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Lower Bounds for the Permutation Flowshop Scheduling Problem

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The Permutation Flowshop Scheduling Problem, PFM

The *Permutation Flowshop Scheduling Problem with Makespan objective*, or PFM:

- A classic machine scheduling problem that has been widely addressed in the *OR* and *OM*.
- Consists of finding a sequence of n jobs or tasks to be processed on m resources or machines.
- The time to process job j on machine i is deterministic and denoted by p_{ij}
- The objective is to minimise the completion time of the last job on the last machine: *makespan*.

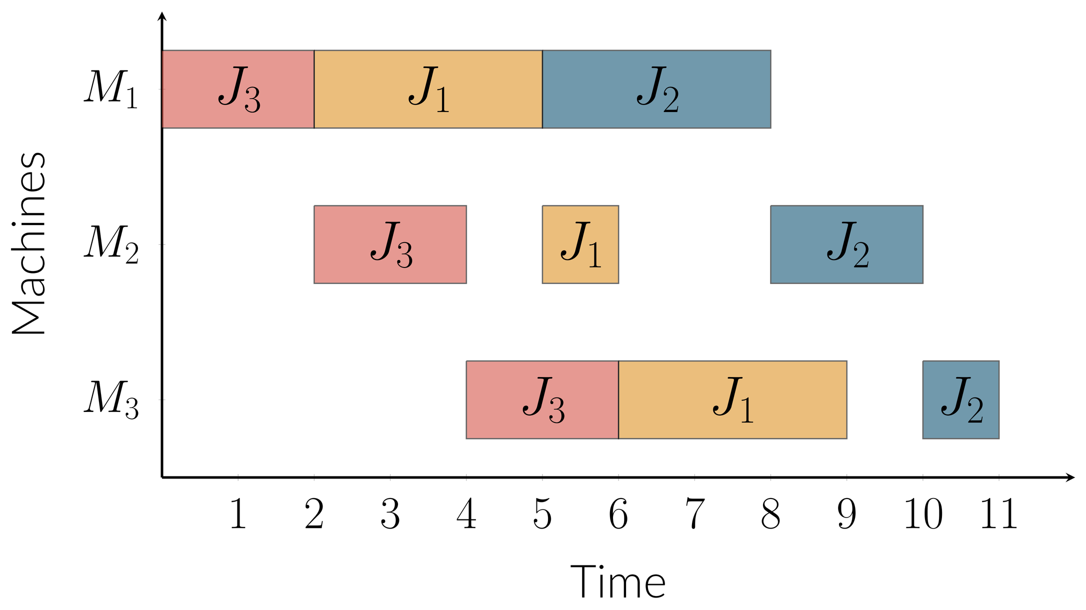

Figure 1. Gantt chart for the sequence $\{3, 1, 2\}$

Figure 1 shows a Gantt chart for the PFM instance ($n = m = 3$), where the makespan is 11.

Some Lower Bounds for the PFM

Improved machine-based bound, (L_M^+): [1]

$$L_M^+ = \max_{1\le i\le m} \left\{ \sum_{j=1}^{n} p_{ij} + \min_{1\le j\le n} \left\{ \sum_{r<i} p_{rj} \right\} + \min_{1\le j\le n} \left\{ \sum_{r>i} p_{rj} \right\} \right\}$$

Improved job-based bound, (L_J^+): [3]

$$L_J^+ = \max_{j=1}^{n} \left\{ \sum_{i=1}^{m} p_{ij} + \sum_{s\neq j} \min \{p_{1s}, p_{ms}\} \right\}.$$

Kumar *et al.* bound, (γ_{mn}): [2]

- σ_{ik} is the minimum time needed for machine i to process k jobs.
- For all k, γ_{1k} is set to σ_{1k}.
- For all i, $\gamma_{i1} = \min_j \left\{ \sum_{i'=1}^{i} p_{i',j} \right\}$
- γ_{ik} is the larger of:

$$\beta_{ik}^1 = \sigma_{ik} + \gamma_{i-1,1}, \qquad \beta_{ik}^2 = \sigma_{i,k-1} + \gamma_{i1}$$

$$\beta_{ik}^3 = \max_{1\le i'\le i} \left\{ \gamma_{i',k-1} + \min_j \left\{ \sum_{i''=i'}^{i} p_{i'',j} \right\} \right\}$$

$$\beta_{ik}^4 = \max_{1\le i'<i} \left\{ \gamma_{i',k} + \min_j \left\{ \sum_{i''=i'+1}^{i} p_{i'',j} \right\} \right\}$$

Linear Programming (LP) bound, (L_c):

Relaxation of the binary variables of the mixed-integer program by Stafford *et al.* [4].

Analysis of the Lower Bounds

Through a series of lemmas and theorems, we proved the following chains of inequalities:

$$OPT \ge L_c \ge \gamma_{mn} \ge L_M^+ \ge L_M \ge OPT/m$$

and

$$OPT \ge L_J^+ \ge L_J \ge OPT/n.$$

We also showed that L_c (and therefore also γ_{mn}) can be weaker than L_J. Specifically:

For any $m, n \ge 2$, and any small $\epsilon > 0$, there exists a PFM instance such that $L_c/L_J < \epsilon + (m+n-1)/mn$.

Finally, we conjectured that $L_c \ge OPT/n$.

Improved Lower Bounds

Efficient implementation of the Kumar *et al.* bound

If the Kumar et al. procedure is implemented in a naive way, it takes $O(m^2 n^2)$ time.

We have found an implementation that runs in $O(m^2 n + mn \log n)$ time. It consists of two sub-routines:

- **Algorithm 1**: Sort the processing times for each machine and compute the σ values. Also, compute the cumulative processing times:

$$\alpha_{ij} = \sum_{i'=1}^{i} p_{i',j} \qquad (i = 1, \ldots, m;\ j = 1, \ldots, n)$$

- **Algorithm 2**: Use the σ and α values to compute the β and γ values.

Strengthened Kumar *et al.* bound

We devised a strengthened version of the Kumar *et al.* procedure.

Let $\lambda(i, j) = \min \{p_{1j}, p_{ij}\}$ for all i and j.

Then

$$\beta_{ik}^5 = \max_j \left\{ \alpha_{mj} + \sum_{j'\neq j} \lambda(i, j') \right\}$$

is a lower bound on the time at which machine i finishes the k-th job.

We then let γ_{ik}^+ be the maximum of $\beta_{ik}^1, \ldots, \beta_{ik}^5$.

We have $\gamma_{mn}^+ \ge \max\{\gamma_{mn}, L_J^+\}$, and we have examples where the inequality is strict.

Strengthened LP bound

Finally, we strengthened the LP relaxation by adding the constraint $f_{ik} \ge \gamma_{ik}^+$, for all i and k, to the Stafford *et al.* LP.

We called the resulting lower bound L_c^+.

We have:

$$L_c^+ \ge L_c \ge \gamma_{mn} \ge L_M^+$$

and

$$L_c^+ \ge \gamma_{mn}^+ \ge L_J^+.$$

Computational Results

The results for Taillard [5] instances are:

Table 1. Average percentage gaps for *Taillard* instances

m	n	L_J^+	L_M^+	γ_{mn}	γ_{mn}^+	L_c	L_c^+
5	20	13.88	2.35	2.35	2.35	1.70	1.68
	50	26.39	0.80	0.80	0.80	0.59	0.59
	100	28.82	1.06	1.06	1.06	0.64	0.64
10	20	16.07	8.48	8.48	8.13	6.48	6.31
	50	21.84	2.10	2.10	2.10	1.68	1.65
	100	26.88	0.80	0.80	0.80	0.56	0.55
	200	28.49	0.66	0.66	0.66	0.45	0.45
20	20	14.73	17.00	17.00	14.25	13.30	12.50
	50	22.17	8.17	8.17	8.17	6.99	6.99
	100	25.79	3.74	3.74	3.74	3.00	3.00
	200	28.60	1.54	1.54	1.54	1.18	1.18
	500	31.12	0.54	0.54	0.54	0.44	0.44

The results for Vallada *et al.* [6] instances are:

Table 2. Average percentage gaps for *Vallada et al.* instances

m	n	L_V	L_J^+	L_M^+	γ_{mn}	γ_{mn}^+	L_c	L_c^+
5	10	18.72	6.16	19.33	19.33	5.64	14.09	5.51
	20	7.06	16.14	7.16	7.16	7.08	4.85	4.85
	30	3.72	24.29	3.78	3.78	3.78	2.73	2.73
	40	3.08	20.47	3.32	3.32	3.32	2.06	2.06
	50	2.17	26.12	2.18	2.18	2.18	1.34	1.34
	60	1.78	26.30	1.78	1.78	1.78	1.18	1.18
10	10	25.41	7.26	27.23	27.23	7.18	18.44	7.12
	20	14.68	14.32	15.32	15.32	12.15	10.54	9.79
	30	10.50	17.56	10.65	10.65	10.65	7.46	7.46
	40	6.70	19.39	6.76	6.76	6.76	4.99	4.99
	50	5.44	21.75	5.48	5.48	5.48	3.66	3.66
	60	4.39	20.20	4.58	4.58	4.58	2.88	2.88
15	10	27.76	9.14	29.62	29.58	7.43	18.48	7.37
	20	19.31	15.58	20.24	20.24	13.60	13.60	11.60
	30	14.97	17.04	15.53	15.53	14.11	11.65	11.38
	40	11.41	20.51	11.63	11.63	11.63	8.84	8.84
	50	9.04	21.16	9.20	9.20	9.20	6.98	6.98
	60	7.63	23.93	7.77	7.77	7.77	5.91	5.91
20	10	26.18	11.07	27.61	27.61	9.80	17.20	9.79
	20	22.00	13.27	23.08	23.08	12.27	16.20	12.08
	30	17.86	17.38	17.92	17.92	15.68	13.65	12.99
	40	15.69	19.55	15.97	15.97	15.21	12.40	12.40
	50	13.08	22.19	13.40	13.40	13.40	10.61	10.61
	60	10.76	22.31	10.87	10.87	10.87	8.76	8.76

References

[1] E. Ignall and L. Schrage. Application of the branch and bound technique to some flow-shop scheduling problems. *Oper. Res.*, 13: 400–412, 1965.

[2] S.S. Kumar, Rajendran, C., and R. Leisten. Bounding strategies for obtaining a lower bound for N-job and M-machine flowshop scheduling problem with objective of minimising the total flowtime of jobs. *Int. J. Oper. Res.*, 41:244–269, 2021.

[3] G.B. McMahon and P.G. Burton. Flow-shop scheduling with the branch-and-bound method. *Oper. Res.*, 15:473–481, 1967.

[4] E.F. Stafford, F.T. Tseng, and J.N. Gupta. Comparative evaluation of milp flowshop models. *J. Oper. Res. Soc.*, 56:88–101, 2005.

[5] E. Taillard. Benchmarks for basic scheduling problems. *Eur. J. Oper. Res.*, 64:278–285, 1993.

[6] E. Vallada, R. Ruiz, and J.M. Framinan. New hard benchmark for flowshop scheduling problems minimising makespan. *Eur. J. Oper. Res.*, 240:666–677, 2015.

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