_{r s} \alpha_{r} \wedge \alpha_{s}
$$

whose coefficients are entries of the $(4 N+3) \times(4 N+3)$ exchange matrix

$$
B_{A_{2 N}}=\left(\begin{array}{c|c}
\mathbf{A}_{2 N} & \mathbf{B}_{2 N}  \tag{79}\\
\hline-\mathbf{B}_{2 N}^{T} & \mathbf{C}_{2 N}
\end{array}\right)
$$

which is composed of four block skew-symmetric matrices: a $(2 N+3) \times(2 N+3)$ matrix $\mathbf{A}_{2 N}$,
a $(2 N+3) \times 2 N$ matrix $\mathbf{B}_{2 N}$ and a $2 N \times 2 N$ matrix $\mathbf{C}_{2 N}$ where


Here, the block matrix has $\left(\mathbf{A}_{N}\right)_{i j}=\left(\mathbf{A}_{4}\right)_{i j}$ for $N-2<i, j<N+6,\left(\mathbf{B}_{N}\right)_{m n}=\left(\mathbf{B}_{4}\right)_{m n}$ for $N<m<N+6$ and $3(N-2)+7<n<3(N-2)+12$ and $\left(\mathbf{C}_{N}\right)_{r s}=\left(\mathbf{C}_{4}\right)_{r s}$ for $3(N-2)+7<r, s<3(N-2)+12$, and
$\mathbf{A}_{4}=\left(\begin{array}{ccccccc}0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0\end{array}\right), \quad \mathbf{B}_{4}=\left(\begin{array}{cccc}0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0\end{array}\right), \quad \mathbf{C}_{4}=\left(\begin{array}{cccc}0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{array}\right)$
Based on the pattern of local expansion from $Q_{A_{4}}$ to $Q_{A_{6}}$, we introduce frozen variables and extend the matrix by extra rows $\mathbf{b}_{1}$ and $\mathbf{b}_{2}$ in which entries are

$$
\begin{align*}
& \left(\mathbf{b}_{1}\right)_{i}=\delta_{i, N+1}+\delta_{i, N+2}-\delta_{i, N+3}-\delta_{i, N+4} \\
& \left(\mathbf{b}_{2}\right)_{i}=-\delta_{i, N}-\delta_{i, N+1}+\delta_{i, 3 N+3}+\delta_{i, 3 N+4} \tag{80}
\end{align*}
$$

so that


By this calculation and formalisation, we have obtained a description of local expansion in terms of exchange matrices, corresponding to the graphical description in terms of quivers.

For the example of $A_{4}$ to $A_{6}$, the following exchange matrix

$$
\tilde{B}_{A_{4}}=\left(\begin{array}{c|c}
\mathbf{A}_{4} & \mathbf{B}_{4}  \tag{82}\\
\hline-\mathbf{B}_{4}^{T} & \mathbf{C}_{4}
\end{array}\right)
$$

represents a quiver in which the edges of a subquiver (four-cycle) have been removed i.e. the beginning and end of off-diagonal entries in each block matrix are set to be zero. The local expansion in Figure 6 is equivalent to inserting new columns and rows in each block matrix i.e.

$$
\begin{aligned}
& \mathbf{A}_{6}=\left(\begin{array}{c|cccc|c}
0 & 1 & 0 & \cdots & 0 & 1 \\
\hline-1 & & & & & 0 \\
0 & & & \mathbf{A}_{4} & & 0 \\
\vdots & & & & & \vdots \\
0 & & & & & 1 \\
\hline-1 & 0 & 0 & \cdots & -1 & 0
\end{array}\right), \quad \mathbf{B}_{6}=\left(\begin{array}{c|ccccc|c}
0 & -1 & 0 & \cdots & 0 & -1 \\
\hline 1 & & & & & 0 \\
0 & & \mathbf{B}_{4} & & & 0 \\
\vdots & & & & & \vdots \\
0 & & & & -1 \\
\hline 1 & 0 & 0 & \cdots & 1 & 0
\end{array}\right) \\
& \mathbf{C}_{6}=\left(\begin{array}{c|cccc|c}
0 & 1 & 0 & \cdots & 0 & 1 \\
\hline-1 & & & & & 0 \\
0 & & & \mathbf{C}_{4} & & 0 \\
\vdots & & & & & \vdots \\
0 & & & & & 1 \\
\hline-1 & 0 & 0 & \cdots & -1 & 0
\end{array}\right)
\end{aligned}
$$

Such extension leads to the exchange matrix (70), now written as

$$
\tilde{B}_{A_{6}}=\left(\begin{array}{c|c}
\mathbf{A}_{6} & \mathbf{B}_{6}  \tag{83}\\
\hline-\mathbf{B}_{6}^{T} & \mathbf{C}_{6}
\end{array}\right)
$$

### 3.4 The type $A_{2 N}$ deformed periodic cluster map

Earlier we proved that the Laurent property of type $A_{6}$ deformed map can be restored by lifting the map into a higher-dimensional cluster map defined on the space of tau functions, which is done by finding the particular sequence of mutations preserving the structure of the quiver up to shifting the labels. In this section, we use a similar procedure to show that there exists a sequence of mutations such that the structure of the candidate deformed quiver $Q_{A_{2 N}}$ (or exchange matrix $\tilde{B}_{A_{2 N}}$, as constructed in the previous section) is preserved and show that the corresponding cluster variables can be produced by a two-parameter family of deformed cluster maps corresponding to type $A_{2 N}$.

Example 3.8 (Deformed quiver $Q_{A_{8}}$ ). Let us consider the initial seed formed by the initial $\operatorname{cluster}\left(\tilde{x}_{1}, \tilde{x}_{2}, \ldots, \tilde{x}_{4 N+3}\right)=\left(q_{2}, q_{1}, q_{0}, \tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \sigma_{1}, \ldots, \sigma_{9}, p_{0}, p_{1}, p_{2}, a_{1}, a_{2 N}\right)$ and exchange
matrix

$$
\left(\begin{array}{cccccccccccccccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1  \tag{84}\\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{array}\right)
$$

where
$\mathbf{A}_{8} / \mathbf{C}_{8}=\left(\begin{array}{cc|cccc|cc}0 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \hline 0 & -1 & & & & & 0 & 0 \\ 0 & 0 & & & \mathbf{A}_{4} / \mathbf{C}_{4} & & 0 & 0 \\ 0 & \vdots & & & & & \vdots & \vdots \\ 0 & 0 & & & & & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & -1 & 0\end{array}\right)$,
$\mathbf{B}_{8}=\left(\begin{array}{cc|cccc|cc}0 & -1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \hline 0 & 1 & & & & & 0 & 0 \\ 0 & 0 & & \mathbf{B}_{4} & & & 0 & 0 \\ 0 & \vdots & & & & \vdots & \vdots \\ 0 & 0 & & & & & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0\end{array}\right)$
$\mathbf{b}_{8}^{(1)}+\mathbf{b}_{8}^{(2)}=\left(\begin{array}{ccccccccccccccccccc}0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0\end{array}\right)$

Reading off the exchange matrix, the corresponding deformed quiver $Q_{A_{8}}$ can be drawn and this is depicted in Figure 8. Recall that the composition of mutations $\mu_{3} \mu_{2} \mu_{1} \mu_{15} \mu_{14} \mu_{6}$ main-


Figure 8: (Candidate) deformed quiver $Q_{A_{8}}$
tains the form of the deformed quiver $Q_{A_{6}}$ except that the particular labellings of the nodes are permuted. Such mutation periodicity was already observed in the type $A_{4}$ case, in which the relevant sequence of mutations is $\mu_{2} \mu_{1} \mu_{11} \mu_{5}$. Comparing the cases, one can deduce the pattern
of mutations for the type $A_{8}$, which is given by $\mu_{4} \mu_{3} \mu_{2} \mu_{1} \mu_{19} \mu_{18} \mu_{17} \mu_{7}$. Then performing the iteration of matrix mutations above gives rise to the following exchange matrix:

Let us compare the mutated matrix (85) with (84). Then one can see that there have been changes in certain regions in the matrix, highlighted above. In the submatrix defined by the orange region, the transformation described by mutations is equivalent to cyclic permuting of the matrix, i.e. for the 13 -cycle $\rho=(4,5,6,7,8,9,10,11,12,13,14,15,16)$ cyclic permutation, the entries satisfy

$$
b_{\rho(i) \rho(j)}=b_{i+1, j+1} \quad \text { for } 4 \leq i, j \leq 16,
$$

with

$$
\begin{equation*}
b_{i, 17}=b_{i, 1}\left(\text { or } b_{17, j}=b_{1, j}\right) \text { for } i, j \in\{4,5, \ldots, 16\} \tag{86}
\end{equation*}
$$

As for the green highlighted submatrices, the entries in the upper and lower rows are shifted to the right by 1. For the left and right end column vectors, the entries are moved downwards by 1, i.e letting $I_{1}=\{1,2,3\}$ and $I_{2}=\{17,18,19\}$ and then

$$
b_{l, \rho(j)}=b_{l, j+1} \quad \text { for } l \in I_{1} \cup I_{2}, j \in\{4, \ldots, 16\}
$$

in agreement with (86). Such a transformation is seen to be the cyclic permutation of the labels $\{4,5, \ldots, 16\}$ of the deformed quiver $Q_{A_{8}}$.

Thus by using the same procedure that appeared in the type $A_{6}$ case, one can construct the cluster map $\psi_{A_{8}}=\rho_{A_{8}}^{-1} \mu_{4} \mu_{3} \mu_{2} \mu_{1} \mu_{19} \mu_{18} \mu_{17} \mu_{7}$ which generates the set of cluster variables

$$
\begin{align*}
\tau_{n+2} \sigma_{n} & =\sigma_{n+2} \tau_{n}+a_{1} p_{n} \\
p_{0, n+1} p_{0, n} & =\sigma_{n+3} \sigma_{n+2} \tau_{n} \tau_{n+1}+p_{1, n} \sigma_{n+1} \tau_{n+2} \\
p_{1, n+1} p_{1, n} & =\sigma_{n+4} \sigma_{n+3} \tau_{n} \tau_{n+1}+p_{2, n} p_{0, n+1} \\
p_{2, n+1} p_{2, n} & =\sigma_{n+5} \sigma_{n+4} \tau_{n} \tau_{n+1}+q_{2, n} p_{1, n+1}  \tag{87}\\
q_{2, n+1} q_{2, n} & =\sigma_{n+6} \sigma_{n+5} \tau_{n} \tau_{n+1}+q_{1, n} p_{2, n+1} \\
q_{1, n+1} q_{1, n} & =\sigma_{n+7} \sigma_{n+6} \tau_{n} \tau_{n+1}+q_{1, n} q_{1, n+1} \\
q_{0, n+1} q_{0, n} & =\sigma_{n+8} \sigma_{n+7} \tau_{n} \tau_{n+1}+\sigma_{n+9} q_{1, n+1} \tau_{n-1} \\
\sigma_{n+9} \tau_{n-1} & =\sigma_{n+7} \tau_{n+1}+a_{8} q_{0, n+1}
\end{align*}
$$

As we are aware, the exchange matrix (84) is constructed through pullback of the original
symplectic form by the map which is analogous to the variable transformation

$$
\begin{align*}
& x_{1, n}=\frac{\sigma_{n} \tau_{n+1}}{\sigma_{n+1} \tau_{n}} \quad x_{2, n}=\frac{p_{0, n}}{\sigma_{n+2} \tau_{n}} \quad x_{3, n}=\frac{p_{1, n}}{\sigma_{n+3} \tau_{n}} \quad x_{4, n}=\frac{p_{2, n}}{\sigma_{n+4} \tau_{n}} \\
& x_{5, n}=\frac{q_{2, n}}{\sigma_{n+5} \tau_{n}} \quad x_{6, n}=\frac{q_{1, n}}{\sigma_{n+6} \tau_{n}} \quad x_{7, n}=\frac{q_{0, n}}{\sigma_{n+7} \tau_{n}} \quad x_{8, n}=\frac{\sigma_{n+9} \tau_{n-1}}{\sigma_{n+8} \tau_{n}} \tag{88}
\end{align*}
$$

Then by manipulation of equations in (87) and imposing (88), we obtain the original twoparameter family of maps $\varphi_{A_{8}}$ defined on the initial variables $\left(x_{1}, x_{2}, \ldots, x_{8}\right)$. Hence it has been shown that the candidate deformed quiver/exchange matrix that emerged from the constructive approach allows us to define the cluster map $\psi_{A_{8}}$ which is a Laurentified deformed map $\varphi_{A_{8}}$.

As we have seen from the example, the deformed map $\varphi_{A_{8}}$ can be Laurentified to the cluster map $\psi_{A_{8}}$ which preserves the deformed exchange matrix (84). We now extend this procedure to show that the deformation of type $A_{2 N}$ cluster maps $\varphi_{A_{2 N}}$ can be lifted to a cluster map $\psi_{A_{2 N}}$

Following from the cases of type $A_{4}, A_{6}$ and $A_{8}$, there exists a specific sequence of mutation equivalent to a permutation of the vertices:

$$
\mu_{\tau_{-1}} \tilde{\mu} \mu_{\sigma_{0}}\left(Q_{6}\right)=\rho\left(Q_{6}\right)
$$

where $\tilde{\mu}=\mu_{q_{0}} \mu_{q_{1}} \mu_{p_{1}} \mu_{p_{2}}$ and $\rho$ is the inverse cyclic permutation of the vertices $\left(\tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \ldots, \sigma_{5}\right)$. This begins with a mutation in the direction of $\sigma_{0}, \mu_{\sigma_{0}}$, followed by the mutations $\tilde{\mu}$, then ends with $\mu_{\tau_{-1}}$. Recall that the local expansion on the deformed quiver introduces the new vertices $\sigma, p_{i}$ and $q_{i}$ and new edges. The position of the other vertices remains the same.

Proposition 3.9. For each deformed quiver $Q_{A_{2 N}}$ with vertices

$$
\left(q_{N-2}, \ldots, q_{1}, q_{0}, \tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \sigma_{1}, \sigma_{2}, \ldots, \sigma_{2 N}, \sigma_{2 N+1}, p_{0}, p_{1}, \ldots, p_{N-2}\right)
$$

we have invariance up to cyclic permutation under mutation:

$$
\begin{equation*}
\mu_{\tau_{-1}} \tilde{\mu} \mu_{\sigma_{0}}\left(Q_{A_{2 N}}\right)=\rho\left(Q_{A_{2 N}}\right) \quad \text { for } \tilde{\mu}=\mu_{q_{0}} \cdots \mu_{q_{N-2}} \mu_{p_{N-2}} \cdots \mu_{p_{0}} \tag{89}
\end{equation*}
$$

Proof. To see such a phenomenon explicitly, it is convenient for us to approach it from the exchange matrix perspective instead of quivers. As we are aware from the matrix (81), the local expansion in the matrix includes new entries on the surrounding block matrices. The non-zero entries $b_{i j}$ in the direction of the first two mutations $\mu_{p_{0}} \mu_{\sigma_{0}}=\mu_{3 N+5} \mu_{N+3}$ are positioned in the block matrices and their adjacent columns and rows. The matrix mutation replaces old entries with new, $\mu_{k}(B)$, if $b_{i k} b_{k j}<0$. Therefore the the transformation only occurs at the block matrices and their adjacent entries. This implies the first two mutations give the same result as the lower-rank cases (e.g. type $A_{8}$ ).

For the successive mutations, $\mu_{p_{1}} \mu_{p_{0}} \mu_{\sigma_{0}}=\mu_{3 N+6} \mu_{3 N+5} \mu_{N+3}$, the entries in the direction of the forthcoming mutation, $\mu_{k}$,

$$
\begin{equation*}
b_{i, k}=\left(\delta_{i, N+1}+\delta_{i, N+2}\right)+\left(-\delta_{i, k+1}-\delta_{i, k-1}+\delta_{i, k-2 N}+\delta_{i, k-2 N+1}\right) \tag{90}
\end{equation*}
$$

and its adjacent entries are written as follows

$$
\begin{align*}
b_{i, k-1} & =-\left(\delta_{i, N+1}+\delta_{i, N+2}\right)+\left(\delta_{i, k}+\delta_{i, k-2}-\delta_{i, k-2 N-1}-\delta_{i, k-2 N}\right)  \tag{91}\\
b_{i, k+1} & =\left(\delta_{i, k-2 N+2}-\delta_{i, k-2 N}\right)+\left(\delta_{i, k}-\delta_{i, k+2}\right)
\end{align*}
$$

From (90), one can see there are only two negative entries and the rest of the entries are positive. This implies that $b_{i k} b_{k j}<0$ if $i$ or $j$ is $k-1$ or $k+1$. Therefore the mutation $\mu_{k}$ does not affect the other entries except the adjacent entries in the direction $k$. The matrix mutation $\mu_{k}$ can be written as

$$
b_{i j}^{(n)}=b_{i j}^{(n-1)}-2 \delta_{j, k} b_{i, j}^{(n-1)}-2 \delta_{i, k} b_{k, j}^{(n-1)}+\beta_{i j}^{(n-1)}
$$

for

$$
\beta_{i j}^{(n-1)}=\left(\delta_{j, k+1}+\delta_{j, k-1}\right)\left(\delta_{i, N+1}+\delta_{i, N+2}+\delta_{i, k-2 N}+\delta_{i, k-2 N+1}\right)
$$

and it gives rise to the new columns in the $k-1$ and $k+1$ directions, which are expressed by $b_{i, k}$ in (90) and $b_{i, k+1}$ in (91) respectively with the subscript of entries in the second bracket of each column shifted by 1 . From $(79)$, one can observe that the exchange matrix consists of columns, in which the entries are written as

$$
b_{i, v}=\delta_{i, v-2 N+1}-\delta_{i, v-2 N-1}+\delta_{i, v-1}-\delta_{i, v+1}
$$

for $3 N+6 \leq v \leq 4 N+2$. By comparison, the situation of the next mutation is similar to the previous mutation. Thereby applying matrix mutations inductively in the direction $p_{0+l}=(3 N+5)+l$, for $l \in\{1, \ldots, N-2\}$, we obtain the following columns:

$$
\begin{equation*}
b_{i,(3 N+5+m)}=\delta_{i, N+7+m}-\delta_{i, N+5+m}+\delta_{i, 3 N+4+m}-\delta_{i, 3 N+6+m} \tag{92}
\end{equation*}
$$

for $m \in\{1, \ldots, l-1\}$. Furthermore, the mutation brings changes to the last and first columns, which are given by

$$
\begin{align*}
& b_{i, 4 N+3}=-\left(\delta_{i, N+1}+\delta_{i, N+2}\right)+\left(-\delta_{i, 2 N+3}-\delta_{i, 2 N+4}+\delta_{i, 1}+\delta_{i, 4 N+2}\right)  \tag{93}\\
& b_{i, 1}=\left(\delta_{i, N+1}+\delta_{N+2}\right)+\left(-\delta_{i, 2}-\delta_{i, 4 N+3}+\delta_{i, 2 N+4}+\delta_{i, 2 N+5}\right)
\end{align*}
$$

Now the mutations take place on the left side of the exchange matrix, which begins from $\mu_{q_{N-2}}=\mu_{1}$. The columns $b_{i, l}$ in the exchange matrix (39) for $l \in\{2, \ldots, N-2\}$ are structured as

$$
\begin{equation*}
b_{i, l}=\delta_{i, l+2 N+4}-\delta_{i, l+2 N+2}+\delta_{i, l-1}-\delta_{i, l+1} \tag{94}
\end{equation*}
$$

Under the mutation in the direction of the first column, we have the last column $b_{i, 4 N+3}$ written in the form of 92 with setting $\delta_{i, 4 N+4}=\delta_{i, 1}$, and its subsequent column $b_{i, 2}$ is

$$
\begin{equation*}
b_{i, 2}=\left(\delta_{i, N+1}+\delta_{i, N+2}\right)+\left(-\delta_{i, 1}-\delta_{i, 3}+\delta_{i, 2 N+5}+\delta_{i, 2 N+6}\right) \tag{95}
\end{equation*}
$$

Notice that we have the same situation as in the case of the sequence of the mutations on the right side of the matrix. To be specifically, the entries in $b_{i, 2}$ and its adjacent columns are arranged in the same way as (90) and (91). Computing the matrix mutations $\mu_{1} \cdots \mu_{N-2}=$ $\mu_{q_{N-2}} \cdots \mu_{q_{0}}$ subsequently, we find new columns $\left(b_{i, s}\right)(1 \leq i \leq 4 N+3,1 \leq s \leq N-2)$,

$$
b_{i . s}=\delta_{i, s-1}-\delta_{i, s+1}-\delta_{i, s+2 N+3}+\delta_{s+2 N+5}
$$

and the next adjacent column $b_{i, N-1}$ is given by

$$
b_{i, N-1}^{(2 N-4)}=-\delta_{i, N-2} \underbrace{-\delta_{i, N}+\delta_{i, N+1}+\delta_{i, N+2}}_{\left(\mathbf{A}_{4}\right)_{i, 1}}+\underbrace{\delta_{i, 3 N+2}+\delta_{i, 3 N+3}-\delta_{i, 3 N+4}}_{\left(-\mathbf{B}_{4}^{T}\right)_{i, 1}}
$$

The mutation in the direction $N-1$ gives the column $b_{i, N}$ :

$$
b_{i, N}^{(2 N-3)}=\underbrace{-\delta_{i, N-1}+\delta_{i, N+2}}_{\left(\mathbf{A}_{4}\right)_{i, 2}}+\underbrace{\delta_{i, 3 N+3}}_{\left(-\mathbf{B}_{4}^{T}\right)_{i, 2}}
$$

Therefore, combining all the results above, we can say that the composition of mutations in (89) transforms the type $A_{2 N}$ deformed exchange matrix (79) into the following

From the transformed matrix (96), one can see the same phenomenon as in Example 3.8 such that 1) the components in the $(2 N-1) \times(2 N-1)$ submatrix which is positioned in the centre of the matrix are cyclically permuted and 2) entries in the middle of the rows and columns, embedded next to the central submatrix, are shifted one downwards and upwards by one respectively. Hence we conclude that the quiver arising from the exchange matrix (96) exhibits symmetry such that the structure of the quiver is the same except that the labels $\{N, N-1, \ldots, 3 N+5\}$ are cyclically permuted i.e.

$$
\mu_{N} \tilde{\mu} \mu_{N+3}\left(Q_{2 N}\right)=\rho\left(Q_{2 N}\right) \quad \text { for } \tilde{\mu}=\mu_{1} \mu_{2} \cdots \mu_{N-2} \mu_{4 N+3} \mu_{4 N+2} \cdots \mu_{3 N+7} \mu_{3 N+6}
$$

where $\rho_{A_{2 N}}=(N, N+1, \ldots, 3 N+5)$.

The statement above enables us to define the cluster map which is associated with the exchange matrix (81), that is, we define $\psi_{A_{2 N}}=\rho_{A_{2 N}}^{-1} \mu_{N} \tilde{\mu} \mu_{N+3}$ and see that

$$
\begin{equation*}
\psi_{A_{2 N}}=\rho_{A_{2 N}}^{-1} \mu_{N} \tilde{\mu} \mu_{N+3}\left(B_{A_{2 N}}\right)=B_{A_{2 N}} \tag{97}
\end{equation*}
$$

Let us consider the cluster variables generated by the $\psi_{A_{2 N}}$. Recall that the new cluster variables are expressed by the exchange relation, in which the old cluster variables and coefficients appearing in the expression entirely depend on the matrix entries in the direction of mutation.

In the proof of the proposition above, we see that the transformation given by the first two mutations $\mu_{p_{0}} \mu_{\sigma_{0}}=\mu_{3 N+5} \mu_{N+3}$ only occurs in the block matrices and their surrounding entries. The corresponding matrix entries are written as

$$
\begin{align*}
& b_{i, N+3}=\delta_{i, N+1}+\delta_{i, N+5}-\delta_{i, 3 N+5}-\delta_{i, 4 N+4}  \tag{98}\\
& b_{i, 3 N+5}=\delta_{i, N+1}+\delta_{i, N+2}-\delta_{i, N+3}-\delta_{i, N+4}+\delta_{i, N+5}+\delta_{i, N+6}-\delta_{i, 3 N+6}
\end{align*}
$$

From the results above, we can see that the column $b_{i, k}$ in the direction of mutation $\mu_{k}$, for $k \in\{1,2, \ldots, N-2,3 N+6, \ldots, 4 N+3\}$ are arranged as following

$$
\begin{equation*}
b_{i, k}=\left(\delta_{i, N+1}+\delta_{i, N+2}\right)+\left(-\delta_{i, k+1}-\delta_{i, k-1}+\delta_{i, k-2 N}+\delta_{i, k-2 N+1}\right) \tag{99}
\end{equation*}
$$

At the latter end of the sequence of mutations,

$$
\begin{align*}
& b_{i, N-1}=-\delta_{i, N-2}-\delta_{i, N}+\delta_{i, N+2}+\delta_{i, 3 N+2}+\delta_{i, 3 N+3}-\delta_{i, 3 N+4}  \tag{100}\\
& b_{i, N}=-\delta_{i, N-1}+\delta_{i, N+2}+\delta_{i, 3 N+3}-\delta_{i, 4 N+5}
\end{align*}
$$

Thus starting from the initial seed ( $\tilde{\mathbf{x}}_{0}, B_{A_{2 N}}$ ), where $B_{A_{2 N}}$ is (81) and $\tilde{\mathbf{x}}_{0}$ is the initial cluster

$$
\left(q_{N-2,0}, \ldots, q_{1,0}, q_{0,0}, \tau_{-1}, \tau_{0}, \tau_{1}, \sigma_{0}, \sigma_{1}, \sigma_{2}, \ldots, \sigma_{2 N}, \sigma_{2 N+1}, p_{0,0}, p_{1,0}, \ldots, p_{N-2,0}\right)
$$

the $n$-th iterate of the cluster map $\psi_{A_{2 N}}$ generates cluster variables defined by the relations

$$
\begin{aligned}
\tau_{n+2} \sigma_{n}= & \sigma_{n+2} \tau_{n}+a_{1} p_{0, n} \\
p_{0, n+1} p_{0, n}= & \sigma_{n+3} \sigma_{n+2} \tau_{n} \tau_{n+1}+p_{1, n} \sigma_{n+1} \tau_{n+2} \\
p_{1, n+1} p_{1, n}= & \sigma_{n+4} \sigma_{n+3} \tau_{n} \tau_{n+1}+p_{2, n} p_{0, n+1} \\
& \vdots \\
p_{N-2, n+1} p_{N-2, n}= & \sigma_{n+N+1} \sigma_{n+N} \tau_{n} \tau_{n+1}+q_{N-2, n} p_{N-3, n+1} \\
q_{N-2, n+1} q_{N-2, n}= & \sigma_{n+N+2} \sigma_{n+N+1} \tau_{n} \tau_{n+1}+q_{N-3, n} p_{N-2, n+1} \\
q_{N-3, n+1} q_{N-3, n}= & \sigma_{n+N+3} \sigma_{n+N+2} \tau_{n} \tau_{n+1}+q_{N-4, n} q_{N-2, n+1} \\
& \vdots \\
q_{0, n+1} q_{0, n}= & \sigma_{n+2 N} \sigma_{n+2 N-1} \tau_{n} \tau_{n+1}+\sigma_{n+2 N+1} q_{1, n+1} \tau_{n-1} \\
\sigma_{n+2 N+2} \tau_{n-1}= & \sigma_{n+2 N} \tau_{n+1}+a_{2 N} q_{0, n+1}
\end{aligned}
$$

We then impose the variable transformations in (78) and we find the exchange relations

$$
\begin{aligned}
x_{1, n} x_{1, n+1} & =1+a_{1} x_{2, n} \\
x_{2, n} x_{2, n+1} & =1+x_{3, n} x_{1, n+1} \\
\vdots & \\
x_{2 N-1, n} x_{2 N-1, n+1} & =1+x_{2 N, n} x_{2 N-2, n+1} \\
x_{2 N, n} x_{2 N, n+1} & =1+a_{N} x_{2 N-1, n+1}
\end{aligned}
$$

In conclusion,
Theorem 3.10. The deformed map $\tilde{\varphi}_{A_{2 N}}$ arising from the cluster map $\varphi_{A_{2 N}}$ can be Laurentified to a cluster map $\psi_{A_{2 N}}$.

This implies that the variables induced by the deformed map can be generated by cluster mutation. However, the question of finding a general proof of integrability of the deformed map of $\varphi_{A_{2 N}}$ for $N>3$ is an open problem. The best way to resolve this question would be to find a general formula for the associated first integrals, for any $N$.

So far we were able to find the first integrals for the type $A_{6}$ case and show these are invariant and Poisson-commuting functions. However, as we move on to the higher rank cases, it is extremely difficult to find the corresponding integrals; in type $A_{8}$ we have the sum and product expressions analogously to the other types but no explicit expressions for the remaining two first integrals we require. This remains a topic of further study.

Related to this, together with Kouloukas and Vanhaecke, the third author is currently investigating a way to find a Lax pair for the deformation of type $A_{4}$ via the singularity
analysis of the deformed $A_{4}$ map, and an associated family of Abelian surfaces. We expect that finding such a Lax pair and combining this with our inductive local expansion approach will suggest a way to identify Lax pairs for the general case of type $A_{2} N$, which will prove Liouville integrability.

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