Incentivizing variety in innovation contests with specialized suppliers

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Abstract

We study the optimal design of an innovation contest where a buyer seeks product variety and faces a moral hazard problem. The suppliers are specialized and may differ in their flexibility to adopt approaches outside their areas of expertise. If the specializations are sufficiently different and suppliers are otherwise symmetric, the buyer attains the first-best with a fixed-prize contest (FPC). If one supplier is inherently advantaged or the specializations are sufficiently close, the first-best is unattainable with an FPC. In all cases, an auction is an optimal contest and implements the first-best, provided the buyer can discriminate within the contest; if not, the buyer may prefer an FPC.

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1 Introduction

Contests are commonly used by public and private entities to inspire new ideas and procure innovations.[1] While a contest designer may have some broad objective they hope to achieve, there are often different means to the same end. As the 'best' approach is not always known ahead of time, a designer may wish to encourage a variety of different approaches. Take, for example, NASA's 'Watts on the Moon Challenge', which sought the development of new technology to transmit power on the moon. The broad objective is clear, but there are many different ways this could be achieved, and NASA explicitly sought different approaches: "NASA has a significant interest in both wired and wireless transmission, and the challenge seeks to incentivize and demonstrate both types of solutions"[2]

To elicit product variety, it is only natural to seek participation from innovators with different areas of expertise. An important question, then, is how to structure a contest to elicit product variety when participants are specialized. In this paper, we address this question by extending the literature on contest design as a means of incentivizing variety (Ganuza & Hauk, 2006; Erat & Krishnan, 2012; Letina & Schmutzler, 2019)[3] to an environment in which innovators are specialized and may differ in their flexibility to adopt approaches outside their areas of expertise. Contrasting existing results in the literature, we find that fixed-prize contests (FPCs) can be highly effective tools for eliciting product variety. However, their efficacy depends on the relative positions of the specializations and the competitive balance within the contest. Specifically, FPCs perform best when the participants' specializations are sufficiently different and no participant is inherently advantaged.

Our model builds on the framework of Letina & Schmutzler (2019) (L&S) and is inspired by Hotelling (1929): A buyer seeks an innovation from one of two suppliers. There are different approaches available to the suppliers, which correspond to points in the unit interval. *Ex ante*, the ideal approach is unknown, but after the innovations are developed, the uncertainty is resolved. The quality (buyer's value) of a supplier's innovation then depends on the distance between the approach taken and the ideal approach. This reflects the idea that once the buyer *sees* the innovations, she is better able to assess the extent to which they suit her needs. The suppliers' approaches are unobservable to the buyer, and the realized qualities are observable but non-verifiable. Therefore, neither the approaches nor the qualities are contractible, but the buyer's purchasing decision depends on the qualities.

[1] For a survey on the contest theory literature, see Konrad (2009).

[2] https://www.challenge.gov/?challenge=watts-on-the-moon-phase-2)

[3] See, also, Lemus & Temnyalov (2024), which explores information design in a related context.

Each supplier has a specialization, which is an approach that can be adopted at no cost. The cost incurred by a supplier is an increasing function of the distance between her chosen approach and her specialization. These cost functions may differ, capturing the idea that the suppliers may differ in their flexibility to adopt approaches outside their areas of expertise.

To provide a concrete interpretation of the model, consider a hypothetical mobile phone design competition: A mobile phone company (the buyer) contracts with two product designers (the suppliers) to develop prototypes for a redesigned mobile phone. Broadly speaking, mobile phones can be large – ideal for streaming/gaming – or small – better suited for portability. The core competencies of the suppliers may differ in this regard: One might specialize in discreet, wearable technology, while the other is experienced in designing larger tablets with high-quality screens. The buyer, uncertain about which approach is best, seeks a range of options, and her ideal design is most likely to fall between the two extremes – smaller than a tablet, but more functional than an earpiece.

While the context of this particular mobile phone competition is hypothetical, procurement design competitions are commonplace in both the public and private sectors. The Department of Defense in the U.S., for example, frequently runs such competitions to procure innovative weapons and defense systems (Lichtenberg, 1988). Similarly, firms, including Apple, Telstar, and IBM, have been noted to use such competitions (see Lu et al., 2022, and the references therein). The question is, how should the buyer optimally structure the contest to elicit the 'right' level of variety?

Given their prevalence in the real world and in the Economics literature, our baseline model focuses on FPCs and, in line with Fullerton & McAfee (1999), we allow entry fees. In an FPC, the buyer commits *ex ante* to a prize (or price) for each supplier. After the qualities are realized, the buyer purchases from the innovator offering the greatest surplus.[4]

L&S find that when all approaches are equally costly, an FPC is incapable of eliciting *any* product diversity. Instead, adhering to the Principle of Minimum Differentiation (Hotelling, 1929; Downs, 1957), an FPC leads both innovators to cluster at the approach that maximizes individual expected quality.[5] In our model, the optimal prize structure depends on the relative locations of the specializations, but there is never a lack of diversity in equilibrium.

If the specializations are sufficiently different, the optimal FPC calls for positive prizes for both suppliers, who subsequently choose more conservative approaches (closer to the

[4]In the baseline model, we allow the buyer to set different prizes and entry-fees/subsidies for the two suppliers. But we also consider a variant in which the buyer faces an anonymity constraint, requiring that she offer the same contract to both suppliers. We further discuss this issue in Section 6.

[5]Similar clustering is found by Erat & Krishnan (2012).

median). As these approaches are costly, the buyer maintains variety by limiting the size of the prizes. If, in addition, no supplier is inherently advantaged – through having lower marginal cost or a more central specialization – the buyer attains the first-best with an FPC. If one supplier is inherently advantaged, there is a distortion from the social optimum since the first-best approaches cannot be implemented with equal prizes. When the prizes differ, the supplier offering the greatest surplus may not be the supplier with the highest quality. As a result, the buyer's *ex post* purchasing decision may not be socially efficient. This *ex post* inefficiency gives rise to an *ex ante* 'distortion cost' for the buyer.[6] In response, the buyer reduces the prize spread, relative to the prizes that implement the first-best, which reduces the likelihood that she later purchases the lower-quality innovation. In equilibrium, the advantaged supplier adopts an approach that is 'too far' from her specialization, while the disadvantaged supplier adopts an approach that is 'too close' to her specialization. To emphasize the role of the buyer's limited commitment power in driving this result, we also consider a variant in which the buyer can credibly commit to purchasing the highest quality innovation. Reminiscent of results on rank-order tournaments (Lazear & Rosen, 1981), the buyer attains the first-best with an FPC.[7]

When the specializations are sufficiently close, the buyer's primary objective is to increase variety, and this cannot be done in an FPC with two positive prizes. Instead, the buyer awards a positive prize to exactly one supplier, say i. Supplier j is not excluded, but is awarded a "prize" of zero and subsequently produces at her specialization. With this prize structure, the buyer commits to purchasing from i only if her innovation is of sufficiently higher quality than j's. Since similar approaches yield similar qualities, i is led to differentiate her approach from j. This scenario can be interpreted as one in which the buyer engages in a bilateral negotiation with supplier i, committing to a price and using the threat of acquiring j's default innovation as leverage to induce differentiation.

Finally, for some configurations of the specializations, the buyer cannot improve upon the default scenario using an FPC: The optimal FPC calls for prizes of zero and leaves the suppliers to produce at their specializations. In this case, there is, in effect, no contest at all. Instead, the buyer leaves the suppliers to produce according to their own core competencies and subsequently acquires the innovation of higher quality.[8] This occurs when the specializations are moderately close – close enough that the buyer would prefer greater

[6] A similar distortion emerges in Maskin & Riley (2000) and Che & Gale (2003).

[7] See Galasso (2020) for another example of how limited commitment can lead to welfare distortions in the context of innovation incentives.

[8] Equivalently, the buyer acquires both innovations and consumes the one of higher quality.

variety, but sufficiently different that the prize needed to incentivize differentiation imposes a prohibitively large distortion cost.

We then study a more general contest design problem, à la Che & Gale (2003). In this setting, the buyer commits to a menu of allowable prices for each supplier; after the qualities are realized, the suppliers compete by choosing prices from their menus. L&S show that an auction, in which prices are unconstrained, always implements the first-best, but may be inferior (for the buyer) to an FPC if entry fees are not permissible. We extend the first point and obtain a similar finding to the second. First, we show that the auction implements the first-best independently of the details of suppliers' specializations/costs and the distribution of the ideal. Driving the result is that the location-choice subgame is a potential game (Monderer & Shapley, 1996), with a potential equal to the objective function of the social planner. Second, if the buyer cannot offer identity-dependent prizes/entry-fees, she may prefer an FPC, due to the auction's tendency to leave greater rents to the suppliers.

Our results add to the literature on innovation contest design.[9] Much of this literature focuses on providing the right incentives for contestants to exert costly effort, which either raises the probability/speed of successful innovation or increases the expected quality of the innovation (see, e.g., Taylor, 1995; Fullerton & McAfee, 1999; Che & Gale, 2003; Schöttner, 2007).[10] In either case, what defines 'success' or 'high quality' must be specified ahead of time. This is not possible in our model, given the uncertainty over the ideal approach.

Gretschko & Wambach (2016) compare the welfare properties of different mechanisms for procurement in an environment in which sellers are horizontally differentiated and the buyer is uncertain of her preferences. But in their model, the specifications of the sellers are fixed exogenously, and there is no scope for the buyer to induce different specifications.

Relative to the small number of studies that have considered contest design in the context of eliciting product diversity, our contribution is the inclusion of supplier specializations. This added feature is a relevant concern in real-world innovation contests. Indeed, the consulting firm, McKinsey, recommends using prizes when the optimal approach is uncertain, precisely to encourage participation from innovators with different areas of expertise.[11] As a concrete example, take the Mobile Design Competition organized in 2019 by Samsung and

[9] For studies on the optimal design of contests more broadly see, e.g., Moldovanu & Sela (2001), Liu et al. (2018) or Olszewski & Siegel (2020).

[10] Baye & Hoppe (2003) show that, in some cases, patent races (innovators compete to innovate first) are strategically equivalent innovation tournaments (innovators compete to produce the best innovation).

[11] "By attracting *diverse talent* and a range of potential solutions, prizes draw out many possible solutions..." (Bays et al., 2009, p. 49).

Dezeen, which sought the development of new accessories for Galaxy smartphones.[12] One of Samsung's stated goals with the competition was to foster "…collaboration with a *wide variety of different designers* to inform the future development of its products" (emphasis added).

Moreover, introducing supplier specializations has important economic consequences for the optimal design of incentives for variety. Contrasting prior results, we find that with specialized suppliers, FPCs can be highly effective tools for eliciting variety, particularly when there is sufficient differentiation between innovators' specializations and no supplier is inherently advantaged. Additionally, if one supplier has an inherent advantage, the buyer may benefit (*ex ante*) from committing to purchase based solely on quality, without regard to price. Finally, we extend results from the literature and demonstrate that, quite generally, an auction provides just the right incentives for variety. At the same time, its attractiveness to a buyer depends on her ability to extract surplus from the innovators. If this cannot be done effectively – e.g., because suppliers are asymmetric but the buyer is constrained to treat them equally – an FPC may be preferable.[13]

Our results also add to the extensive literature on innovation incentives. A related strand of this literature concerns the length and breadth of patent protection. Waterson (1990), in particular, emphasizes the role of patents in influencing product differentiation. He shows that the patent system can raise social welfare (relative to free-entry) through its influence on firms' variety choices. Our study complements these general insights and shows how prize competitions can be optimally structured to improve sellers' variety choices for a buyer.

Finally, our model relates to the spatial models used in political economy to explain the policy choices of political candidates/lobbyists.[14] Our finding that FPCs induce variety when innovators are specialized relates to the finding that, when candidates/lobbyists are ideological, equilibria can be supported in which policies differ (see, e.g., Osborne, 1995; Epstein & Nitzan, 2004). Differentiating our model from this stream of literature is the endogeneity of the rewards for victory, which is our main focus.

[12] https://www.dezeen.com/samsung-mobile-design-competition-brief/

[13] The inclusion of supplier specializations and costly location choices also has technical implications for the model. Reisinger et al. (2023) consider a duopoly Hotelling model in which firms first commit to prices, then choose locations in product space. The second stage of this game is similar to the location-choice subgame in our model but differs in that location choices are costless. The authors show that in many circumstances, a PSNE does not exist. In our model, for many prize configurations, a PSNE does exist. The reason is that the sort of mimicry that destroys PSNE when location choices are costless need not be profitable when location choices are costly.

[14] For recent contributions to this literature, see, e.g., Münster (2006) or Balart et al. (2022).

2 The model

There are two suppliers, called zero and one, from whom a buyer wishes to procure a single unit of innovation. Neither the suppliers nor the buyer is certain of the ideal approach, y, but all parties believe y is distributed according to a CDF, F, with full support on $[0,1]$. We assume that F is smooth on $(0,1)$ with PDF $f = F'$, which is also smooth on $(0,1)$, symmetric about $\frac{1}{2}$, and increasing on $[0, \frac{1}{2}]$. Formally:

Assumption 1. *For all* $y \in [0,1]$, $f(y) = f(1-y) > 0$. *For* $y \in \left(0, \frac{1}{2}\right)$, $f'(y) \geq 0$.

Prior to the realization of the ideal, the suppliers simultaneously choose their approaches. We let $\ell_i \in [0,1]$ denote supplier i's approach and let $\ell = (\ell_0, \ell_1)$ denote a profile of approaches. Using the mobile phone design competition discussed in the introduction as an example, ℓ_i can be interpreted as the screen size chosen by supplier i, where $\ell_i = 0$ corresponds to a very small, wearable device, and $\ell_i = 1$ corresponds to a device as large as a tablet. If i chooses approach ℓ_i and the ideal is y, the buyer's monetary value of i's innovation is $Q_i(\ell_i, y) = q - |\ell_i - y|$. We refer to $Q_i(\cdot)$ as the quality of i's innovation.

Supplier i has a specialization, $s_i \in (0,1)$, where $s_0 \leq s_1$.[15] If i chooses approach ℓ_i, she incurs the cost $C_i(\ell_i - s_i)$. Each C_i is symmetric about 0: $C_i(x) = C_i(-x)$ for all $x \in [-1,1]$, and thrice differentiable with $C_i(0) = C_i'(0) = 0$, $C_i'' > 0$, and $C_i''' \geq 0$. Thus, i can produce an innovation at her specialization at zero cost, while $C_i(d) > 0$ for $d \neq 0$. Each supplier pays her cost regardless of the buyer's purchasing decision. Abstracting from concerns of adverse selection, we assume the specializations and cost functions are known by the buyer.

In our baseline model, we focus on FPCs: Prior to the suppliers choosing their approaches, the buyer commits to prizes, $v = (v_0, v_1)$, where $v_i \geq 0$ is the prize to be received by supplier i if she wins the contest. The buyer may also charge entry fees (or offer subsidies), $t = (t_0, t_1)$. As a convention, we let $t_i > 0$ denote an entry fee paid by supplier i and $t_i < 0$ denote a subsidy received by i. The approaches taken by the two suppliers are unobservable by the buyer, while the realized qualities are observable but non-verifiable. Therefore, neither prize may depend on the suppliers' approaches or realized qualities, but the buyer's purchasing decision will be a function of the realized qualities. If the contest parameters are (v, t) and the buyer purchases from supplier i, the buyer's payoff is $Q_i(\ell_i, y) - v_i + t_0 + t_1$ and supplier j's payoff is $\mathbb{1}(i = j)v_j - C_j(d_j) - t_j$. All parties are risk-neutral, and we assume that q is sufficiently large such that the buyer will always purchase from one of the two suppliers.

[15] We restrict attention to specializations on the interior of the unit interval for technical convenience, as it simplifies some of the proofs. But it has no meaningful impact on our results. Our working paper (see Protopappas & Rietzke, 2023) covers the case where $s_0 = 0$ and $s_1 = 1$.

The timing is summarized as follows: In stage 1, the buyer commits to the contest parameters, (v, t). In stage 2, the suppliers decide whether to participate; the participating suppliers then simultaneously choose their approaches. In stage 3, the quality of each innovation is realized, and the buyer makes her purchasing decision.

We make the following assumptions on the distribution and cost functions:

Assumption 2.

(i) For all $y \in [0,1]$, $|f'(y)| < 2f(0)$.

(ii) For all $d \in [0,1]$ and each $i \in \{0,1\}$, $2f\left(\frac{1}{2}\right) < C_i''(d)$.

Assumption 2(i) ensures that even the least-likely states occur with sufficiently high probability. Assumption 2(ii) implies that each supplier's cost function is "sufficiently convex"; in Section 6, we discuss its relevance in more detail. Assumptions 1-2 imply the following additional properties on the distribution and cost functions:

Lemma 1. *For all $y, y', d \in [0,1]$, and each $i = 0, 1$,*

(i) $f(y) < 2f(y')$

(ii) $C_i'(d) > 2df\left(\frac{1}{2}\right)$

(iii) $|f'(y)| < C_i''(d)$.

Terminology

We define key terms used in the analysis: Supplier i's *default innovation* corresponds to an approach choice equal to her specialization. If both suppliers produce their default innovations, we call this the *default scenario* (or simply, *default*). Approaches closer to $\frac{1}{2}$ are called more *conservative*, while those further from $\frac{1}{2}$ are more *radical.* If i's specialization is more conservative than j's, we say that i is more conservative (equivalently, j is more radical).

Supplier i is more *flexible* than j if $C_i' < C_j'$.[16] The suppliers are *symmetric* if they are equally flexible and conservative: $C_0 = C_1$ and $s_0 = 1 - s_1$. Supplier i has a *cost advantage* if the suppliers are equally conservative but i is more flexible: $s_0 = 1 - s_1$ and $C_i' < C_j'$. Supplier i has a *quality advantage* if the suppliers are equally flexible, but i is more conservative: $C_i = C_j$ and $|s_i - \frac{1}{2}| < |s_j - \frac{1}{2}|$. It is obvious that greater flexibility provides an inherent advantage, as it enables i to choose approaches different from her specialization at

[16] When we write $C_i' < C_j'$, we mean $C_i'(d) < C_j'(d)$ for all $d > 0$.

a lower cost than j. But i is also advantaged when she is more conservative, as this implies that the expected quality of her default innovation is higher than j's.

We use the terms *diversity* or *variety* to refer to the spread between the chosen approaches, $|\ell_1 - \ell_0|$. Finally, we sometimes consider a simplified model in which $y \sim U[0,1]$ and $C_i(d) = \frac{c_i}{2}d^2$; we refer to this as the *uniform/quadratic model.*

3 Preliminary results

In this section, we characterize the first-best approaches; we then study the stages 2-3 subgames.

3.1 Benchmark solution – The first-best

As a benchmark, we consider the problem of a social planner whose objective is to maximize total surplus. The planner's problem proceeds in 2 stages: In stage 1 – facing uncertainty over the ideal – she chooses the suppliers' approaches. In stage 2 – after the uncertainty is resolved – she allocates an innovation to the buyer. Efficiency requires that the planner allocates the highest quality innovation to the buyer in stage 2. When the allocation is *ex post* efficient, the *ex ante* expected surplus is,

$$S_{FB}(\ell) = \overline{Q}_{FB}(\ell) - C_0 - C_1,$$

where $\overline{Q}_{FB}(\ell)$ is the expected highest quality innovation: $\overline{Q}_{FB}(\ell) = \mathbb{E}\left[\max\{Q_0(\ell_0, y), Q_1(\ell_1, y)\}\right]$.

Since $s_0 \leq s_1$, it is always optimal for the planner to choose approaches such that $\ell_0 \leq \ell_1$. Then, $Q_0(\ell_0, y) > Q_1(\ell_1, y)$ if $y < m(\ell)$ and $Q_0(\ell_0, y) < Q_1(\ell_1, y)$ if $y > m(\ell)$, where $m(\ell) = \frac{\ell_0+\ell_1}{2}$. Thus, $\overline{Q}_{FB}(\ell) = q - \kappa_{FB}(\ell)$, where

$$\kappa_{FB}(\ell) = \int_0^{m(\ell)} |\ell_0 - y| \, \mathrm{d}F(y) + \int_{m(\ell)}^1 |\ell_1 - y| \, \mathrm{d}F(y).$$

Letting $L_{FB} = \kappa_{FB} + C_0 + C_1$, the planner's stage-1 problem can be expressed,

$$\min_{\ell \in [0,1]^2} L_{FB}(\ell). \tag{FB-P}$$

Before characterizing the solution to this problem, we mention two related benchmarks. First, Assumption 1 implies that the expected quality of i's individual innovation is maximized at an approach of $\frac{1}{2}$. Second, $\overline{Q}_{FB}(\ell)$ is maximized at approaches $(s^*, 1 - s^*)$, where,

$$F(s^*) = \frac{1}{4}. \tag{1}$$

Moving both suppliers' approaches closer to $\frac{1}{2}$ increases the expected quality of each individual innovation, but decreases diversity. Given the uncertainty over the ideal, there is inherent value in maintaining variety, as this generates an option value for the buyer. The approaches $(s^*, 1-s^*)$ balance this trade-off to maximize the expected highest quality.

Proposition 1. *There is a unique solution to* (FB-P) *characterized by the following first-order conditions:*

$$F\left(m\left(\ell^{FB}\right)\right) - 2F\left(\ell_0^{FB}\right) = C_0'\left(\ell_0^{FB} - s_0\right)$$
$$1 + F\left(m\left(\ell^{FB}\right)\right) - 2F\left(\ell_1^{FB}\right) = C_1'\left(\ell_1^{FB} - s_1\right).$$

Moreover, if $s_0, 1-s_1 < s^*$*, then,* $s_0 < \ell_0^{FB}$ *and* $\ell_1^{FB} < s_1$*. If* $s^* < s_0, 1-s_1$*, then,* $\ell_0^{FB} < s_0$ *and* $s_1 < \ell_1^{FB}$*.*

The first-best approaches characterized in Proposition 1 balance the trade-off between raising the expected highest quality and limiting costs.

To provide the intuition for the two cases ($s_0, 1-s_1 < s^*$ and $s^* < s_0, 1-s_1$) mentioned in Proposition 1, first note that Assumptions 1-2 imply $s^* < \frac{1}{3}$ (see Lemma A.1 in the Appendix). Thus, the condition $s_0, 1-s_1 < s^*$ implies $s_0 < \frac{1}{3} < \frac{2}{3} < s_1$, which means that the specializations are sufficiently different and radical – i.e., positioned at opposite ends of the unit interval. In this case, the default results in an inefficiently high level of diversity. Relative to the default, it is optimal for the planner to reduce diversity and choose more conservative approaches. Conversely, the condition $s^* < s_0, 1-s_1$ implies, $\frac{1}{3} < s_0 \leq s_1 < \frac{2}{3}$, which means that the specializations are relatively close to one another and positioned towards the center of the unit interval. In this case, the supplier's specializations are 'too conservative' and the default results in an inefficiently low level of diversity. Relative to the default, the planner chooses more radical approaches and increases variety.

3.2 The buyer's purchasing decision

Here, we examine the buyer's *ex post* purchasing decision. Given the prizes and realized qualities, the buyer purchases from i if she offers a greater surplus than j: $Q_i(\ell_i, y) - v_i > Q_j(\ell_j, y) - v_j$. If both offer the same surplus, $Q_0(\ell_0, y) - v_0 = Q_1(\ell_1, y) - v_1$, we adopt the tie-breaking rules in Che & Gale (2003) and L&S: We assume that the buyer has a preference for quality and purchases from supplier i if $Q_i(\ell_i, y) > Q_j(\ell_j, y)$. If $Q_0(\ell_0, y) - v_0 = Q_1(\ell_1, y) - v_1$ and $Q_0(\ell_0, y) = Q_1(\ell_1, y)$, the buyer purchases from each supplier with probability $\frac{1}{2}$.

It is useful to note that if $|v_1 - v_0| \leq \ell_1 - \ell_0$ – i.e., the approaches are sufficiently different, relative to the prize spread – then supplier zero [one] wins with probability $F(y_0(\ell, v))$ $[1 - F(y_0(\ell, v))]$, where,

$$y_0(\ell, v) = \frac{\ell_0 + \ell_1 + v_1 - v_0}{2}.$$

However, if the approaches are sufficiently close, relative to the prize spread – in particular, $|\ell_1 - \ell_0| < v_i - v_j$ – then the supplier with the smaller prize, j, wins with probability 1. In this case, the difference in qualities is always less than the difference in prizes; hence, j offers greater surplus in any state of the world.

3.3 The suppliers' equilibrium approaches

In this section, we take the prize values as given and study the stage-2 approach choice subgame, assuming both suppliers have entered. We focus on pure-strategy Nash equilibrium in this subgame (henceforth, "equilibrium").[17] Letting $p_i(\ell, v)$ denote i's probability of victory, her expected payoff is $\pi_i(\ell, v) = p_i(\ell, v)v_i - C_i(\ell_i - s_i)$. Supplier i solves $\max_{\ell_i \in [0,1]} \pi_i(\ell, v)$.

The functional form of p_i differs, depending on the prizes. If $v_0 = v_1$, then,

$$p_0(\ell, v) = \begin{cases} F(m(\ell)), & \ell_0 < \ell_1 \\ \frac{1}{2}, & \ell_0 = \ell_1 \\ 1 - F(m(\ell)), & \ell_0 > \ell_1 \end{cases}$$

Letting $\tilde{y}_0(\ell, v) = \frac{\ell_0 + \ell_1 + v_0 - v_1}{2}$ and $\mathbb{1}(\cdot)$ be an indicator function; if $v_0 \neq v_1$, then,

$$p_0(\ell, v) = \begin{cases} F(y_0(\ell, v)), & \ell_1 - \ell_0 \geq |v_1 - v_0| \\ \mathbb{1}(v_0 < v_1), & |\ell_1 - \ell_0| < |v_1 - v_0|, \\ 1 - F\left(\tilde{y}_0(\ell, v)\right), & \ell_0 - \ell_1 \geq |v_1 - v_0| \end{cases}$$

And $p_1(\ell, v) = 1 - p_0(\ell, v)$. Let

$$u_0(\ell, v) = F\left(y_0\left(\ell, v\right)\right) v_0 - C_0(\ell_0 - s_0)$$

[17] For tractability, we do not consider mixed strategy Nash equilibria (MSNE). For a treatment of MSNE in a game closely related to the 2nd stage of our game, see Reisinger et al. (2023).

and

$$u_1(\ell, v) = [1 - F(y_0(\ell, v))]\, v_1 - C_1(\ell_1 - s_1).$$

For (ℓ, v) such that $|v_1 - v_0| \leq \ell_1 - \ell_0$, $\pi_i(\ell, v) = u_i(\ell, v)$. Note that each u_i is differentiable in (ℓ, v) and twice differentiable whenever $y_0(\cdot) \in (0, 1)$. Moreover, Lemma 1(iii) implies $\frac{\partial^2 u_i}{\partial \ell_i^2} < \frac{v_i}{4} |f'(y_0(\cdot))| - |f_i'(y_0(\cdot))|$; thus, if $v_i \leq 4$ then $\frac{\partial^2 u_i(\ell, v)}{\partial \ell_i^2} < 0$.

While the u_i's are relatively well-behaved, the π_i's are not generally continuous or quasi-concave. As a consequence, for a given pair of prizes, different types of equilibria can emerge; moreover, there may be multiple equilibria, or an equilibrium may not exist. Propositions A.1 and A.2 (in the Appendix) provide several necessary equilibrium conditions and give sufficient conditions for existence. To avoid cumbersome technicalities, we present a parsimonious result, which emphasizes just the most important ideas for our later analysis. In what follows, we let $\Phi(v)$ denote the set of stage-2 equilibria when the prizes are v.

Proposition 2. *Let $v_0, v_1 \geq 0$ and suppose $\ell^* \in \Phi(v)$. Let $d_i^* = |\ell_i^* - s_i|$.*

(i) If $d_0^*, d_1^* > 0$ then $s_0 < \ell_0^* < \ell_1^* < s_1$, $|v_1 - v_0| < \ell_1^* - \ell_0^*$, and $\frac{\partial u_i(\ell^*, v)}{\partial \ell_i} = 0$, $i = 0, 1$.

(ii) If $s_1 - s_0 < v_i$ and $v_j = 0$ then $\ell_j^* = s_j$ and $\ell_i^* \in \{s_i, s_j \pm v_i\}$.

Part (i) shows that FPCs tend to reduce diversity relative to the default. More precisely, in any equilibrium in which both suppliers choose approaches different from their specializations, the approaches lie between the specializations. Moreover, the approaches remain sufficiently far apart that the equilibrium occurs in the region of the strategy space where $\pi_i = u_i$.

Part (ii) shows that, despite their tendency to reduce variety, an FPC can induce greater variety. To explain, suppose $v_j = 0$ and $s_1 - s_0 < v_i$. Clearly, j has a dominant strategy to choose s_j. Moreover, i's prize is sufficiently large that, if she were to choose s_i, or any other approach that is too close to s_j (specifically, any approach in $(s_j - v_i, s_j + v_i)$), the difference in qualities is never large enough for the buyer to purchase from i. Depending on the cost of doing so, i may therefore choose to differentiate her approach from j, which increases the potential difference in qualities, and gives i *some* chance of victory. The next example illustrates.

Example 1. *Consider the uniform/quadratic model, and suppose $s_0 = \frac{2}{5}$, $s_1 = \frac{3}{5}$, $v_0 = 0$ and $v_1 = \frac{3}{10}$. In any equilibrium, $\ell_0^* = s_0$, so we focus on the choice of supplier one. To win with non-zero probability, supplier one must choose an approach outside the interval, $(s_0 - v_1, s_0 + v_1) = \left(\frac{1}{10}, \frac{7}{10}\right)$. It is straightforward to compute that either $\ell_1^* = s_1$ or $\ell_1^* = \frac{7}{10}$,* *depending on c_1. In particular, for $c_1 < 18$, there is a unique equilibrium with $\ell_1^* = \frac{7}{10}$. For $c_1 > 18$, there is a unique equilibrium with $\ell_1^* = s_1$. For $c_1 = 18$ there are two equilibria; one in which $\ell_1^* = s_1$ and one in which $\ell_1^* = \frac{7}{10}$.*

The next result provides sufficient conditions guaranteeing uniqueness of equilibrium, and gives necessary and sufficient conditions characterizing this equilibrium, should it exist.

Lemma 2. *Suppose $s_0, 1 - s_1 < s^*$ and either $0 < v_0 \leq v_1 \leq 2s_1 - s_0 - 1$ or $0 < v_1 < v_0 \leq s_1 - 2s_0$. Then there is at most one equilibrium in the stage-2 subgame and $\ell^* \in \Phi(v)$ if and only if,*

$$|v_1 - v_0| < \ell_1^* - \ell_0^* \tag{2}$$
$$\frac{\partial u_0(\ell^*)}{\partial \ell_0} = \frac{v_0}{2} f\left(y_0\left(\ell^*, v\right)\right) - C_0'(\ell_0^* - s_0) = 0 \tag{3}$$
$$\frac{\partial u_1(\ell^*)}{\partial \ell_1} = -\frac{v_1}{2} f\left(y_0\left(\ell^*, v\right)\right) - C_1'(\ell_1^* - s_1) = 0. \tag{4}$$
$$v_i < v_j \implies u_i(\ell^*, v) \geq v_i - C_i\left(\ell_j^* + v_0 - v_1 - s_i\right) \tag{5}$$

Recall from the discussion following Proposition 1 that $s_0, 1 - s_1 < s^*$ implies $s_0 < \frac{1}{3} < \frac{2}{3} < s_1$, meaning that the specializations are sufficiently different and radical. Under this condition, Lemma 2 shows that if the prizes are not too large, then any equilibrium is unique and characterized by (2)-(5). Condition (2) says that the equilibrium occurs in the region of the strategy space where $\pi_i = u_i$. Conditions (3)-(4) are local optimality conditions, ensuring each $\pi_i(\cdot, \ell_j^*, v)$ attains a local maximum at ℓ_i^*. Condition (5) is a global optimality condition, which ensures the low-prize innovator does not have an incentive to deviate to a point at which her payoff jumps upwards.

Thus far in this section, we have treated the prizes as exogenous. In the next section, we analyze the buyer's optimal choice of these prizes. While a stage-2 equilibrium need not exist for every possible prize configuration, it is important to note that there are always prize configurations for which a stage-2 equilibrium *does* exist. Trivially, if $v_0 = v_1 = 0$ then $(s_0, s_1) \in \Phi(v)$. However, our next result shows that there are also non-trivial prize configurations for which an equilibrium exists.

Lemma 3. *If $v_i = 0$ for some i then $\Phi(v) \neq \emptyset$ for all $v_j \geq 0$. Furthermore, if $s_0 < s_1$, then there exist $v_0, v_1 > 0$ such that $\Phi(v) \neq \emptyset$ and $\ell^* \in \Phi(v) \implies \ell_i^* \neq s_i$ for each i.*

Lemma 3 shows that, for any configuration of the specializations, an equilibrium always exists if $v_i = 0$ for some i. Moreover, as long as the suppliers have different specializations, there are strictly positive prizes for which an equilibrium exists and, in any equilibrium, both suppliers choose approaches different from their specializations.

4 The optimal FPC

We now study properties of the optimal FPC. To begin, we set up the buyer's problem.

4.1 The buyer's problem

As discussed in Section 3.3, certain prize configurations may lead to multiple equilibria. In such cases, we assume the suppliers play the equilibrium that is preferred by the buyer. We also note that it is always optimal for the buyer to ensure participation from both suppliers. This is because each supplier's default innovation can be produced at no cost; therefore, the buyer can costlessly benefit from supplier i's participation in the contest.

Given a pair of approaches and prizes, we let $\overline{Q}_b(\ell, v)$ denote the *ex ante* expected quality of the innovation purchased by the buyer:

$$\overline{Q}_b(\ell, v) = q - \sum_i p_i(\ell, v)\mathbb{E}\left[|\ell_i - y| \;\mid i \text{ wins the contest}\right].$$

The buyer's stage-1 expected payoff is, $\tilde{\pi}_b(\ell, v, t) = \overline{Q}_b(\ell, v) - p_0(\ell, v)v_0 - p_1(\ell, v)v_1 + t_0 + t_1$; her objective is to induce an equilibrium $\ell \in \Phi(v)$ by choosing (v, t), so as to maximize $\tilde{\pi}_b$; formally: $\max_{\ell,v,t} \tilde{\pi}_b(\ell, v, t)$ s.t. $v_i \geq 0, \pi_i(\ell, v) - t_i \geq 0, i = 0, 1$ and $\ell \in \Phi(v)$.

Note that in any optimal FPC, the individual rationality constraints, $\pi_i \geq t_i, i = 0, 1$, must bind. If not, the buyer could increase t_i and strictly increase her payoff. We formalize this below (we omit the proof, as it is immediate from the text):

Lemma 4. *In any optimal FPC, $t_i^* = \pi_i(\ell^*, v^*)$, and the suppliers capture no rent.*

Lemma 4 is consistent with moral hazard models under risk neutrality, where agents typically capture no rent, absent additional frictions such as wealth constraints or adverse selection. Consequently, our baseline model avoids the typical rent-extraction/efficiency trade-off, allowing us to focus on a less conventional trade-off concerning *ex ante* vs. *ex post* efficiency. However, in Section 5, we study a variant of the model in which the buyer is subject to an anonymity constraint. In that setting, one supplier may capture a positive rent, and the standard trade-off arises.

Given Lemma 4, we can express the buyer's payoff as

$$\pi_b(\ell, v) = \overline{Q}_b(\ell, v) - C_0(\ell_0 - s_0) - C_1(\ell_1 - s_1),$$

and we may formulate the buyer's problem as

$$\max_{\ell,v} \pi_b(\ell, v) \text{ s.t. } v_0, v_1 \geq 0, \text{ and } \ell \in \Phi(v). \tag{P}$$

To elucidate the *ex ante*/*ex post* efficiency trade-off, it is instructive to relate the buyer's objective function with that of the planner in Section 3.1. See that π_b can be written,

$$\pi_b(\ell, v) = S_{FB}(\ell) - T(\ell, v),$$

where $T(\ell, v) = \overline{Q}_{FB}(\ell) - \overline{Q}_b(\ell, v)$. We refer to T as the buyer's quality distortion cost, and it captures the loss of *ex ante* surplus that results from the buyer's (potentially) socially inefficient *ex post* purchasing decision. To elaborate, note that by definition of $\overline{Q}_{FB}$, $T(\ell, v) \geq 0$. That is, the *ex ante* expected quality of the innovation purchased by the buyer can be no greater than the expected highest-quality innovation. If $v_0 = v_1$, the buyer always purchases the highest-quality innovation, and in this case, $T(\ell, v) = 0$. On the other hand, if $v_0 \neq v_1$ and $\ell_0 \neq \ell_1$, then, with strictly positive probability, the buyer purchases the lower-quality, cheaper, innovation; in this case, $T(\ell, v) > 0$. Moreover, the greater the spread of the prizes, $|v_1 - v_0|$, the more likely it is that the buyer subsequently purchases the lower-quality innovation. For this reason, T, is increasing in the prize spread, $|v_1 - v_0|$.

4.2 Optimal prize structures

This section provides some general features of the structure of the prizes in the optimal FPC. We let $\mathcal{F}_P = \{(\ell, v) | \ell \in \Phi(v), v_0, v_1 \geq 0\}$ denote the buyer's set of feasible choices and we note that by Lemma 3, $\mathcal{F}_P$ is non-empty. We also let $\mathcal{A}_P = \arg\max_{(\ell,v)\in\mathcal{F}_P} \pi_b(\ell, v)$ denote the set of optimal FPCs for the buyer.

Before stating the main results in this section, we provide two preliminary results; the first one shows that a solution to the buyer's problem exists.

Lemma 5. *$\mathcal{A}_P \neq \emptyset$ for all $s_0, s_1 \in (0, 1)$.*

The next result shows that the buyer offers a non-zero prize to i if and only if i's subsequent equilibrium approach is distinct from her specialization.

Lemma 6. *Let* $(\ell^*, v^*) \in \mathcal{A}_P$. *Then,* $v_i^* > 0$ *if and only if* $\ell_i^* \neq s_i$.

Our first main result in this section shows that if the specializations are sufficiently different and radical, the buyer awards a non-zero prize to each supplier. On the other hand, if both specializations are conservative enough, at least one supplier's prize is zero.

Proposition 3. *If the specializations are sufficiently different, the buyer awards a non-zero prize to each supplier, and equilibrium variety decreases relative to the default. If both suppliers' specializations are sufficiently conservative, then at least one supplier's prize is zero. Formally, if* $s_0, 1 - s_1 < s^*$ *then* $(\ell^*, v^*) \in \mathcal{A}_P$ *implies* $v_0^*, v_1^* > 0$ *and* $s_0 < \ell_0^* < \ell_1^* < s_1$. *If* $s^* \leq s_0, 1 - s_1$ *then* $(\ell^*, v^*) \in \mathcal{A}_P$ *implies* $v_j^* = 0$ *for some* j.

When the specializations are sufficiently different ($s_0, 1 - s_1 < s^*$), the buyer wants to elicit more conservative approaches; she can implement such an equilibrium by offering positive prizes to both suppliers. On the other hand, if the suppliers are both sufficiently conservative ($s^* < s_0, 1 - s_1$), the buyer wants greater diversity. As an FPC with two positive prizes tends to *reduce* diversity, there is a conflict between the buyer's objective and the supplier's incentives. But recall from Proposition 2 and Example 1 that the buyer may be able to elicit greater diversity by offering *one* positive prize. As our next result shows, when the specializations are sufficiently close, this is indeed the structure of the optimal FPC.

Proposition 4. *If the specializations are sufficiently close, the buyer awards a positive prize to exactly one supplier and equilibrium variety increases relative to the default. Formally, for each* $s_1 \in (0, 1)$ *there exists* $\epsilon > 0$ *such that for all* $s_0 \in (s_1 - \epsilon, s_1]$, $(\ell^*, v^*) \in \mathcal{A}_P$ *implies* $v_j^* = 0 < v_i^*$ *and* $s_1 - s_0 < |\ell_i^* - s_j| = v_i^*$.

When the specializations are close, the buyer benefits from increased variety, and she can accomplish this by awarding one positive prize that exceeds the spread between the specializations. As outlined in the Introduction, we can interpret this scenario as one in which the buyer negotiates solely with i, using j's default innovation as leverage to induce i to differentiate her approach. It is then natural to ask with which supplier the buyer chooses to negotiate. The next result sheds light on this question.

Proposition 5.

(i) Suppose supplier i *has a cost advantage and the specializations are sufficiently close. Then the buyer awards a non-zero prize only to supplier* i. *Formally, suppose* $C_i' < C_j'$

and $s_0 = 1 - s_1 = s$. *There exists* $\epsilon > 0$ *such that for all* $s \in \left(\frac{1}{2} - \epsilon, \frac{1}{2}\right]$, $(\ell^*, v^*) \in \mathcal{A}_P$ *implies* $v_j^* = 0 < v_i^*$.

(ii) *Consider the uniform/quadratic model and suppose supplier* i *has a quality advantage and the specializations are sufficiently close. Then the buyer awards a non-zero prize only to supplier* i. *Formally, suppose* $C_0 \equiv C_1$. *For each* $s_1 \in (0, 1)$ *there exists* $\epsilon > 0$ *such that for all* $s_0 \in (s_1 - \epsilon, s_1)$ *with* $\left|s_i - \frac{1}{2}\right| < \left|s_j - \frac{1}{2}\right|$, $(\ell^*, v^*) \in \mathcal{A}_P(s_0, s_1)$ *implies* $v_j^* = 0 < v_i^*$.

Part (i) of Proposition 5 shows that if i is more flexible and the specializations are symmetric and sufficiently close, the buyer negotiates only with i. The intuition is straightforward: The buyer wants to induce one supplier to choose an approach different from her specialization, and the more flexible supplier can do so at a lower cost. As the buyer internalizes these costs, she is best off motivating the more flexible supplier.

Part (ii) of Proposition 5 shows that if i has a quality advantage and the specializations are sufficiently close, then in the uniform/quadratic model, the buyer negotiates only with i. This result is more subtle, and there are competing forces at play.

To explain, suppose $s_0 < s_1$ and both specializations are close to 1. Here, the buyer wants one supplier to choose a more conservative approach. To implement a given level of variety, it is more costly to do so by awarding the prize to supplier one since she would necessarily need to adopt an approach further from her specialization. All else equal, this favors awarding the prize to supplier zero. On the other hand, for a given level of variety, more centrally-located approaches tend to generate higher quality for the buyer. More precisely, for any $k \geq 0$ and any two approaches such that $|\ell_1 - \ell_0| = k$, the expected maximum quality is higher when the midpoint between those approaches is closer to $\frac{1}{2}$. All else equal, this favors awarding the prize to supplier one, as the resulting approaches would be more centrally located. For the uniform distribution, the second force is more subdued, and the first force dominates. However, this second force is stronger when the ideal is strictly less likely to occur in more radical regions of the approach space.

To close this section, we first present an example illustrating how the prize structure changes as the specializations vary. We then discuss configurations of specializations not covered by the results in this section.

Example 2. *Consider the uniform/quadratic model and suppose* $C_i(d) = \frac{1}{2}d^2$ *and* $s_0 = 1 - s_1 = s \leq \frac{1}{2}$. *We will examine how the structure of the optimal FPC varies with* s. *First, let* $s < s^* = \frac{1}{4}$. *As will be shown in Lemma 7, the solution to the buyer's problem is the* *solution to* (P') *, which is characterized in the FOCs in* (17)*-*(19)*. Solving this system, we obtain,* $v_0^* = v_1^* = \frac{1}{3} - \frac{4}{3}s > 0$*, and the equilibrium approaches are,* $\ell_0^* = 1 - \ell_1^* = s + \frac{v_i^*}{2} = \frac{1}{6} + \frac{s}{3}$*.*

Next, let $s \geq \frac{1}{4}$*. Using Propositions 3 and 4, we can compute the optimal FPC by comparing the buyer's payoff in two scenarios: The first is her payoff in the default scenario; we denote this payoff by* $\pi_b^{def}(s)$*. It holds,*

$$\pi_b^{def}(s) = q - \int_0^{\frac{1}{2}} |y - s| dy - \int_{\frac{1}{2}}^1 |y - (1 - s)| dy = q + s - 2s^2 - \frac{1}{4}.$$

In the second scenario, the buyer chooses $v_j = 0 < s_1 - s_0 < v_i$ *and* i*'s equilibrium approach satisfies* $|\ell_i - s_i| = v_i$*. As the suppliers are symmetric, it is irrelevant whether the prize is awarded to supplier zero or one, but for concreteness, let's suppose* $v_0 > 0$*. Assuming supplier zero then adopts an approach different from her specialization, there is an equilibrium in which* $\ell_0 = \ell_0(v_0) = s_1 - v_0$*.*[18] *The buyer's payoff is then,*

$$q - \int_0^{\ell_0(v_0)} (\ell_0(v_0) - y) dy - \int_{\ell_0(v_0)}^1 |y - (1 - s)| dy - \frac{1}{2}(\ell_0(v_0) - s)^2. \tag{6}$$

The buyer chooses v_0 *to maximize* (6) *subject to the constraints,* $v_0 \geq s_1 - s_0$ *and* $u_0(\ell_0(v_0), s_1) \geq 0$*. The first constraint ensures that the prize provides the right incentive for supplier zero to differentiate her approach from supplier one. The second constraint ensures that supplier zero is better off choosing an approach different from her specialization (at the optimum, neither constraint binds). Computing the value function associated with this problem and comparing it with* π_b^{def}*, we find the optimal prize structure: Letting* $\overline{s} = \frac{1}{\sqrt{6}} \approx .408$*, for* $s \in \left[\frac{1}{4}, \overline{s}\right)$*,* $v_0^* = v_1^* = 0$ *and* $\ell_i^* = s_i$*. For* $s \in \left(\overline{s}, \frac{1}{2}\right]$*,* $v_1^* = 0 < s_1 - s_0 < v_0^* = \frac{2}{3} - s$*,* $\ell_1^* = s_1$*, and* $\ell_0^* = s_1 - v_0^* = \frac{1}{3}$*.*[19]

Example 2 examines a scenario in which the suppliers are symmetric with $s_0 = 1 - s_1 = s \leq \frac{1}{2}$. As s increases from 0 to $\frac{1}{2}$, the specializations move closer to one another, and the structure of the optimal FPC changes. Consistent with Proposition 3, for $s < \frac{1}{4} = s^*$, $v_0^*, v_1^* > 0$, but when $\frac{1}{4} < s$, $v_j^* = 0$ for at least one supplier, j. In particular, there is $\overline{s} \in \left(\frac{1}{4}, \frac{1}{2}\right)$ such that for $s \in \left[\frac{1}{4}, \overline{s}\right)$, $v_0^* = v_1^* = 0$, whilst for $s \in \left(\overline{s}, \frac{1}{2}\right]$, $v_j^* = 0 < v_i^*$.

To understand the distinction between the latter two sub-cases, recall from the discussion surrounding Proposition 3 that, when the buyer wants to encourage greater diversity, she

[18] For $s < \frac{1}{2}$, this equilibrium is unique. For $s = \frac{1}{2}$, supplier zero is indifferent between choosing $s_1 - v_0$ or $s_1 + v_0$; this is not important, since the buyer would also be indifferent between these two choices.

[19] If $s = \overline{s}$, the buyer is exactly indifferent between choosing $v_0^* = v_1^* = 0$ or $v_1^* = 0 < v_0^*$.

must offer a strictly positive prize to exactly one supplier and that prize must exceed the spread between the specializations: $s_1 - s_0 = 1 - 2s < v_i$. For s close to $\frac{1}{2}$, the specializations are close and the buyer can elicit greater diversity with a small prize, which imposes a small quality distortion cost. In addition, the buyer's default payoff is low. In this case, the benefit from increased diversity outweighs the cost of achieving it. As s decreases towards $\frac{1}{4}$, the buyer's default payoff increases. Additionally, the minimum prize needed to elicit greater variety increases, which increases the quality distortion cost. For s close enough to $\frac{1}{4}$, the benefit from increased variety is outweighed by the cost, and the buyer is better off at the default. The threshold, $\overline{s}$, leaves the buyer indifferent between the default and implementing greater variety.

Finally, it is worth discussing the parameter configurations not covered by the results in this section (for example, if $1 - s_1 < s^* < s_0$ and the specializations are not "too close"). The qualitative conclusions drawn from Proposition 3 and Example 2 extend to such scenarios. For instance, if the specializations are sufficiently different, the optimal FPC specifies positive prizes for both suppliers, and equilibrium variety decreases relative to the default. If the specializations are moderately close, then the structure of the prizes depends on whether the buyer is most concerned with increasing or decreasing variety (relative to the default).

Figure 1 illustrates how the structure of the FPC changes in Example 2 for all possible configurations of the specializations in which supplier zero has a quality advantage; i.e., $s_0 \geq 1 - s_1$ (the case $s_0 \leq 1 - s_1$ is symmetric). The relevant region of the parameter space is the triangle bounded on the left by the line $s_1 = 1 - s_0$ and on the right by the line $s_1 = s_0$. Hence, as we move from left to right in the triangle, the specializations move closer to one another.

In Region 1, the specializations are sufficiently different and the buyer awards a positive prize to both suppliers. In Region 4, the specializations are sufficiently close and the buyer awards a positive prize to only one supplier – supplier zero in this example – and induces greater variety. In Regions 2 and 3, the specializations are moderately close. In Region 2, the buyer awards a positive prize only to supplier one. Different from the single-prize scenario covered in Proposition 4, this prize is sufficiently small (specifically, $v_i < s_1 - s_0$) that it provides supplier one an incentive to adopt a more conservative approach, and equilibrium variety decreases relative to the default. Intuitively, in this region, supplier one's default innovation is of sufficiently low quality that the buyer's primary motivation is to encourage her to adopt a more conservative approach. This can be accomplished by offering her a sufficiently small positive prize. Ideally, the buyer would also like supplier zero to adopt a more radical approach (closer to 0), but such an equilibrium cannot be implemented with two positive prizes. Instead, the buyer awards a prize of zero to supplier zero. In Region 3, both prizes are 0, and the intuition is analogous to the case $s \in [\frac{1}{4}, \overline{s})$ in example 2. That is, the buyer would like to increase diversity, relative to the default, but to do so would require a prize spread that imposes too large of a quality distortion cost on the buyer.

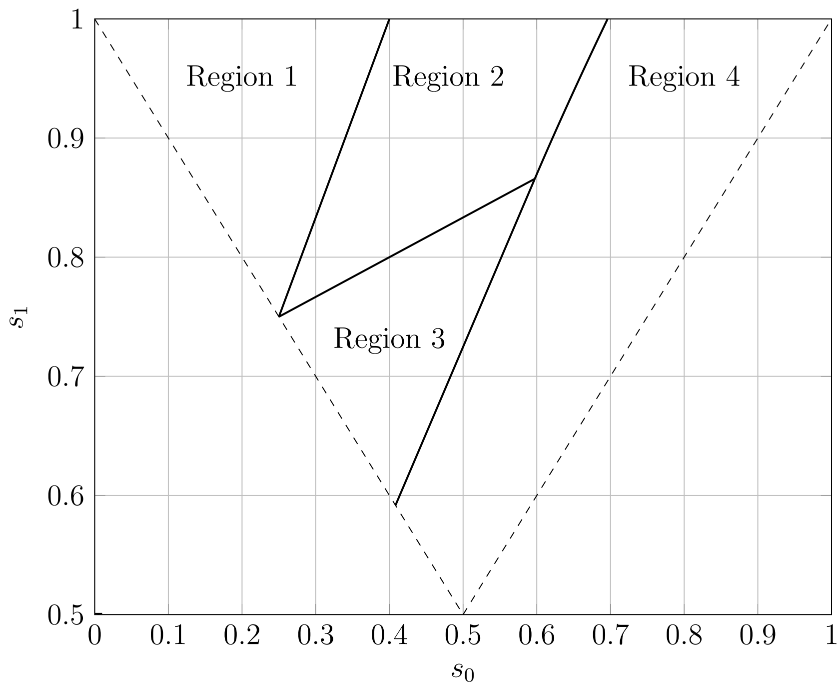

Figure 1: An illustration of how the structure of the prizes in the optimal FPC changes as the specializations vary in Example 2. In Region 1, $v_0^*, v_1^* > 0$. In Region 2, $v_0^* = 0 < v_1^* < s_1 - s_0$. In Region 3, $v_0^* = v_1^* = 0$. And in Region 4, $v_1^* = 0 < s_1 - s_0 < v_0^*$.

4.3 Further analysis and welfare implications

We now examine the welfare implications of moral hazard in our model, and we explore further properties of the optimal FPC when the buyer awards positive prizes to both suppliers. In what follows, we let $\pi_b^* = \pi_b(\ell^*, v^*)$ denote the buyer's *ex ante* expected payoff in the optimal FPC, and let $S^{FB} = S_{FB}(\ell^{FB})$ denote maximized *ex ante* total surplus. We say that the buyer attains the first-best if $\pi_b^* = S^{FB}$. Our first result shows the circumstances under which the buyer can attain the first-best.

Proposition 6. *The following statements are equivalent:*

(i) The buyer attains the first-best with an FPC.

(ii) $C_0'(\ell_0^{FB} - s_0) = C_1'(s_1 - \ell_1^{FB}) \geq 0$.

(iii) There exists $v_0^{FB} = v_1^{FB} \geq 0$ *such that* $\ell^{FB} \in \Phi(v^{FB})$.

Proposition 6 shows that the buyer attains the first-best if and only if the first-best approaches can be implemented by equal prizes; equivalently, marginal costs are equal at the first-best approaches and $s_0 \leq \ell_0^{FB}$, $\ell_1^{FB} \leq s_1$. Intuitively, when the prizes are equal, the quality distortion cost is zero. So, if the first-best can be implemented by equal prizes, the buyer offers such prizes, appropriates the suppliers' surplus through the entry fees, and attains the first-best payoff. The structure of the optimal FPC here is similar to the typical 'franchise contract' that emerges in moral hazard models with risk neutrality (see, e.g., Bolton & Dewatripont, 2004). On the other hand, if ℓ^{FB} cannot be implemented with equal prizes, the buyer must either offer unequal prizes and/or implement different approaches. Either way, she cannot attain the first-best. We now provide a more economically meaningful corollary, which shows one set of circumstances under which the first-best is attainable.

Corollary 1. *If the specializations are sufficiently different and the suppliers are symmetric, the buyer attains the first-best with an FPC. That is, if* $s_0 = 1 - s_1 < s^*$ *and* $C_0 \equiv C_1$*, then,* $\pi_b^* = S^{FB}$.

When the suppliers are symmetric, the first-best can be implemented with equal prizes and, as a result, there is no distortion from the social optimum. Note that symmetry of the distribution is important for Corollary 1, but is not critical for Proposition 6. That is, whenever the first-best can be implemented with equal prizes, the optimal FPC will be socially optimal. But with an asymmetric distribution, this need not correspond to a setting with symmetric suppliers. In the remainder of this section, we provide further insights into the design of the optimal FPC when the specializations are sufficiently different but the first-best cannot be attained.

To begin, we discuss some technical points that will greatly simplify the analysis. By Proposition 2, if $\ell \in \Phi(v)$ and $\ell_i \neq s_i$ for each i then $|v_1 - v_0| < \ell_1 - \ell_0$. This means $\overline{Q}_b(\ell, v) = q - \kappa(\ell, v)$, where

$$\kappa(\ell, v) = \int_0^{y_0(\ell,v)} |\ell_0 - y| dF(y) + \int_{y_0(\ell,v)}^1 |\ell_1 - y| dF(y).$$

In addition, ℓ must satisfy the first-order conditions given in equations (3) and (4). Motivated by these facts, let

$$L(\ell, v) = \kappa(\ell, v) + C_0(\ell_0 - s_0) + C_1(\ell_0 - s_0),$$

and consider the following auxiliary problem for the buyer:

$$\max_{\ell, v} q - L(\ell, v) \text{ s.t. } v_0, v_1 \geq 0, |v_1 - v_0| \leq 1 \text{ and } \frac{\partial u_i(\ell, v)}{\partial \ell_i} = 0, \quad i = 0, 1. \tag{P'}$$

Similar to the first-order approach used to characterize optimal contracts in principal-agent models, the problem (P') simplifies (P) by replacing the equilibrium constraint with the first-order conditions in (3)-(4).[20] Our next result shows that the solutions to (P') and (P) coincide when $s_0, 1 - s_1 < s^*$.

Lemma 7. *If $s_0, 1 - s_1 < s^*$, then, (ℓ^*, v^*) solves (P) if and only if (ℓ^*, v^*) solves (P').*

Lemma 7 greatly simplifies the analysis, as we can study the buyer's problem, (P), by analyzing the simpler problem, (P'). This is not obvious because the feasible sets (and, in general, the objective functions) in the two problems do not coincide. Nevertheless, we exploit this fact to characterize the solution to the buyer's problem in Appendix B.1 (see equations (17)-(19)). Here, we focus on the key ideas.

Proposition 7. *Suppose $s_0, 1 - s_1 < s^*$ and $C_i'\left(\left|\ell_i^{FB} - s_i\right|\right) < C_j'\left(\left|\ell_j^{FB} - s_j\right|\right)$. Then $(\ell^*, v^*) \in \mathcal{A}_P$ implies $v_i^* < v_j^*$ and $\left|\ell_i^* - \frac{1}{2}\right| < \left|\ell_j^* - \frac{1}{2}\right|$. Moreover, $\left|\ell_i^* - \frac{1}{2}\right| < \left|\ell_i^{FB} - \frac{1}{2}\right|$ while $\left|\ell_j^{FB} - \frac{1}{2}\right| < \left|\ell_j^* - \frac{1}{2}\right|$.*

Proposition 7 describes some key features of the optimal FPC and shows how the approaches are distorted when the social optimum is unattainable. In particular, when $C_i'\left(\left|\ell_i^{FB} - s_i\right|\right) < C_j'\left(\left|\ell_j^{FB} - s_j\right|\right)$, supplier i is awarded a smaller prize and adopts a more conservative approach than supplier j. Moreover, relative to the first-best, i adopts an approach that is too conservative and too far from her specialization, while j adopts an approach that is too radical and too close to her specialization.

To provide the intuition, first note that under the hypotheses of Proposition 7, the buyer *could* implement the first-best approaches by offering unequal prizes, $v_i^{FB} < v_j^{FB}$. However, doing so requires a prize spread that results in a high-quality distortion cost. Instead, relative to the first-best approaches, the buyer induces i to choose an approach further from her specialization and induces j to choose an approach closer to her specialization. Doing

[20] The additional constraint that $|v_1 - v_0| \leq 1$ is made purely for technical reasons. It is shown in Appendix B that this constraint never binds.

so enables the buyer to reduce the prize spread, which limits the distortion cost. Also note that since one supplier's approach is more conservative than her first-best approach, while the other's is more radical, it is not clear how the equilibrium level of diversity compares with the first-best. Indeed, the comparison is generally ambiguous, and it is not difficult to construct examples wherein equilibrium diversity is greater, lower, or equal to the first-best level.

Following on from Proposition 7, the next corollary provides a more economically meaningful statement revealing the nature of the distortion that arises. In what follows, we say that supplier i has an advantage over j if she is both more flexible and conservative than j: $C_i' \leq C_j'$ and $\left|s_i - \frac{1}{2}\right| \leq \left|s_j - \frac{1}{2}\right|$, where at least one inequality is strict.

Corollary 2. *If the specializations are sufficiently different and supplier i has an advantage over j then i receives a smaller prize and chooses a more conservative approach than j. Moreover, i's equilibrium approach is more conservative than her first-best approach, whilst j's is more radical. Formally, suppose $s_0, 1 - s_1 < s^*$. If $\left|s_i - \frac{1}{2}\right| \leq \left|s_j - \frac{1}{2}\right|$, and $C_i' \leq C_j'$, where at least one of the inequalities is strict, then $(\ell^*, v^*) \in \mathcal{A}_P$ implies, $v_i^* < v_j^*$ and $\left|\ell_i^* - \frac{1}{2}\right| < \left|\ell_j^* - \frac{1}{2}\right|$. Moreover, $\left|\ell_i^{FB} - \frac{1}{2}\right| < \left|\ell_i^* - \frac{1}{2}\right|$ while $\left|\ell_j^* - \frac{1}{2}\right| < \left|\ell_j^{FB} - \frac{1}{2}\right|$.*

As Corollary 2 suggests, if i has an advantage over j then $C_i'\left(\left|\ell_i^{FB} - s_i\right|\right) < C_j'\left(\left|\ell_j^{FB} - s_j\right|\right)$, and the conclusion of Proposition 7 follows.

Corollary 2 is reminiscent of a common theme in contest design, which is 'leveling the playing field' – i.e., placing contestants on equal footing to raise the intensity of competition (see, e.g., Chowdhury et al., 2023). However, the motivation behind awarding a larger prize to the disadvantaged supplier in our context is driven by very different considerations; namely, the buyer's motivations to balance a trade-off between raising individual expected qualities, maintaining variety, and limiting costs. Suppose, for example, i has a quality advantage over j. By definition, this means i's default innovation is closer to $\frac{1}{2}$ – the innovation that maximizes expected individual quality. Due to diminishing returns, at the default, a marginally more conservative approach for i yields a lower marginal increase in her expected quality, as compared to j. Driven by this fact, the buyer wants to motivate j to move further from her specialization than i, and this calls for a greater prize for j.

4.4 Contractible rank-order

As we showed in the last section, when one supplier is advantaged, the buyer is unable to attain the first-best with an FPC. Driving the inefficiency is not moral hazard *per se*. Rather, it is the buyer's inability to commit to an *ex post* socially efficient purchasing rule. In this section, we formalize this intuition. In the spirit of Lazear & Rosen (1981), we examine a model in which the rank-order of qualities is contractible, and the buyer can commit to a purchasing rule. We show that the buyer can achieve the first-best with an FPC for any specializations such that $s_0, 1 - s_1 < s^*$.

Formally, given the realized qualities, let $z_0(Q) = \mathbb{1}(Q_0 > Q_1)$ be an indicator function equal to 1 if $Q_0 > Q_1$ and equal to 0 otherwise. We assume that $z_0(\cdot)$ is contractible. In stage 1, the buyer chooses the contest parameters, (v, t), and commits to a purchasing rule, $\omega : \{0, 1\} \to \{0, 1\}$, which is a mapping from the realizations of z_0 to the identity of the supplier from whom the buyer will purchase.[21]

Proposition 8. *If $s_0, 1 - s_1 < s^*$ and the rank-order of qualities is contractible, then the buyer attains the first best with an FPC.*

Consider the context of a procurement design competition, as discussed in the Introduction. Proposition 8 suggests that buyers might benefit from committing to evaluate design submissions anonymously – that is, without regard to the identities (and therefore the prize values) attached to the submissions.

5 Optimal contests

In this section, we study the design of optimal contests, where a contest now takes the more general form considered by Che & Gale and L&S. Specifically, we allow the buyer to offer each supplier a menu of allowable prices, $\mu_i \subseteq \mathbb{R}_+$. After the approaches are chosen, the qualities are realized and observed by both suppliers, who then simultaneously choose prices from their respective menus. After the prices are chosen, the buyer purchases from the supplier offering her the greatest surplus.

The game proceeds in four stages: In stage 1, the buyer chooses the menus and entry fees/subsidies. In stage 2, the suppliers make their entry decisions, and each participating supplier chooses their approach. In Stage 3, the suppliers observe the realized qualities and choose a price from their menus. In stage 4, the buyer chooses which innovation to purchase.

We maintain the same tie-breaking rules as in the baseline model, and we will consider two versions of the model regarding the buyer's ability to discriminate (i.e., specify different

[21] Note that the buyer's purchasing rule does not distinguish between the case where $Q_0 < Q_1$ and $Q_0 = Q_1$. Moreover, we have restricted attention to deterministic purchasing rules. Certainly, both of these simplifications can be relaxed. But for our purposes, restricting attention to these simple rules is sufficient to show that the buyer can attain the first best.

contest parameters for the suppliers). First, we allow for identity-dependent subsidies/entry fees, and we show that, quite generally, an auction is an optimal contest and the buyer attains the first-best. We then consider a variant in which the buyer faces an anonymity constraint and is required to set the same contest parameters for the two suppliers. In that environment, although the auction implements the first-best, it may *not* be an optimal contest; in particular, the buyer may be better off using an FPC.

5.1 Discriminatory contests

To begin, we allow the buyer to set different entry-fees for the two suppliers; we will show that, for any pair of specializations, an auction in which $\mu_i = \mathbb{R}_+$ is an optimal contest and the buyer attains the first-best. In fact, as will be seen, this result is quite robust and holds independently of several earlier assumptions on the distribution of the ideal and suppliers' costs. In particular, symmetry of the distribution, as well as differentiability, convexity, and continuity of the cost functions, are not crucial for the result.

So let $\mu_i = \mathbb{R}_+$ for each i. We begin with the stage-3 and stage-4 subgames, noting that in these subgames, the suppliers' costs are sunk and the ideal has been realized. Hence, the equilibrium in this subgame is independent of the cost functions and distribution of the ideal. Then, the equilibrium is exactly as characterized by L&S: The stage-3 equilibrium pricing strategy of supplier i is, $\sigma_i(\ell, y) = \max\{Q_i(\ell_i, y) - Q_j(\ell_j, y), 0\}$. And in stage 4, the buyer purchases from i whenever $Q_i(\cdot) > Q_j(\cdot)$.

Now consider stage 2. Assuming equilibrium behavior in subsequent stages, supplier i's expected payoff is, $\pi_i(\ell) = \mathbb{E}[\sigma_i(\ell, y)] - C_i(\ell_i - s_i)$. Letting $p_i(\ell) = \text{pr}(Q_i(\cdot) > Q_j(\cdot))$, and, suppressing the arguments of all functions for ease of exposition, we have $\mathbb{E}[\sigma_i] = p_i\mathbb{E}\left[Q_i | Q_i > Q_j\right] - p_i\mathbb{E}\left[Q_j | Q_i > Q_j\right]$. Adding and subtracting $p_j\mathbb{E}[Q_j|Q_j > Q_i]$, $\mathbb{E}[\sigma_i]$ can be written, $\mathbb{E}[\sigma_i] = \mathbb{E}[\max\{Q_0, Q_1\}] - \mathbb{E}[Q_j]$; hence, i's payoff can be written,

$$\pi_i(\ell) = \overline{Q}_{FB}(\ell) - \mathbb{E}\left[Q_j\left(\ell_j, y\right)\right] - C_i(\ell_i - s_i) = S_{FB}(\ell) - \mathbb{E}\left[Q_j\left(\ell_j, y\right)\right].$$

As $Q_j(\cdot)$ is independent of ℓ_i, it is easy to see that the stage-2 subgame is an exact potential game with potential, S_{FB}; that is, $\pi_i(\ell'_i, \ell_j) - \pi_i(\ell_i, \ell_j) = S_{FB}(\ell'_i, \ell_j) - S_{FB}(\ell_i, \ell_j)$ for all $\ell'_i, \ell_i, \ell_j \in [0, 1]$. It follows that if $\ell^{FB} \in \arg\max_\ell S_{FB}(\ell)$, then, ℓ^{FB} is an equilibrium profile in the stage-2 subgame (see Monderer & Shapley, 1996). By setting $t_i = \pi_i(\ell^{FB})$, the buyer appropriates each supplier's surplus, and the next result is immediate:

Proposition 9. *An auction is an optimal contest, and the buyer attains the first-best.*

L&S show that an auction implements the socially optimal approaches when all approaches share the same fixed cost (see their Proposition 2). Here we have shown why the auction implements the socially optimal approaches, and why this is independent of the specializations, cost functions, and distribution of the ideal. More intuitively, in the auction – as in the FPC – each supplier benefits from having a higher-quality innovation, as this raises her chance of victory. At the same time, the auction incentivizes the suppliers to differentiate their approaches, as similar approaches lead to small quality differences and more intense price competition at stage 3. Consequently, each supplier chooses their approach to strike a balance between raising their individual quality, maintaining diversity, and limiting costs – precisely the trade-offs resolved by the first-best approaches.

L&S do not allow for entry fees and show that, although the auction implements the first-best, it tends to leave large, positive rents to the suppliers, rendering it inferior to the FPC in some circumstances. In the next section, we study a related issue that arises when the buyer faces an anonymity constraint.

5.2 Auctions vs FPCs under anonymity

As we elaborate upon in Section 6, unequal treatment of the suppliers may not be feasible for a number of practical and legal reasons. To understand the implications, in this section, we consider a variant of the model in which the buyer faces an anonymity constraint, requiring that the prizes/menus/entry-fees are the same for both suppliers. First, we focus on environments where the specializations are sufficiently different, but one supplier is advantaged. We discuss how the anonymity constraint affects the buyer in both the auction and FPC, and we then compare the performance of the two. In order to obtain clear-cut welfare comparisons, the results in this section are based on the linear/quadratic model, but many of the insights discussed below hold more generally.

Adding additional constraints clearly cannot help the buyer, regardless of the contest structure she uses. However, their impact differs in the auction and the FPC. In the auction, the buyer loses the ability to fully extract rents from the suppliers, but, as the menus are unaffected by the constraint, total surplus remains at the first-best level. The buyer sets the entry fee to fully extract the rent from the disadvantaged supplier, leaving a positive rent for the advantaged supplier, equal to the difference in the suppliers' payoffs.

In the FPC, the entry fees are set in an analogous manner to the auction, but as the relinquished rent is an increasing function of the prize, this leads to a distortion from the social optimum. In particular, the prize is suboptimally small (from a welfare perspective), and the suppliers choose approaches that are closer to their specializations than is socially optimal. In the linear/quadratic model, the magnitudes of these distortions are greater than the distortions that emerge without an anonymity constraint. As a consequence, total surplus falls, relative to that scenario.

Comparing the performance of the FPC and the auction under anonymity, the auction generates a greater total surplus, but it leaves a greater rent to the advantaged supplier than does the FPC. Intuitively, in the FPC, the buyer limits the relinquished rent by reducing the size of the prize. On balance, in the linear/quadratic model, the buyer prefers the FPC.

The next result formalizes the points made in the discussion thus far. In what follows, we say that i has a singular advantage over j if she has either a cost or quality advantage.

Proposition 10. *Consider the uniform-quadratic model. Suppose $s_0, 1-s_1 < s^*$ and supplier i has a singular advantage over j. Under an anonymity constraint, in both the auction and FPC, the disadvantaged supplier captures no rent, while the advantaged supplier captures a positive rent. In addition,*

(i) In the auction: Total surplus is equal to the first-best.

(ii) In the FPC: Total surplus is less than the first-best and each supplier's approach is closer to her specialization than is her first-best approach (i.e., $|\ell_i^* - s_i| < \left|\ell_i^{FB} - s_i\right|$ for each i). Moreover, total surplus decreases, relative to the discriminatory FPC.

(iii) The buyer prefers an FPC to the auction.

Whilst the FPC outperforms the auction in some circumstances, we note that the comparison between the two is, in general, ambiguous. Consider, for instance, a setting in which $s_0 = s_1 = \frac{1}{2}$ and the suppliers are equally flexible. As an anonymous FPC cannot induce greater variety, the buyer awards prizes of zero, and both suppliers produce at their specializations. In the auction, the suppliers choose the first-best approaches, and their expected payoffs are equal. The buyer can therefore attain the first-best by setting a single entry fee/subsidy, which fully extracts the suppliers' rent.

6 Conclusion and Discussion

This paper adds to a new literature on the use of contests to encourage product diversity by exploring an environment in which suppliers are specialized. Our main results show: (1) An

FPC is an optimal means of eliciting variety when the specializations are sufficiently different and no supplier is inherently advantaged. (2) If one supplier has an inherent advantage, an FPC leads to a distortion from the social optimum. In this case, the prizes are chosen in such a way as to reduce the likelihood that the buyer's *ex post* purchasing decision is socially inefficient. This distortion can be eliminated if the buyer is able to commit to purchasing the highest-quality innovation *ex post*. (3) Provided that unequal treatment of the suppliers is feasible, auctions are an optimal means of eliciting variety. But if the buyer faces an anonymity constraint, an FPC may be preferable, so long as the specializations are sufficiently different.

We've seen that when the specializations are close, FPCs can be used to induce greater variety, but their efficacy is limited. In practice, this may not pose a significant hindrance to their use. This is because buyers often limit participation in the final competition through an initial screening process (Fullerton & McAfee, 1999). If product variety is a first-order concern, buyers might use such processes to identify suppliers with different areas of expertise. Indeed, an interesting avenue for future work might be to explore optimal mechanisms for selecting the 'right' set of suppliers in this context.

We now briefly discuss the relevance of some key modeling assumptions:

Unequal treatment of the suppliers: In the baseline model, we allow the prizes/entry-fees to depend on the identities of the suppliers. This assumption may be more realistic in some settings than others. Identity-dependent prizes are unlikely to be found in large open-call contests for a variety of practical reasons. But, as discussed by Lu et al. (2022), in the context of procurement design competitions, it is entirely reasonable for buyers to offer different prices to different suppliers.[22] Design competitions are also commonly used in the public sector, and government agencies may face greater scrutiny on the unequal treatment of suppliers. Nevertheless, it may be legal if based on inherent differences between suppliers. In the UK, for example, the Procurement Act of 2023 states that, "...a contracting authority must treat suppliers the same *unless a difference between the suppliers justifies different treatment*" (emphasis added).[23] Moreover, some countries explicitly grant preferential treatment to certain firms – namely, small businesses – in the procurement process. In China, for example, small businesses are granted a price preference between 6-10%, and similar provisions exist in Mexico and Korea (OECD, 2018). In the U.S., preferential treatment is afforded to small businesses through the use of targets – which specify a certain fraction of contracts that

[22] See also Gürtler & Kräkel (2010) for concrete examples in the context of labor market tournaments.
[23] See, Section 12 https://www.legislation.gov.uk/ukpga/2023/54

should be awarded to small businesses (currently 23%) – and set-asides – competitions that are only open to small businesses. However, these practices are country-specific and are prohibited in some jurisdictions, including the European Union (OECD, 2018).

Entry fees: In our model, the optimal FPC involves entry fees, which are not commonly observed in practice. Without these entry fees, the buyer cannot extract the suppliers' rents, and must then balance the usual trade-off between rent extraction and efficiency.

On the other hand, subsidies are commonly observed, particularly in public procurement (Lichtenberg, 1990). The fact that entry fees, rather than subsidies, emerge in our model is driven by our assumption that there are no fixed costs of entry; that is, suppliers can costlessly produce at their specialization. If we were to introduce positive fixed costs and allow subsidies (but not entry fees), our results would be unaffected provided that (i) it is always optimal for the buyer to induce entry from both suppliers; and (ii) the optimal subsidy is strictly positive for both suppliers.

However, for sufficiently high fixed costs, it may not be optimal for the buyer to subsidize the entry of both suppliers, and the optimal contest must account for the possibility of exclusion. Unlike the case with two entrants/one positive prize that emerges in our model, when one supplier, say j, is excluded entirely from the contest, the buyer loses access to her default innovation. This eliminates the buyer's leverage over supplier i, and there is no way for her to induce i to choose an approach different from her specialization. Hence, if it is optimal for the buyer to exclude a supplier from the contest, then she simply purchases one supplier's default innovation at cost.

Symmetry of the distribution: Symmetry of the distribution is important for our finding that FPCs induce the first-best approaches when suppliers are symmetric (Corollary 1), but is not critical for Proposition 6. That is, with an asymmetric distribution, the buyer could attain the first-best payoff whenever the first-best approaches can be implemented with equal prizes. If the suppliers were symmetric but the distribution was asymmetric, there would be no reason to think that the first-best approaches could be induced with equal prizes.

Assumption on costs: Assumption 2(ii) is important for establishing the equivalence between (P) and (P') in Lemma 7. In particular, Assumption 2(ii) ensures that, at the solution to (P'), the equilibrium condition (5) is non-binding. Without this assumption, the Buyer's optimal FPC (when $s_0, 1-s_1 < s^*$) could be characterized by solving an alternative problem in which the problem (P') is augmented by adding the additional constraint corresponding to (5). In general, it is challenging to determine if and when this constraint binds. Assumption 2(ii) is not relevant for the circumstances under which the FPC implements the first-best, nor is it crucial to our conclusion that the buyer can always elicit variety with an FPC. Indeed, so long as costs are not identically zero, both of these conclusions hold.

Adverse Selection: In this study, we followed L&S and focused on moral hazard as the primary contracting friction, abstracting from adverse selection. In particular, in our model, the suppliers' specializations are known by the buyer. We believe this assumption is most reasonable in the context of a long-term business relationship, as one would expect a buyer to have an understanding of the core competencies of their suppliers. In other contexts – e.g., that of a new business relationship – it is not difficult to imagine that a supplier's specialization may be her private information. In this case, we would expect further distortions from the first-best as a consequence of the usual rent extraction/efficiency trade-off. But how this trade-off interacts with the *ex ante* vs. *ex post* efficiency trade-off that arises in our model is not obvious, and is an interesting question for future research.

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A Proofs

Proof of Lemma 1. We prove parts (ii)-(iii); part (i) follows from Lemma A1 in L&S. Let $d \in [0,1]$; see that $C_j'(d) = \int_0^d C_j''(x)dx > \int_0^d 2f\left(\frac{1}{2}\right)dx = 2df\left(\frac{1}{2}\right)$, where the inequality holds by Assumption 2(ii). Next, see that, $|f'(y)| < 2f(0) \leq 2f\left(\frac{1}{2}\right) \leq C_j''(d)$, where the first inequality holds by Assumption 2(i); the second by Assumption 1 and the third by Assumption 2(ii).

□

Proof of Proposition 1. We first show that there is a unique solution to (FB-P). Note that this problem can be equivalently written, $\min_\ell L_{FB}(\ell)$, where

$$\begin{aligned} L_{FB}(\ell) = &\int_0^{\ell_0} (\ell_0 - y)\,\mathrm{d}F(y) + \int_{\ell_0}^{m(\ell)} (y - \ell_0)\,\mathrm{d}F(y) + \int_{m(\ell)}^{\ell_1} (\ell_1 - y)\,\mathrm{d}F(y) \\ &+ \int_{\ell_1}^{1} (y - \ell_1)\,\mathrm{d}F(y) + C_0(\ell_0 - s_0) + C_1(\ell_1 - s_1). \end{aligned}$$

Let $H(\ell)$ denote the Hessian of L_{FB} evaluated at ℓ:

$$H(\ell) = \begin{bmatrix} 2f(\ell_0) - \frac{1}{2}f\left(m\left(\ell\right)\right) + C_0''(\ell_0 - s_0) & -\frac{1}{2}f\left(m\left(\ell\right)\right) \\ -\frac{1}{2}f\left(m\left(\ell\right)\right) & 2f(\ell_1) - \frac{1}{2}f\left(m\left(\ell\right)\right) + C_1''(\ell_1 - s_1) \end{bmatrix}.$$

By Lemma 1, the terms on the main diagonal of $H(\ell)$ are strictly positive. Lemma 1 also implies that for each i, $2f(\ell_i) - \frac{1}{2}f\left(m\left(\ell\right)\right) + C_i''(d_i) > \frac{1}{2}f\left(m\left(\ell\right)\right)$, which, in turn, implies $H(\ell)$ is positive definite and, hence, L_{FB} is strictly convex in (ℓ_0, ℓ_1). Therefore, there is a unique solution to (FB-P). It is also straightforward to verify that $\frac{\partial L_{FB}(\ell)}{\partial \ell_i}|_{\ell_i=0} < 0 < \frac{\partial L_{FB}(\ell)}{\partial \ell_i}|_{\ell_i=1}$, which means that $\ell_0^{FB}, 1-\ell_1^{FB} \in (0,1)$. Then, the following FOCs are necessary and sufficient for characterizing ℓ^{FB}:

$$\frac{\partial L_{FB}}{\partial \ell_0} = 2F(\ell_0^{FB}) - F\left(m\left(\ell^{FB}\right)\right) + C_0'(\ell_0^{FB} - s_0) = 0 \tag{7}$$

$$\frac{\partial L_{FB}}{\partial \ell_1} = 2F(\ell_1^{FB}) - 1 - F\left(m\left(\ell^{FB}\right)\right) + C_1'(\ell_1^{FB} - s_1) = 0. \tag{8}$$

Let

$$\Gamma(x,z) = 2F(x) + 2F(z) - 2F(m(x,z)) - 1. \tag{9}$$

It is useful to note that for all $x, z \in (0,1)$, Γ is strictly increasing in (x,z) and that

$$\operatorname{sgn}(\Gamma(x,z)) = \operatorname{sgn}(x+z-1). \tag{10}$$

Now suppose $s_0, 1-s_1 < s^*$. We show that $s_0 < \ell_0^{FB}$ and $\ell_1^{FB} < s_1$. Proceeding by contradiction, first suppose $s_0 \geq \ell_0^{FB}$ and $\ell_1^{FB} \geq s_1$. Then, we have $\ell_0^{FB} < s^*$ and $\ell_1^{FB} > 1-s^*$. (7) implies, $0 = 2F(\ell_0^{FB}) - F(m(\ell^{FB}) + C_0'(\ell_0 - s_0) < 2F(s^*) - F\left(\frac{1}{2}\right) = 0$, where the inequality follows by strict convexity of $L_{FB}(\cdot, \ell_1)$ and since $\frac{\partial L_{FB}}{\partial \ell_0}$ is strictly decreasing in ℓ_1. The final equality holds by definition of s^*. We have a contradiction; therefore, we cannot have $s_0 \geq \ell_0^{FB}$ and $\ell_1^{FB} \geq s_1$.

Next, suppose $\ell_0^{FB} \leq s_0$ and $\ell_1^{FB} < s_1$. Adding (7) and (8), re-arranging, and using the fact that $C'(x) = -C'(-x)$ we have, $\Gamma(\ell) = C_1'(s_1 - \ell_1^{FB}) + C_0'(s_0 - \ell_0^{FB}) > 0$. (10) then implies $m(\ell^{FB}) > \frac{1}{2}$. (7) then implies, $0 = 2F(\ell_0^{FB}) - F(m(\ell^{FB}) + C_0'(\ell_0^{FB} - s_0) < 2F(s^*) - \frac{1}{2} = 0$, yielding a contradiction. The logic for the case where $\ell_0^{FB} > s_0$ and $\ell_1^{FB} \geq s_1$ is similar. This establishes that $\ell_0^{FB} > s_0$ and $\ell_1^{FB} < s_1$. The proof for the case where $s_0, 1-s_1 > s^*$ is analogous.

□

Proofs for Section 3.3

Before proving the main results in this section, we provide two auxiliary results. We also point out that Proposition 2 follows by Proposition A.1.

Proposition A.1. *Let $v_0, v_1 \geq 0$ and suppose $\ell^* \in \Phi(v)$; let $d_i^* = |\ell_i^* - s_i|$. (i) If $|v_1 - v_0| \geq 1$, then, $d_0^* = d_1^* = 0$. (ii) If $d_i^* > 0$ for some i, then, $|v_1 - v_0| \leq |\ell_1^* - \ell_0^*|$. (iii) If $d_0^*, d_1^* > 0$, then, $|v_1 - v_0| < \ell_1^* - \ell_0^*$, $\ell_0^* \in (s_0, s_0 + \frac{v_0}{4})$, $\ell_1^* \in (s_1 - \frac{v_1}{4}, s_1)$ and $\frac{\partial u_i(\ell^*, v)}{\partial \ell_i} = 0$ for $i = 0, 1$. (iv) If $v_0, v_1 > 0$ and $d_i^* > 0 = d_j^*$, then, $v_0 = v_1$ and $s_j = \frac{1}{2} = \ell_i^*$ (v) If $v_0, v_1 > 0$, $s_0, s_1 \in (0,1) \setminus \left\{\frac{1}{2}\right\}$ and $|v_1 - v_0| \geq s_1 - s_0$, then $d_0^* = d_1^* = 0$. (vi) If $v_i > 0 = v_j$ and $v_i \geq s_1 - s_0$, then, $\ell_j^* = s_j$ and $\ell_i^* \in \{s_i, s_j - v_i, s_j + v_i\}$. (vii) If $d_0^* > 0$ $[d_1^* > 0]$, $s_0, 1 - s_1 \in \left(0, \frac{1}{2}\right)$, and $|v_1 - v_0| < s_1 - s_0$, then, $|v_1 - v_0| \leq \ell_1^* - \ell_0^*$ and $\frac{\partial u_0(\ell^*)}{\partial \ell_0} \geq 0$ $\left[\frac{\partial u_1(\ell^*)}{\partial \ell_1} \leq 0\right]$.*

Proof. Part (i): Suppose $|v_1 - v_0| \geq 1$; let j denote the identity of the player with the smaller prize. For all $\ell_0, \ell_1 \in [0,1]$ and $y \in (0,1)$, it holds, $|\ell_j - y| + v_j < |\ell_i - y| + v_i$, which means $p_j(\ell, v) = 1$. It follows that $\ell_0 = s_0$ and $\ell_1 = s_1$ are dominant strategies for suppliers 0 and 1, respectively.

Part (ii): Let $\ell^* \in \Phi(v)$ and suppose $d_i^* > 0$. Proceed by contradiction and suppose $|\ell_1^* - \ell_0^*| < |v_1 - v_0|$. We will consider the case where $v_0 \leq v_1$; the case where $v_0 > v_1$ is analogous. First see that $|\ell_1^* - \ell_0^*| < v_1 - v_0$ implies $p_1(\ell^*) = 0$. Suppose $d_1^* > 0$; since supplier one wins with probability zero, she has a profitable deviation to $\ell_1' = s_1$, as this will not reduce her expected earnings, but will strictly lower her cost. Next, suppose $d_0^* > 0$. Then, there exists $\ell_0' \neq \ell_0^*$ such that $|\ell_0' - s_0| < d_0^*$ and $|\ell_0' - \ell_1^*| < v_1 - v_0$. A deviation to ℓ_0' does not affect supplier zero's earnings but strictly reduces her cost, and therefore strictly increases her payoff. At least one player has a profitable deviation, which contradicts the hypothesis that $\ell^* \in \Phi(v)$.

Part (iii): Let $\ell^* \in \Phi(v)$ with $d_0, d_1^* > 0$. We'll first show that $|v_1 - v_0| < |\ell_1^* - \ell_0^*|$. WLOG suppose $v_1 \geq v_0$. Proceed by contradiction and suppose $v_1 - v_0 \geq |\ell_1^* - \ell_0^*|$. Part (ii) implies $v_1 - v_0 = |\ell_1^* - \ell_0^*|$. Now if $\ell_1^* \in \{0, 1\}$, then, $p_1(\ell^*, v) = 0$, and supplier one has a profitable deviation to $\ell_1' = s_1$, as this does not decrease here expected earnings but strictly reduces her cost. This contradicts the definition of equilibrium, so it must be that $\ell_1^* \in (0, 1)$. Then, suppose $\ell_0^* \leq \ell_1^*$. Then, $p_0(\ell^*) < 1$. And, $\pi_0(\ell^*) = u_0(\ell^*) = v_0 p_0(\ell^*) - C_0(\ell_0^* - s_0) < v_0 - C_0(\ell_0^* - s_0) = \lim_{\ell_0 \downarrow \ell_0^*} \pi_0(\ell_0, \ell_1^*)$, which means supplier zero has a profitable deviation to some $\ell_0' > \ell_0^*$, contradicting the definition of equilibrium. A similar contradiction is reached if $\ell_0^* > \ell_1^*$, as supplier zero has a profitable deviation to some $\ell_0' < \ell_0^*$. This establishes that $|v_1 - v_0| < |\ell_1^* - \ell_0^*|$.

Next, we show that $\ell_1^* \geq \ell_0^*$. Proceed by contradiction and suppose $\ell_0^* > \ell_1^*$. Since $s_0 \leq s_1$, we must have either $\ell_0^* > s_0$ or $\ell_1^* < s_1$. Suppose $\ell_1^* < s_1$. By the first part of this proof, we must have $|v_1 - v_0| < \ell_0^* - \ell_1^*$ and so in some neighborhood of ℓ_1^*, $\pi_1(\ell_0^*, \ell_1) = v_1 F(\tilde{y}_0(\ell_0^*, \ell_1, v)) - C_1(\ell_1 - s_1)$, where $\tilde{y}_0(\ell, v) = \frac{\ell_0 + \ell_1 + v_0 - v_1}{2}$. See that $\frac{\partial \pi_1(\ell^*, v)}{\partial \ell_1} = \frac{v_1}{2} f(\tilde{y}_0(\ell^*, v)) - C_1'(\ell_1^* - s_1) > 0$, where the inequality follows since $\ell_1^* < s_1 \implies C_1'(\ell_1^* - s_1) < 0$. Hence, supplier one has a profitable deviation to some $\ell_1' > \ell_1^*$, contradicting the definition of equilibrium. If $\ell_0^* > s_0$, then a similar line of reasoning reveals that supplier zero has a profitable deviation to some $\ell_0' < \ell_0^*$. This establishes that $\ell_1^* \geq \ell_0^*$.

Finally, we show that $\ell_0^* \in (s_0, s_0 + \frac{v_0}{4})$, $\ell_1^* \in (s_1 - \frac{v_1}{4}, s_1)$ and, for each i, $\frac{\partial u_i(\ell^*, v)}{\partial \ell_i} = 0$. Consider supplier zero: We first show that $\ell_0^* \in (s_0, 1)$. Obviously, since $\ell_1^* - \ell_0^* > |v_1 - v_0| \geq 0$, we have $\ell_0^* < \ell_1^* \leq 1$. Then, proceeding by contradiction, suppose $\ell_0^* \leq s_0$. Since $\ell_1^* - \ell_0^* > |v_1 - v_0|$, for ℓ_0 in some neighborhood of ℓ_0^*, $\pi_0(\ell_0, \ell_1^*, v) = u_0(\ell_0, \ell_1^*, v)$. Then, $\frac{\partial \pi_0(\ell^*, v)}{\partial \ell_0} = \frac{v_0}{2} f(y_0(\ell^*, v)) - C_0'(\ell_0^* - s_0) > 0$, where the inequality holds since $\ell_0^* \leq s_0$ implies $C_0'(\ell_0^* - s_0) \leq 0$. This means supplier zero has a profitable deviation to some $\ell_0' > \ell_0^*$, contradicting the definition of equilibrium. Hence, $\ell_0^* \in (s_0, 1)$, which means $\ell_0^* \in (0, 1)$. Since $\pi_0(\ell_0, \ell_1^*) = u_0(\ell_0, \ell_1^*)$ for ℓ_0 in some neighborhood of ℓ_0^* and $u_0(\cdot, \ell_1^*, v)$ is differentiable, ℓ_0^* must satisfy the FOC, $\frac{\partial u_0(\ell^*, v)}{\partial \ell_0} = 0$. Finally, see that $0 = \frac{\partial u_0(\ell^*, v)}{\partial \ell_0} = \frac{v_0}{2} f(y_0(\ell^*)) - C_0'(\ell_0^* - s_0) <$

$\frac{v_0}{2}f(\frac{1}{2}) - 2f(\frac{1}{2})(\ell_0^* - s_0) \implies \ell_0^* < s_0 + \frac{v_0}{4}$. Similar reasoning reveals that $\ell_1^* \in (s_1 - \frac{v_1}{4}, s_1)$ and that ℓ_1^* must satisfy, $\frac{\partial u_1(\ell^*, v)}{\partial \ell_1} = 0$.

Part (iv): Let $v_0, v_1 > 0$ and suppose $\ell^* \in \Phi(v)$ with $d_i^* > 0 = d_j^*$ WLOG, suppose $d_0^* > 0 = d_1^*$. We first show that $|\ell_1^* - \ell_0^*| = |v_1 - v_0|$. Proceed by contradiction and suppose $|\ell_1^* - \ell_0^*| \neq |v_1 - v_0|$. Since $d_0^* > 0$, Part (ii) of this lemma implies $|\ell_1^* - \ell_0^*| > |v_1 - v_0|$. First suppose $\ell_1^* > \ell_0^*$. Then, for ℓ_1 in some neighborhood of ℓ_1^*, $\pi_1(\ell_0^*, \ell_1) = u_1(\ell_0^*, \ell_1)$, and $\frac{\partial \pi_1(\ell^*)}{\partial \ell_1} = -\frac{v_1}{2} f(y_0(\ell^*)) < 0$. Then, supplier one has a profitable deviation to some $\ell_1' < \ell_1^*$, contradicting the definition of equilibrium. Next, suppose $\ell_0^* > \ell_1^*$. Then, for ℓ_0 in some neighborhood of ℓ_0^*, $\pi_0(\ell_0, \ell_1^*) = v_0(1 - F(\tilde{y}_0(\ell^*))) - C_0(\ell_0^* - s_0)$, and $\frac{\partial \pi_0(\ell^*)}{\partial \ell_0} = -\frac{v_0}{2} f(\tilde{y}_0(\ell^*)) - C_0'(\ell_0^* - s_0) < 0$, where the inequality holds since $\ell_0^* > s_1 \geq s_0$ implies $C_0'(\ell_0^* - s_0) > 0$. Then, supplier zero has a profitable deviation to some $\ell_0' < \ell_0^*$, contradicting the definition of equilibrium. We have now established that $|\ell_1^* - \ell_0^*| = |v_1 - v_0|$.

Next, we show that $\ell_1^* = s_1 = \frac{1}{2}$. We proceed by contradiction and suppose, WLOG, $s_1 > \frac{1}{2}$. If $v_1 = v_0$, then, since $|\ell_1^* - \ell_0^*| = |v_1 - v_0| = 0$, we must have $\ell_0^* = \ell_1^* = s_1$. Then, $\pi_1(\ell^*) = \frac{v_1}{2} < v_1 F(s_1) = \lim_{\ell_1 \uparrow \ell_1^*} \pi_1(\ell_0^*, \ell_1)$. So, supplier one has a profitable deviation to some $\ell_1' < s_1$, contradicting the definition of equilibrium. Next, suppose $v_0 < v_1$. Then, either $p_0(\ell^*) = F(s_1) < 1$ (if $\ell_0^* < s_1$) or $p_0(\ell^*) = 1 - F(s_1) < 1$ (if $\ell_0^* > s_1$). In either case, $\pi_0(\ell^*) = v_0 p_0(\ell^*) - C_0(\ell_0^* - s_0) < v_0 - C_0(\ell_0^* - s_0) = \limsup_{\ell_0 \to \ell_0^*} \pi_0(\ell_0, \ell_1^*)$; thus supplier zero has a profitable deviation to some ℓ_0' slightly closer to s_1. Finally, suppose $v_0 > v_1$. Since $d_0^* > 0$, it must be that $p_0(\ell^*) > 0$ and so $p_1(\ell^*) < 1$. Then, $\pi_1(\ell^*) = v_1 p_1(\ell^*) < v_1 = \limsup_{\ell_1 \to \ell_1^*} \pi_1(\ell_0^*, \ell_1)$ and supplier one has a profitable deviation to some ℓ_1' slightly closer to ℓ_0^*. This establishes that $s_1 = \frac{1}{2}$.

Next, we show that $\ell_0^* = \frac{1}{2}$ and $v_0 = v_1$. Proceed by contradiction and suppose $\ell_0^* \neq \frac{1}{2}$. Since $|\ell_1^* - \ell_0^*| = |\frac{1}{2} - \ell_0^*| = |v_1 - v_0|$, we have that $v_0 \neq v_1$. Note that if $v_0 < v_1$, then, $p_0(\ell^*) = \frac{1}{2}$; it follows that $\pi_0(\ell^*) = \frac{v_0}{2} - C_0(\ell_0^* - s_0) < v_0 - C_0(\ell_0^* - s_0) = \limsup_{\ell_0 \to \ell_0^*} \pi_0(\ell_0, \ell_1^*)$, and supplier zero has a profitable deviation to some ℓ_0' slightly closer to $\frac{1}{2}$. If $v_1 < v_0$, then, since $d_0^* > 0$, it must be that $p_0(\ell^*) > 0$, which means $p_1(\ell^*) < 1$. Then, $\pi_1(\ell^*) = v_1 p_1(\ell^*) < v_1 = \limsup_{\ell_1 \to \frac{1}{2}} \pi_1(\ell_0^*, \ell_1)$, and supplier one has a profitable deviation to some ℓ_1' slightly closer to ℓ_0^*, which contradicts the definition of equilibrium. This establishes that $\ell_0^* = \frac{1}{2}$. Since $|\ell_1^* - \ell_0^*| = |v_1 - v_0|$, this implies $v_0 = v_1$, and establishes the lemma.

Part (v): Let $v_0, v_1 > 0$, $s_0, s_1 \in (0, 1) \setminus \{\frac{1}{2}\}$ and suppose $|v_1 - v_0| \geq s_1 - s_0$. Let $\ell^* \in \Phi(v)$ and suppose, contrary to the statement of the lemma, $d_i^* > 0$ for some i. First note that since $s_0, s_1 \neq \frac{1}{2}$, Part (iv) implies that we must have $d_0^*, d_1^* > 0$. Then, by Part (iii), $\ell_1^* - \ell_0^* > |v_1 - v_0|$, $\ell_0^* > s_0$ and $\ell_1^* < s_1$. It follows that $s_1 - s_0 > \ell_1^* - \ell_0^* > |v_1 - v_0|$, contradicting the hypothesis that $|v_1 - v_0| \geq s_1 - s_0$.

Part (vi): WLOG, let $v_1 > 0 = v_0$ and $v_1 \geq s_1 - s_0$. Let $\ell^* \in \Phi(v)$. Obviously, since $v_0 = 0$, $d_0^* = 0$. So to establish the lemma it suffices to show that if $d_1^* > 0$, then $\ell_1^* \in \{s_0 - v_1, s_1 + v_1\}$. So suppose $d_1^* > 0$; part (iii) implies $v_1 \leq |\ell_1^* - s_0|$. Suppose $v_1 < \ell_1^* - s_0$. Since $s_1 - s_0 \leq v_1$, we must have $\ell_1^* > s_1$. Moreover, for ℓ_1 in some neighborhood of ℓ_1^*, $\pi_1(s_0, \ell_1) = u_1(s_0, \ell_1)$ and $\frac{\partial \pi_1(s_0, \ell_1^*)}{\partial \ell_1} = -\frac{v_1}{2} f(y_0) - C_1'(\ell_1^* - s_1) < 0$, where the inequality follows since $v_1 > 0$ and $\ell_1^* > s_1$. This means that supplier one has a profitable deviation to some $\ell_1' < \ell_1^*$, contradicting the definition of equilibrium. Similar reasoning shows that if $v_1 < s_0 - \ell_1^*$, then supplier one has a profitable deviation to some $\ell_1' > \ell_1^*$. So, if $d_1^* > 0$, then, it must be that $v_1 = |\ell_1^* - s_0|$, which means $\ell_1^* \in \{s_0 - v_1, s_0 + v_1\}$.

Part (vii): If $d_0^*, d_1^* > 0$ the result follows by Part (iii), so suppose, WLOG, $d_0^* > 0 = d_1^*$. Since $s_1 \neq \frac{1}{2}$, Part (iv) implies that we must have $v_1 = 0$. We now show that $\ell_0^* \in [0, s_1 - v_0]$. First see that for $\ell_0 \in (s_1 - v_0, s_1 + v_0)$, $\pi_0(\ell_0, s_1) < 0 = \pi_0(s_0, s_1)$. Next, note that $\pi_0(\cdot, s_1)$ is strictly decreasing on $[s_1 + v_0, 1]$; moreover, $p_0(s_1 + v_0, s_1) = 1 - F(s_1 + v_0)$. And since $s_1 > \frac{1}{2}$, symmetry of the distribution implies $1 - F(s_1 + v_0) < F(s_1 - v_0) = p_0(s_1 - v_0, s_1)$. And since supplier zero's cost of choosing the point $s_1 - v_0$ is strictly less than the cost of choosing $s_1 + v_0$, we have that $\pi_0(s_1 - v_0, s_1) > \pi_0(s_1 + v_0, s_1)$. Thus, it must be that $\ell_0^* \in [0, s_1 - v_0]$. It follows that for any $\ell_0 \leq \ell_0^*$, $\pi_0(\ell_0, s_1) = u_0(\ell_0, s_1)$. If it were the case that $\frac{\partial u_0(\ell^*)}{\partial \ell_0} < 0$, then, for some $\ell_0' < \ell_0^*$, we have $\pi_0(\ell_0', \ell_1^*) = u_0(\ell_0', \ell_1^*) > u_0(\ell^*)$, contradicting the definition of equilibrium. Thus, $\frac{\partial u_0(\ell^*)}{\partial \ell_0} \geq 0$. □

Before proving Lemma 2, we state and prove the following.

Lemma A.1. *Let s^* satisfy $F(s^*) = \frac{1}{4}$. Then $s^* < \frac{1}{3}$.*

Proof. See that $F\left(\frac{1}{3}\right) - F(s^*) = \frac{1}{2}\int_0^{\frac{1}{3}} f(x)dx - \frac{1}{2}\int_{\frac{1}{3}}^{\frac{1}{2}} f(x)dx \geq \frac{1}{6}f(0) - \frac{1}{12}f\left(\frac{1}{2}\right) > 0$, where the inequality holds since f is increasing on $\left[0, \frac{1}{2}\right]$ and by Lemma 1(i). □

Proof of Lemma 2. Let s_0, s_1 and v_0, v_1 be as given in the lemma. In particular, we consider the case where $0 < v_0 \leq v_1 < 2s_1 - s_0 - 1$; the case where $v_1 < v_0$ is similar.

To begin, we will show that any solution to (3) and (4) is unique. Suppose ℓ^* and ℓ^{**} satisfy (3) and (4); we will show $\ell^* = \ell^{**}$. To do so, we follow arguments similar to Rosen (1965). Let

$$g(\ell) = \begin{bmatrix} \frac{\partial u_0}{\partial \ell_0} \\ \frac{\partial u_1}{\partial \ell_1} \end{bmatrix} = \begin{bmatrix} \frac{v_0}{2} f\left(y_0\left(\ell\right)\right) - C_0'(\ell_0 - s_0) \\ -\frac{v_1}{2} f\left(y_0\left(\ell\right)\right) + C_1'(s_1 - \ell_1) \end{bmatrix}$$

and for ℓ such that $y_0(\ell, v) \in (0,1)$, let $D_\ell g(\ell)$ denote the Jacobian of $g(\ell)$ with respect to ℓ:

$$D_\ell g(\ell) = \begin{bmatrix} \frac{v_0}{4} f'\left(y_0\left(\ell\right)\right) - C_0''(\ell_0 - s_0) & \frac{v_0}{4} f'\left(y_0\left(\ell\right)\right) \\ -\frac{v_1}{4} f'\left(y_0\left(\ell\right)\right) & -\frac{v_1}{4} f'\left(y_0\left(\ell\right)\right) - C_1''(s_1 - \ell_1) \end{bmatrix}.$$

We show that $D_\ell g(\ell) + \left(D_\ell g\left(\ell\right)\right)^T$ is negative definite for all ℓ. See that

$$D_\ell g(\ell) + (D_\ell g(\ell))^T = \begin{bmatrix} \frac{v_0}{2} f'\left(y_0\left(\ell\right)\right) - 2C_0''(\ell_0 - s_0) & \frac{f'(y_0(\ell))}{4}(v_0 - v_1) \\ \frac{f'(y_0(\ell))}{4}(v_0 - v_1) & -\frac{v_1}{2} f'\left(y_0\left(\ell\right)\right) - 2C_1''(s_1 - \ell_1) \end{bmatrix}.$$

Since $v_0, v_1 < 2s_1 - s_0 - 1 < 1$, Lemma 1(iii) implies that the terms on the main diagonal of $D_\ell g(\ell) + (D_\ell g(\ell))^T$ are strictly negative. Then, see that

$$\begin{aligned} |D_\ell g(\ell) + (D_\ell g(\ell))^T| &= \left[\frac{v_0}{2} f' - 2C_0''\right]\left[-\frac{v_1}{2} f' - 2C_1''\right] - \left[\frac{f'}{4}(v_0 - v_1)\right]^2 \\ &= -(f')^2 \left[\frac{v_0 + v_1}{4}\right]^2 + 4C_0''C_1'' + f'v_1C_0'' - f'v_0C_1'' \\ &> 3C_0''C_1'' + f'v_1C_0'' - f'v_0C_1'' \\ &> 3C_0''C_1'' - C_0''C_1'' \\ &> 0. \end{aligned}$$

The first inequality holds since $v_0, v_1 < 1$, together with Lemma 1(iii), imply, $-(f')^2\left[\frac{v_0+v_1}{4}\right]^2 > -(f')^2 > -C_0''C_1''$. The second inequality holds since $f'v_1C_0'' - f'v_0C_1'' \geq -|f'|\max\{C_0'', C_1''\}$, and Lemma 1(iii) implies, $-|f'|\max\{C_0'', C_1''\} > -\min\{C_0'', C_1''\}\max\{C_0'', C_1''\} = -C_0''C_1''$. The final inequality holds since $C_0''C_1'' > 0$. This establishes that $D_\ell g(\ell) + (D_\ell g(\ell))^T$ is negative definite.

For $\theta \in [0,1]$, let $\ell^\theta = \theta\ell^* + (1-\theta)\ell^{**}$. As argued in the first part of this proof, $y_0(\ell^*), y_0(\ell^{**}) \in (0,1)$, which means that for all θ, $y_0(\ell^\theta) \in (0,1)$. This ensures that $g(\ell^\theta)$ is differentiable in θ. See that $\frac{dg(\ell^\theta)}{d\theta} = D_\ell g(\ell^\theta)\frac{d\ell^\theta}{d\theta} = D_\ell g(\ell^\theta)(\ell^* - \ell^{**})$ or

$$\vec{0} = g(\ell^*) - g(\ell^{**}) = \int_0^1 D_\ell g(\ell^\theta)(\ell^* - \ell^{**})\, d\theta.$$

Pre-multiplying both sides of the equation above by $(\ell^* - \ell^{**})^T$:

$$\vec{0} = \int_0^1 (\ell^* - \ell^{**})^T D_\ell g\left(\ell^\theta\right)(\ell^* - \ell^{**})\,\mathrm{d}\theta \tag{11}$$
$$= \frac{1}{2}\int_0^1 (\ell^* - \ell^{**})^T \left[D_\ell g\left(\ell^\theta\right) + \left(D_\ell g\left(\ell^\theta\right)\right)^T\right](\ell^* - \ell^{**})\,\mathrm{d}\theta.$$

Since $D_\ell g(\ell) + (D_\ell g(\ell))^T$ is negative definite for each θ, (11) implies $\ell^* = \ell^{**}$. We have now established that any solution to (3) and (4) is unique.

Next, we show that there exists at most one equilibrium. First, we argue that in any equilibrium $d_0^*, d_1^* > 0$. Let $\ell^* \in \Phi(v)$ and, by way of contradiction, suppose $d_i^* = 0$ for some i. Then since $1 - s_1, s_0 < s^* < \frac{1}{2}$, Proposition A.1(iv) implies that $d_0^* = d_1^* = 0$. Then see that $v_0 \leq v_1 < 2s_1 - s_0 - 1$, implies $v_1 - v_0 < s_1 - s_0$, which means that for ℓ_0 in some neighborhood of s_0, $\pi_0(\ell_0, s_1) = u_0(\ell_0, s_1)$. Moreover, $\frac{\partial u_0(\ell_0,s_1)}{\partial \ell_0}|_{\ell_0=s_0} = v_0 f(y_0) > 0$. So, supplier zero has a profitable deviation to some $\ell_0' > s_0$, yielding a contradiction. Thus, $d_0^*, d_1^* > 0$. Proposition A.1(iii) implies that ℓ^* solves (3) and (4). As we've shown, there is a unique solution to this system and, therefore, there is at most one equilibrium.

Now let $\ell^* \in \Phi(v)$; we show that (ℓ^*, v) satisfies (2)-(5). Since $d_0^*, d_1^* > 0$ Proposition A.1(iii) implies ℓ^* satisfies (2)-(4). We now show that (5) must hold. So, suppose $v_0 < v_1$. (2) implies $\pi_0(\ell^*) = u_0(\ell^*)$. Moreover, letting $\bar{\ell}_0 = \ell_1^* + v_0 - v_1$, see that $\lim_{\ell_0 \downarrow \bar{\ell}_0} \pi_0(\ell_0, \ell_1^*) = v_0 - C_0(\bar{\ell}_0 - s_0)$. If it were the case that $u_0(\ell^*) < v_0 - C_0\left(\bar{\ell}_0 - s_0\right)$, then, for some $\ell_0 > \bar{\ell}_0$ sufficiently close to $\bar{\ell}_0$, $\pi_0(\ell_0, \ell_1^*) > \pi_0(\ell^*)$, contradicting the definition of equilibrium. Thus, we must have $u_0(\ell^*) \geq v_0 - C_0\left(\bar{\ell}_0 - s_0\right)$.

Now let (ℓ^*, v) satisfy (2)-(5); we will show that $\ell^* \in \Phi(v)$. We prove the case where $v_0 < v_1 < 2s_1 - s_0 - 1$; the proof for the case where $v_0 = v_1$ is similar. We point out that since $v_i < 1$ for each i, $\frac{\partial^2 u_i}{\partial \ell_i^2} < 0$.

We first establish that each ℓ_i^* is feasible; i.e., $\ell_i^* \in [0,1]$. In fact, we will show that $\ell_0^*, 1-\ell_1^* \in (0, \frac{1}{2})$. To begin, note that we must have $y_0(\ell^*) \in [0,1]$; for otherwise, $f(y_0(\ell^*)) = 0$ and (3)-(4) imply $\ell_i^* = s_i$. But $v_0 < v_1 < 2s_1 - s_0 - 1 \implies v_1 - v_0 < s_1 - s_0 \implies y_0(s_0, s_1) \in (0,1)$, yielding a contradiction. Then, it is straightforward to see from (3)-(4) that $\ell_0^* > s_0 > 0$ and $\ell_1^* < s_1 < 1$. Next, we show that $\ell_0^* < s_0 + \frac{v_0}{4}$. See that $\frac{\partial u_0(\ell_0,\ell_1^*)}{\partial \ell_0}|_{\ell_0=s_0+\frac{v_0}{4}} \leq \frac{v_0}{2} f\left(\frac{1}{2}\right) - C_0'\left(\frac{v_0}{4}\right) < \frac{v_0}{2} f\left(\frac{1}{2}\right) - 2\frac{v_0}{4} f\left(\frac{1}{2}\right) = 0$, where the first inequality holds since $f(x) \leq f\left(\frac{1}{2}\right)$ for all x and the second holds by Lemma 1(ii). By strict concavity of $u_0(\cdot, \ell_1)$, it must be that $\ell_0^* < s_0 + \frac{v_0}{4}$. Then, $s_0 + \frac{v_0}{4} < \frac{3}{4}s_0 + \frac{s_1}{2} - \frac{1}{4} < \frac{3}{4}\left(\frac{1}{3}\right) + \frac{1}{2} - \frac{1}{4} = \frac{1}{2}$, where the first inequality holds since $v_0 < 2s_1 - s_0 - 1$, and the second holds since $s_0 < s^* < \frac{1}{3}$

and $s_1 < 1$. This establishes that $\ell_0^* \in \left(0, \frac{1}{2}\right)$; analogous arguments reveal that $\ell_1^* \in \left(\frac{1}{2}, 1\right)$.

Now let $B_i(\ell_j) = \arg\max_{\ell_i \in [0,1]} \pi_i(\ell)$ denote i's best reply. We show that $\ell_0^* \in B_0(\ell_1^*)$. Let $\bar{\ell}_0 = \ell_1^* + v_0 - v_1$, and see that $\bar{\ell}_0 < s_1 + v_0 - v_1 < 1$ and that (2) implies $\bar{\ell}_0 > \ell_0^*$. Let $\ell_0 \in [0, \bar{\ell}_0]$. Then, $\pi_0(\ell_0, \ell_1^*) = u_0(\ell_0, \ell_1^*) \leq u_0(\ell_0^*, \ell_1^*) = \pi_0(\ell_0^*, \ell_1^*)$, where the inequality follows since $v_0 < 1$, which means u_0 is strictly concave on $[0, 1]$, so (3) implies $\ell_0^* = \arg\max_{\ell_0} u_0(\ell_0, \ell_1^*)$. Next, let $\ell_0 \in (\bar{\ell}_0, 1]$. We have, $\pi_0(\ell_0, \ell_1^*) < v_0 - C_0(\bar{\ell}_0 - s_0) \leq u_0(\ell_0^*, \ell_1^*) = \pi_0(\ell_0^*, \ell_1^*)$, where the strict inequality holds since $p_0(\ell_0, \ell_1^*) \leq 1$ and $\ell_0 > \bar{\ell}_0$ implies $C_0(\ell_0 - s_0) > C_0(\bar{\ell}_0 - s_0)$; the weak inequality follows by (5). This establishes that $\ell_0^* \in B_0(\ell_1^*)$.

Next, we show that $\ell_1^* \in B_1(\ell_0^*)$. Let $\ell_1 \in [0, 1]$ be given. If $\ell_1 \in [\ell_0^* + v_1 - v_0, 1]$ then, $\pi_1(\ell_0^*, \ell_1) = u_1(\ell_0^*, \ell_1) \leq u_1(\ell_0^*, \ell_1^*) = \pi_1(\ell_0^*, \ell_1^*)$, where the inequality follows since strict concavity of $u_1(\ell_0^*, \cdot)$ together with (4) imply $\ell_1^* = \arg\max_{\ell_1} u_1(\ell_0^*, \ell_1)$. If $\ell_1 \in (\ell_0^* + v_0 - v_1, \ell_0^* + v_1 - v_0)$ then, $\pi_1(\ell_0^*, \ell_1) = -C_1(s_1 - \ell_1) < 0 < \pi_1(\ell_0^*, \ell_1^*)$. Finally, let $\bar{\ell}_1 = \ell_0^* + v_0 - v_1$, and suppose $\ell_1 \in [0, \bar{\ell}_1]$. Then, we have the following string of inequalities:

$$\pi_1(\ell_0^*, \ell_1) \leq \pi_1(\ell_0^*, \bar{\ell}_1) < \pi_1(\ell_0^*, \ell_0^* + v_1 - v_0) \leq \pi_1(\ell_0^*, \ell_1^*)$$

The first inequality holds since $\pi_1(\ell_0^*, \cdot)$ is strictly increasing on $[0, \bar{\ell}_1]$. The second inequality holds since $\ell_0^* < \frac{1}{2}$ implies that the approach, $\ell_0^* + v_1 - v_0$, gives supplier one a strictly greater probability of victory at strictly lower cost than $\bar{\ell}_1$. The final inequality holds since, as we've already shown, $\pi_1(\ell_0^*, \ell_1) \leq \pi_1(\ell_0^*, \ell_1^*)$ for all $\ell_1 \in [\ell_0^* + v_1 - v_0, 1]$. This establishes that $\ell_1^* \in B_1(\ell_0^*)$. Therefore, $\ell^* \in \Phi(v)$. $\square$

Proposition A.2. *If* $1 - s_1, s_0 < s^*$, $\frac{\partial u_0(\ell^*)}{\partial \ell_0} = \frac{\partial u_1(\ell^*)}{\partial \ell_1} = 0$, *and either* $0 < v_0 \leq v_1 \leq 2\ell_1^* - \ell_0^* - 1$ *or* $0 < v_1 < v_0 < \ell_1^* - 2\ell_0^*$, *then* ℓ^* *is the unique equilibrium in stage 2.*

Proof. Fix $s_0, 1 - s_1 < s^*$, and suppose ℓ^* satisfies $\frac{\partial u_i(\ell^*)}{\partial \ell_i} = 0$ for each i. Further, suppose $v_0 \leq v_1 \leq 2\ell_1^* - \ell_0^* - 1$. We will show that ℓ^* is the unique equilibrium in stage 2. To do so, we appeal to Lemma 2. The proof for the case, $v_1 < v_0 < \ell_1^* - 2\ell_0^*$ follows analogous arguments.

To begin, we show that $y_0(\ell^*) \in [0, 1]$. If, to the contrary, $y_0(\ell^*) =\notin [0, 1]$, then, $f(y_0(\ell^*)) = 0$ and $\frac{\partial u_i(\ell^*)}{\partial \ell_i} = 0$ implies $\ell_i^* = s_i$ for each i. However, $0 < v_0 \leq v_1 \leq 2s_1 - s_0 - 1$ implies $y_0(s_0, s_1) = \frac{s_1 + s_0 - v_1 - v_0}{2} \in (0, 1)$, yielding a contradiction. Thus, we must have $y_0(\ell^*) \in [0, 1]$.

Now since $v_i f(y_0) > 0$, $\frac{\partial u_0(\ell^*)}{\partial \ell_0} = \frac{\partial u_1(\ell^*)}{\partial \ell_1} = 0$ implies $\ell_0^* > s_0$ and $\ell_1^* < s_1$. So, $v_1 < 2\ell_1^* - \ell_0^* - 1 < 2s_1 - s_0 - 1$. By Lemma 2, there is at most one equilibrium in stage 2. Moreover, Lemma 2 implies that $\ell^* \in \Phi(v)$ if (ℓ^*, v) satisfy (2)-(5). Next, we will show that this is the case.

First, since (3) and (4) hold by assumption, it suffices to show (2) and (5). Then see that, $v_1 - v_0 < v_1 \leq \ell_1^* - \ell_0^* + \ell_1^* - 1 < \ell_1^* - \ell_0^*$, which means (2) holds.

Next, we show that (ℓ^*, v) satisfies (5). To this end, it suffices to show that $\Delta \geq 0$, where $\Delta = v_0 F(y_0^*) - C_0(\ell_0^* - s_0) - v_0 + C_0(\ell_1^* + v_0 - v_1 - s_0)$, where $y_0^* = y_0(\ell^*, v)$. See that

$$\begin{aligned}\Delta &= \int_{\ell_0^*}^{\ell_1^*+v_0-v_1} C_0'(x - s_0)dx - v_0 \int_{y_0^*}^{1} f(x)dx \\ &> \int_{\ell_0^*}^{\ell_1^*+v_0-v_1} 2(x - s_0) f\left(\frac{1}{2}\right) dx - v_0 \int_{y_0^*}^{1} f\left(\frac{1}{2}\right) dx \\ &= f\left(\frac{1}{2}\right)\left[(\ell_1^* + v_0 - v_1 - s_0)^2 - (\ell_0^* - s_0)^2 - v_0\left(1 - \frac{\ell_0^* + \ell_1^* + v_1 - v_0}{2}\right)\right].\end{aligned}$$

The strict inequality holds since $C_0'(z) > 2zf\left(\frac{1}{2}\right)$ and since $f(x) \leq f\left(\frac{1}{2}\right)$ for $x \in [0,1]$. Now, for $x \geq 0$, let $G(x) = (\ell_1^* + x - v_1 - s_0)^2 - (\ell_0^* - s_0)^2 - x\left(1 - \frac{\ell_0^*+\ell_1^*+v_1-x}{2}\right)$; the string of inequalities above shows that $\Delta > f\left(\frac{1}{2}\right) G(v_0)$. We will now show that $G'(x) > 0$:

$$\begin{aligned}G'(x) &\propto \ell_0^* + 5\ell_1^* - 4s_0 + 2x - 3v_1 - 2 \\ &> 4(\ell_0^* - s_0) + 1 - \ell_1^* + 2x \\ &> 0.\end{aligned}$$

Where the first inequality follows since $v_1 < 2\ell_1^* - \ell_0^* - 1$ and the second since $\ell_0^* > s_0$ and $\ell_1^* < 1$. So we have that, $\Delta > f\left(\frac{1}{2}\right) G(v_0) > f\left(\frac{1}{2}\right) G(0) = f\left(\frac{1}{2}\right)[(\ell_1^* - v_1 - s_0)^2 - (\ell_0^* - s_0)^2]$. Using the fact that $v_1 < 2\ell_1^* - \ell_0^* - 1$, it is straightforward to verify that $\ell_1^* - v_1 - s_0 > 0$. So, $(\ell_1^* - v_1 - s_0)^2 > (\ell_0^* - s_0)^2$ if and only if $\ell_1^* - \ell_0^* - v_1 > 0$. Since $v_1 < 2\ell_1^* - \ell_0^* - 1$, we have that $\ell_1^* - \ell_0^* - v_1 > 1 - \ell_1^* > 0$, which means $G(0) > 0$, and establishes that $\Delta > 0$, which completes the proof. □

Proof of Lemma 3. Consider the case where $v_i = 0$ for some i; WLOG, suppose $v_1 = 0$. Since $\ell_1 = s_1$ is a dominant strategy for supplier one, it suffices to show $\arg\max_{\ell_0 \in [0,1]} \pi_0(\ell_0, s_1) \neq \emptyset$. If $v_0 = 0$ or $v_0 \geq 1$, then $s_0 = \arg\max_{\ell_0 \in [0,1]} \pi_0(\ell_0, s_1)$, so consider the case $v_0 \in (0,1)$. We will show that $\pi_0(\cdot, s_1)$ is upper semicontinuous (usc) on $[0,1]$.

See that $\pi_0(\cdot, s_1)$ is piecewise continuous: For $\ell_0 \in [0, s_1 - v_0]$, $\pi_0(\ell_0, s_1) = u_0(\ell_0, s_1)$; for $\ell_0 \in (s_1 - v_0, s_1 + v_0), \pi_0(\ell_0, s_1) = -C_0(\ell_0 - s_0)$; and for $\ell_0 \in [s_1 + v_0, 1]$, $\pi_0(\ell_0, s_1) = \tilde{u}_0(\ell_0, s_1) = [1 - F(\tilde{y}_0(\ell, v))] v_0 - C_0(\ell_0 - s_0)$. Therefore, it suffices to show that $\pi_0(\cdot, s_1)$ is usc at $\ell_0 = s_1 \pm v_0$. WLOG, assume $s_1 \pm v_0 \in (0,1)$. It holds that $u_0(s_1 - v_0, s_1) =$

$v_0F(s_1-v_0)-C_0(s_1-v_0-s_0) > -C_0(s_1-v_0-s_0)$. And for $\epsilon > 0$ sufficiently small, continuity of $u_0(\cdot, s_1)$ implies that for all $\ell_0 \in (s_1-v_1-\epsilon, s_1-v_0+\epsilon)\setminus\{s_1-v_1\}$, $u_0(\ell_0, s_1) > -C_0(\ell_0-s_0)$. It follows that $\limsup_{\ell_0\to s_1-v_1} \pi_0(\ell_0, s_1) = u_0(s_1-v_1, s_1) = \pi_0(s_1-v_1, s_1)$. Similar reasoning reveals that $\limsup_{\ell_0\to s_1+v_1} \pi_0(\ell_0, s_1) = \tilde{u}_0(s_1+v_1, s_1) = \pi_0(s_1+v_1, s_1)$. Hence, $\pi_0(\cdot, s_1)$ is usc on the compact set $[0,1]$; therefore, $\arg\max_{\ell_0\in[0,1]} \pi_0(\ell_0, s_1) \neq \emptyset$.

Next, fix $0 < s_0 < s_1 < 1$. We will show that there exist $v_0, v_1 > 0$ sufficiently small such that $\Phi(v_0, v_1) \neq \emptyset$ and $\ell^* \in \Phi(v_0, v_1) \implies \ell_i^* \neq s_i$. For each $i \in \{0,1\}$, since $0 < C_i(s_1-s_0)$, continuity implies that there exists $\epsilon_i > 0$ such that $0 < v < \epsilon_i$ implies $v < C_i\left(s_1 - s_0 - \frac{v}{4}\right)$. Let $v_0 = v_1 = v^*$, such that $0 < v^* < \min\{\epsilon_0, \epsilon_1, s_1 - s_0\}$.

For $i \neq j$, let $r_i(\ell_j) = \arg\max_{\ell_i\in[0,1]} u_0(\ell) = \arg\max_{\ell_i\in[0,1]} v^*F(m(\ell)) - C_0(\ell_i - s_i)$. Let $r(\ell) = (r_0(\ell_1), r_1(\ell_0))$. We now show that r is a continuous function and that $r_0 : \left[s_0, s_0+\frac{v^*}{4}\right] \times \left[s_1-\frac{v^*}{4}, s_1\right] \to \left(s_0, s_0+\frac{v^*}{4}\right) \times \left(s_1-\frac{v^*}{4}, s_1\right)$.

Let $\ell_1 \in \left[s_1-\frac{v^*}{4}, s_1\right]$. Since $m(\ell) \in (0,1)$ for all $\ell_0 \in [0,1]$, $u_0(\cdot, \ell_1)$ is twice differentiable and strictly concave on $[0,1]$. Therefore, $r_0(\cdot)$ is a continuous function. Moreover, see that $\frac{\partial u_0(\ell, v^*)}{\partial \ell_0}|_{\ell_0=s_0} = \frac{v^*}{2}f(m(s_0,\ell_1)) > 0$ and $\frac{\partial u_0(\ell)}{\partial \ell_0}|_{\ell_0=s_0+\frac{v^*}{4}} = \frac{v^*}{2}f\left(m\left(s_0+\frac{v^*}{4}, \ell_1\right)\right) - C_0'\left(\frac{v^*}{4}\right) < \frac{v^*}{2}f\left(\frac{1}{2}\right) - 2\frac{v^*}{4}f\left(\frac{1}{2}\right) = 0$, where the inequality follows since $f(x) \leq f\left(\frac{1}{2}\right)$ for all x and by Lemma 1(ii). Thus, $r_0(\ell_1) \in \left(s_0, s_0+\frac{v^*}{4}\right)$. Analogous arguments reveal that $r_1(\ell_0) \in \left(s_1-\frac{v_1}{4}, s_1\right)$ for all $\ell_0 \in \left[s_0, s_0+\frac{v^*}{4}\right]$. This establishes that r is a continuous function and $r : \left[s_0, s_0+\frac{v^*}{4}\right] \times \left[s_1-\frac{v^*}{4}, s_1\right] \to \left(s_0, s_0+\frac{v^*}{4}\right) \times \left(s_1-\frac{v^*}{4}, s_1\right)$. By Brouwer's fixed point theorem, r has a fixed point in $\left(s_0, s_0+\frac{v^*}{4}\right) \times \left(s_1-\frac{v^*}{4}, s_1\right)$; i.e., there exists $\ell^* \in \left(s_0, s_0+\frac{v^*}{4}\right) \times \left(s_1-\frac{v^*}{4}, s_1\right)$ such that $r(\ell^*) = \ell^*$.

We now show $\ell^* \in \Phi(v^*)$. First, $\ell^* \in \left(s_0, s_0+\frac{v^*}{4}\right) \times \left(s_1-\frac{v^*}{4}, s_1\right)$ and $v^* < s_1 - s_0$ imply $\ell_0^* < \ell_1^*$, which means $\pi_i(\ell^*) = u_i(\ell^*)$. We now show that $\pi_0(\ell_0, \ell_1^*) \leq u_0(\ell^*)$ for all $\ell_0 \in [0,1]$. Let $\ell_0 \in [0,1]$. If $\ell_0 \in [0, \ell_1^*)$ then $\pi_0(\ell_0, \ell_1^*) = u_0(\ell_0, \ell_1^*) \leq u_0(\ell_0^*, \ell_1^*)$, where the inequality follows by definition of ℓ_0^*. If $\ell_0 \in [\ell_1^*, 1]$ then, $\pi_0(\ell_0, \ell_1^*) < v^* - C_0\left(s_1 - \frac{v^*}{4} - s_0\right) < 0 < u_0(\ell^*)$. The first inequality holds since $p_i(\cdot) \leq 1$ and $\ell_0 \geq \ell_1^* > s_1 - \frac{v^*}{4} > s_0$. The second inequality holds by construction of v^*. We have now established that $\pi_0(\ell_0^*, \ell_1^*) \geq \pi_0(\ell_0, \ell_1^*)$ for all $\ell_0 \in [0,1]$. Analogous reasoning reveals that $\pi_1(\ell_0^*, \ell_1^*) \geq \pi_1(\ell_0^*, \ell_1)$ for all $\ell_1 \in [0,1]$. Thus, $\ell^* \in \Phi(v^*)$.

To complete the proof, we show that there does not exist $\tilde{\ell} \in \Phi(v^*)$ with $\tilde{\ell}_i = s_i$. Proceeding by contradiction, suppose there is $\tilde{\ell} \in \Phi(v)$ with $\tilde{\ell}_i = s_i$; in particular, and WLOG, suppose $\tilde{\ell}_1 = s_1$. Following similar reasoning as given in the previous paragraph, it holds that $r_0(s_1) = \arg\max_{\ell_0\in[0,1]} \pi_0(\ell_0, s_1)$; so, by definition of equilibrium, $\tilde{\ell}_0 = r_0(s_1)$. Since $r_0 : [s_1 - \frac{v^*}{4}, s_1] \to (s_0, s_0 + \frac{v^*}{4})$, we have, $\tilde{\ell}_0 < s_0 + \frac{v^*}{4} < s_1$. Then for ℓ_1 in some neighborhood of s_1, $\pi_1(\tilde{\ell}_0, \ell_1) = u_1(\tilde{\ell}_0, \ell_1)$; moreover, $\frac{\partial u_1(\tilde{\ell}_0,\ell_1)}{\partial \ell_1}|_{\ell_1=s_1} = -\frac{v^*}{2} m(\tilde{\ell}_0, s_1) < 0$. This means that supplier one has a profitable deviation to some $\ell_1' < s_1$, contradicting the hypothesis that $\tilde{\ell} \in \Phi(v^*)$. □

Proofs for Section 4.2

We first state and prove three preliminary results:

Lemma A.2. *If $(\ell^*, v^*) \in \mathcal{A}_P$, then $|v_1^* - v_0^*| \leq |\ell_1^* - \ell_0^*|$.*

Proof. Let $(\ell, v) \in \mathcal{F}_P$ such that $|v_1 - v_0| > |\ell_1 - \ell_0|$; we will show that $(\ell, v) \notin \mathcal{A}_P$. Without loss of generality, let us assume $v_0 < v_1$. First note that Proposition A.1(ii) implies $\ell_i = s_i$. Moreover, $p_0(\ell, v) = 1$. It follows that $\pi_b(\ell, v) = \mathbb{E}[Q_0(s_0, y)]$. We now show that there exists $(\ell', v') \in \mathcal{F}_P$ such that $\pi_b(\ell', v') > \pi_b(\ell, v)$. We examine separately the case where $s_0 = s_1$ and the case where $s_0 < s_1$.

First suppose $s_0 < s_1$. Let $v_0' = v_1' = 0$ and $\ell_i' = s_i$. See that $\ell' \in \Phi(v')$ and $\pi_b(\ell', v') = \mathbb{E}\left[\max\left\{Q_0\left(s_0, y\right), Q_1\left(s_1, y\right)\right\}\right] > \mathbb{E}\left[Q_0\left(s_0, y\right)\right] = \pi_b(\ell, v)$.

Next, suppose $s_0 = s_1 = s$ and, WLOG, assume $s \geq \frac{1}{2}$. Consider any prize profile, v' such that $v_0' > 0$ and $v_1' = 0$. Proposition A.2 implies that $\Phi(v') \neq \emptyset$. Moreover, Proposition A.1(vi) implies that if $\ell' \in \Phi(v')$, then, $\ell_0' \in \{s, s - v_0', s + v_0'\}$ and $\ell_1' = s$. Next, see that there exists $\epsilon > 0$, such that for all $v_0' \in (0, \epsilon)$, $\pi_0\left(s - v_0', \ell_1', v_0', v_1'\right) = v_0' F\left(s - v_0'\right) - C_0(v_0') > 0$. Moreover, since $s \geq \frac{1}{2}$, symmetry of the distribution of y about $\frac{1}{2}$ implies that supplier zero can be no worse off choosing $s - v_0'$ than $s + v_0'$. It follows that for all $v_0' \in (0, \epsilon)$, $(s - v_0', s) \in \Phi(v')$.

Now, when the buyer offers prizes v', where $0 = v_1' < v_0'$, and the suppliers subsequently choose according to $\ell' = (s - v_0', s)$, the buyer's payoff can be written as

$$\pi_b(\ell', v') = G(v_0') = q - \int_0^{s-v_0'} (s - v_0' - y) dF(y) - \int_{s-v_0'}^{s} (s - y) dF(y) - \int_s^1 (y - s) dF(y) - C_0(v_0').$$

Note that $G(0) = \pi_b(\ell, v)$; moreover, $G'(0) = F(s) > 0$, and hence, there exists $\epsilon' > 0$ such that for $v_0' \in (0, \epsilon')$, $G(v_0') > G(0)$, which means $\pi_b(\ell', v') > \pi_b(\ell, v)$. Now fix a profile of prizes v' and a pair of approaches ℓ' such that $v_0' \in (0, \min\{\epsilon, \epsilon'\})$, $v_1' = 0$, $\ell_0' = s - v_0'$ and $\ell_1' = s$. As we've just shown, $v_0' < \epsilon \implies \ell' \in \Phi(v')$; moreover, $v_0' < \epsilon' \implies \pi_b(\ell', v') > \pi_b(\ell, v)$. This establishes that $(\ell, v) \notin \mathcal{A}_P$; hence, $(\ell^*, v^*) \in \mathcal{A}_P$ implies $|\ell_1^* - \ell_0^*| \geq |v_1^* - v_0^*|$. □

Lemma A.3. *For some $\overline{v} > 0$ let H denote the set,*

$$H = \left\{ (\ell, v) \in [0,1]^2 \times [0, \overline{v}]^2 | |\ell_1 - \ell_0| \geq |v_1 - v_0|; \frac{\partial u_i(\ell, v)}{\partial \ell_i} = 0, i = 0, 1 \right\}.$$

Then H is closed.

Proof. Let $(\ell_n, v_n)_{n\in\mathbb{N}}$ be a sequence in H such that $(\ell_n, v_n)_{n\in\mathbb{N}} \to (\ell^*, v^*)$. Clearly, since $(\ell_n, v_n) \in H$ for each n, it holds that $(\ell^*, v^*) \in [0,1]^2 \times [0, \overline{v}]^2$ and $|\ell_1^* - \ell_0^*| \geq |v_1^* - v_0^*|$. We will now show that $\frac{\partial u_i(\ell^*, v^*)}{\partial \ell_i} = 0$.

For each $n \in \mathbb{N}$ let $y_{i,n} = \frac{\partial u_i(\ell_n, v_n)}{\partial \ell_i}$, $i = 0, 1$. For each n, $(\ell_n, v_n) \in H$ implies $y_{i,n} = 0$, which means $(y_{i,n})_{n\in\mathbb{N}} \to 0$. Additionally, for all $(\ell, v) \in [0,1]^2 \times [0, \overline{v}]^2$ with $|\ell_1 - \ell_0| \geq |v_1 - v_0|$, $y_0(\ell, v) \in [0, 1]$. This implies that $\frac{\partial u_i(\cdot)}{\partial \ell_i}$ is continuous on H, which means $(y_{i,n})_{n\in\mathbb{N}} \to \frac{\partial u_i(\ell^*, v^*)}{\partial \ell_i}$. Hence, $\frac{\partial u_i(\ell^*, v^*)}{\partial \ell_i} = 0$, which establishes that H is closed. □

Lemma A.4. *Let $\ell_j \in [0,1]$ and $v \in \mathbb{R}_+^2$ be given. Suppose $y_0(s_i, \ell_j, v) \in (0,1)$ and $\frac{\partial u_i(\ell_i^*, \ell_j, v)}{\partial \ell_i} = 0$ for some $\ell_i^* \in [0,1]$. Then $\ell_i^* = \arg\max_{\ell_i \in [0,1]} u_i(\ell, v)$.*

Proof. Consiser supplier zero. Fix $\ell_1 \in [0,1]$ and $v \in \mathbb{R}_+^2$. As the result is trivial if $v_0 = 0$, we consider the case $v_0 > 0$. Since ℓ_1 and v are fixed throughout this proof, we will suppress these as arguments from any functions, and will write $u_0(\ell_0), y_0(\ell_0)$, etc.

First, we show that if $y_0(\ell_0) \in (0,1)$ and $\frac{\partial u_0(\ell_0)}{\partial \ell_0} = 0$ for some $\ell_0 \in [0,1]$ then $\frac{\partial^2 u_0(\ell_0)}{\partial \ell_0^2} < 0$. See that $\frac{\partial u_0(\ell_0)}{\partial \ell_0} = 0$ means $\frac{v_0}{2} f(y_0(\ell_0)) - C_0(\ell_0 - s_0) = 0$. Then,

$$\begin{aligned}
\frac{\partial^2 u_0(\ell_0)}{\partial \ell_0^2} &= \frac{v_0}{4} f'(y_0(\ell_0)) - C_0''(\ell_0 - s_0) \\
&= \frac{f'(y_0(\ell_0))}{2f(y_0(\ell_0))} C_0'(\ell_0 - s_0) - C_0''(\ell_0 - s_0) \\
&< C_0'(\ell_0 - s_0) - C_0''(\ell_0 - s_0) \\
&\leq 0.
\end{aligned}$$

The first inequality holds by Assumption 2(i). The second holds since $C_0'(x) - C_0''(x) \leq C_0'(x) - xC_0''(x)$ for all $x \in [0,1]$ and since $C_0''' \geq 0$ implies $C_0'(x) - xC_0''(x) \leq 0$.

Next, let $\overline{\ell}_0 = \min\{1, 2 + v_0 - v_1 - \ell_1\}$; i.e., $\overline{\ell}_0 = \min\{1, \ell_0 | y_0(\ell_0) = 1\}$. We will show $\arg\max_{\ell_0 \in [0,1]} u_0(\ell_0) = \arg\max_{\ell_0 \in [s_0, \overline{\ell}_0]} u_0(\ell_0)$. To do so, it suffices to show that $\ell_0 \notin [s_0, \overline{\ell}_0] \implies \ell_0 \notin \arg\max_{\ell_0 \in [0,1]} u_0(\ell_0)$. First consider some $\ell_0 < s_0$. Since $y_0(\cdot)$ is increasing, $u_0(\ell_0) = v_0 F(y_0(\ell_0)) - C_0(\ell_0 - s_0) < v_0 F(y_0(s_0)) = u_0(s_0)$, and so $\ell_0 \notin \arg\max_{\ell_0 \in [0,1]}$. Next, consider some $\ell_0 \in [0,1]$ such that $y_0(\ell_0) > 1$. There exists $\ell_0' < \ell_0$ with $y_0(\ell_0') > 1$, and it holds, $u_0(\ell_0) = v_0 - C_0(\ell_0 - s_0) < v_0 - C_0(\ell_0' - s_0)$, and hence, $\ell_0 \notin \arg\max_{\ell_0 \in [0,1]} u_0(\ell_0)$. This establishes that $\arg\max_{\ell_0 \in [0,1]} u_0(\ell_0) = \arg\max_{\ell_0 \in [s_0, \bar{\ell}_0]} u_0(\ell_0)$.

Now, suppose $y_0(s_0) \in (0,1)$ and $\frac{\partial u_0(\ell_0^*)}{\partial \ell_0} = 0$ for some $\ell_0^* \in [0,1]$. We will show $\ell_0^* = \arg\max_{\ell_0 \in [s_0, \bar{\ell}_0]} u_0(\ell_0)$. We first establish feasibility: $\ell_0^* \in [s_0, \bar{\ell}_0]$. See that $\frac{\partial u_0(\ell_0^*)}{\partial \ell_0} = 0$ implies $y_0(\ell_0^*) \in [0,1]$. For if $y_0(\ell_0^*) \notin [0,1]$, then $f(y_0(\ell_0^*)) = 0$ and the first-order condition implies $\ell_0^* = s_0$. However, since $y_0(s_0) \in (0,1)$, this yields a contradiction. Therefore, $y_0(\ell_0^*) \in [0,1]$, which implies $\ell_0^* \leq \bar{\ell}_1$. Additionally, since $\frac{\partial u_0(\ell_0)}{\partial \ell_0} > 0$ for all $\ell_0 \leq s_0$, it must be that $\ell_0^* > s_0$. Hence, $\ell_0^* \in [s_0, \bar{\ell}_0]$.

Finally, we show $\ell_0^* = \arg\max_{\ell_0 \in [s_0, \bar{\ell}_0]} u_0(\ell_0)$. For $\ell_0 \in (s_0, \bar{\ell}_0)$, we have $y_0(\ell) \in (0,1)$, which means u_0 is twice differentiable on the interior of the choice set; moreover, if $\frac{\partial u_0(\ell_0)}{\partial \ell_0} = 0$ for some $\ell_0 \in (s_0, \bar{\ell}_1)$ then $\frac{\partial^2 u_0(\ell_0)}{\partial \ell_0^2} < 0$. This means that $\frac{\partial u_0}{\partial \ell_0}$ crosses the horizontal axis at most once in the interval $[s_0, \bar{\ell}_1]$, and from above. Thus, for $\ell_0 \in [s_0, \ell_0^*)$, $\frac{\partial u_0(\ell_0)}{\partial \ell_0} > 0$ and if $\ell_0^* < \bar{\ell}_0$ then for $\ell_0 \in (\ell_0^*, \overline{\ell_1}]$, $\frac{\partial u_0(\ell_0)}{\partial \ell_0} < 0$. This means $\ell_0^* = \arg\max_{\ell_0 \in [s_0, \bar{\ell}_1]} u_0(\ell_0)$.

□

Proof of Lemma 5. By Lemma A.2, $(\ell, v) \in \mathcal{A}_P \implies |\ell_1 - \ell_0| \geq |v_1 - v_0|$. Moreover, it is straightforward to show that there is some $\bar{v} > 0$ such that $(\ell, v) \in \mathcal{A}_P \implies v_0, v_1 \leq \bar{v}$. We may then equivalently formulate the buyer's problem as $\max_{(\ell,v) \in \hat{\mathcal{F}}_P} \pi_b(\ell, v)$, where

$$\hat{\mathcal{F}}_P = \mathcal{F}_P \cap \{(\ell, v) \in [0,1]^2 \times [0, \bar{v}]^2 | |\ell_1 - \ell_0| \geq |v_1 - v_0|\}.$$

Lemma 3 implies $\hat{\mathcal{F}}_P \neq \emptyset$. Then, to establish that $\mathcal{A}_P \neq \emptyset$, it suffices to show that $\pi_b(\cdot)$ is continuous on $\hat{\mathcal{F}}_P$ and $\hat{\mathcal{F}}_P$ is compact. We first show π_b is continuous.

For all $(\ell, v) \in \hat{\mathcal{F}}_P$, $y_0(\ell, v), \tilde{y}_0(\ell, v) \in [\min\{\ell_0, \ell_1\}, \max\{\ell_0, \ell_1\}] \subseteq [0,1]$. Additionally, if $\ell_0 \leq \ell_1$ then $\pi_b(\ell, v) = q - L(\ell, v)$, where L is as defined in Section 4.3. If $\ell_1 < \ell_0$ then $\pi_b(\ell, v) = q - \tilde{L}(\ell, v)$, where

$$\tilde{L}(\ell, v) = \int_0^{\tilde{y}_0(\ell,v)} |\ell_1 - y| + \int_{\tilde{y}_0(\ell,v)}^1 |\ell_0 - y| + C_0(\ell_0 - s_0) + C_1(\ell_1 - s_1).$$

Since $L(\cdot)$ and $\tilde{L}(\cdot)$ are continuous, and $\ell_0 = \ell_1$ implies $L(\ell, v) = \tilde{L}(\ell, v)$, $\pi_b(\cdot)$ is continuous on $\hat{\mathcal{F}}_P$.

Next, we show $\hat{\mathcal{F}}_P$ is compact. Clearly, $\hat{\mathcal{F}}_P$ is bounded, so it suffices to show that it is closed. To that end, let $(\ell_n, v_n)_{n \in \mathbb{N}}$ be a sequence in $\hat{\mathcal{F}}_P$ such that $(\ell_n, v_n)_{n \in \mathbb{N}} \to (\ell^*, v^*)$. We will show $(\ell^*, v^*) \in \hat{\mathcal{F}}_P$.

It is straightforward to establish that $(\ell^*, v^*) \in [0,1]^2 \times [0, \bar{v}]^2$ and $|\ell_1^* - \ell_0^*| \geq |v_1^* - v_0^*|$.

So, in the remainder of the proof, we will show $\ell^* \in \Phi(v^*)$. For this part of the proof, we consider the case where $s_0 < s_1$ and $s_0, s_1 \neq \frac{1}{2}$. This helps to limit the number of cases we need to consider, but the cases where $s_0 = s_1$ or $s_i = \frac{1}{2}$ could be handled with similar arguments.

For each $n \in \mathbb{N}$, let $d_{i,n} = |\ell_{i,n} - s_i|$ and note that one of the following must hold:

(a) $d_{0,n}, d_{1,n} > 0$ infinitely often (i.o.)

(b) $0 = d_{j,n} < d_{i,n}$ i.o.

(c) $d_{0,n} = d_{1,n} = 0$ i.o.

We will consider each case in turn.

Case (a): $d_{0,n}, d_{1,n} > 0$ **i.o.**

To limit the length of this proof, we will assume (in this case only) that $v_0^* \neq v_1^*$; the case $v_0^* = v_1^*$ can be handled by similar arguments. Without further loss of generality suppose $v_0^* < v_1^*$.

Since $v_1^* > v_0^*$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $v_{1,n} > v_{0,n}$. Since, in addition, $d_{0,n}, d_{1,n} > 0$ i.o., there exists a subsequence, $(\ell_{k(n)}, v_{k(n)})_{n\in\mathbb{N}}$ of $(\ell_n, v_n)_{n\in\mathbb{N}}$ such that $v_{1,k(n)} > v_{0,k(n)}$ and $d_{0,k(n)}, d_{1,k(n)} > 0$ for all $n \in \mathbb{N}$ (here, $k : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function). Since any subsequence of a convergent sequence converges to the same limit as the original sequence, we have that $(\ell_{k(n)}, v_{k(n)})_{n\in\mathbb{N}} \to (\ell^*, v^*)$.

Moreover, since $\ell_{k(n)} \in \Phi(v_{k(n)})$, Proposition A.1(iii) implies that for each $n \in \mathbb{N}$, $\ell_{0,k(n)} > s_0$, $\ell_{1,k(n)} < s_1$, $\ell_{1,k(n)} - \ell_{0,k(n)} > v_{1,k(n)} - v_{0,k(n)}$ and $\frac{\partial u_i(\ell_{k(n)}, v_{k(n)})}{\partial \ell_i} = 0$, $i = 0, 1$. This means, $\ell_0^* \geq s_0$, $\ell_1^* \leq s_1$, $\ell_1^* - \ell_0^* \geq v_1^* - v_0^*$, which implies $\pi_i(\ell^*, v^*) = u_i(\ell^*, v^*)$. Additionally, Lemma A.3 implies, $\frac{\partial u_i(\ell^*, v^*)}{\partial \ell_i} = 0$, $i = 0, 1$; and since $y_i(s_i, \ell_j^*, v^*) \in (0, 1)$ for each i, Lemma A.4 implies, $\ell_i^* = \arg\max_{\ell_i \in [0,1]} u_i(\ell_i, \ell_j^*, v^*)$. We will now show $\ell_i^* \in \arg\max_{\ell_i \in [0,1]} \pi_i(\ell_i, \ell_j^*, v^*)$, $i = 0, 1$.

Consider supplier zero. Let $\ell_0 \in [0, 1]$; we will show $\pi_0(\ell^*, v^*) \geq \pi_0(\ell_0, \ell_1^*, v^*)$. Let $\ell_0' = \ell_1^* - (v_1^* - v_0^*)$. If $\ell_0 \leq \ell_0'$ then $\pi_0(\ell_0, \ell_1^*, v^*) = u_0(\ell_0, \ell_1^*, v^*) \leq u_0(\ell^*, v^*) = \pi_0(\ell^*, v^*)$, where the inequality holds since $\ell_0^* = \arg\max_{\ell_0 \in [0,1]} u_0(\ell_0, \ell_1^*, v^*)$.

If $\ell_0 > \ell_0'$ then $\pi_0(\ell_0, \ell_1^*, v^*) < v_0 - C_0(\ell_0' - s_0)$. For each $n \in \mathbb{N}$, let $x_n = \ell_{1,k(n)} - (v_{1,k(n)} - v_{0,k(n)})$. Note that $\ell_{1,k(n)} - \ell_{0,k(n)} > v_{1,k(n)} - v_{0,k(n)}$ implies $x_n \in (0, 1)$ for each n. Now choose any $n \in \mathbb{N}$. For all $x \in \left(x_n, \max\left\{\ell_{1,k(n)} + v_{1,k(n)} - v_{0,k(n)}, 1\right\}\right)$, $\ell_{k(n)} \in \Phi(v_{k(n)})$ implies $u_0(\ell_{k(n)}, v_{k(n)}) \geq \pi_0(x, \ell_{1,k(n)}, v_{k(n)}) = v_{0,k(n)} - C_0(x - s_0)$, which implies,

$u_0(\ell_{k(n)}, v_{k(n)}) \geq \lim_{x \downarrow x_n} v_{0,k(n)} - C_0(x - s_0) = v_{0,k(n)} - C_0(x_n - s_0)$. Thus, for all $n \in \mathbb{N}$, $u_0(\ell_{k(n)}, v_{k(n)}) \geq v_{0,k(n)} - C_0(x_n - s_0)$. Since $(\ell_{k(n)}, v_{k(n)})_{n\in\mathbb{N}} \to (\ell^*, v^*)$ and $(x_n)_{n\in\mathbb{N}} \to \ell'_0$, continuity of u_0 and C_0 implies, $u_0(\ell^*, v^*) \geq v_0^* - C_0(\ell'_0 - s_0)$. And since $\pi_0(\ell_0, \ell_1^*, v^*) < v_0^* - C_0(\ell'_0 - s_0)$, and $u_0(\ell^*, v^*) = \pi_0(\ell^*, v^*)$, we have $\pi_0(\ell^*, v^*) > \pi_0(\ell_0, \ell_1^*, v^*)$, which establishes that $\ell_0^* \in \arg\max_{\ell_0}, \pi_0(\ell_0, \ell_1^*, v^*)$.

Next, consider supplier one. Let $\ell_1 \in [0, 1]$. If $\ell_1 \geq \ell_0^* + (v_1^* - v_0^*)$, then by analogous arguments as were made for supplier zero, we have $\pi_1(\ell^*, v^*) = u_1(\ell^*, v^*) \geq u_1(\ell_0^*, \ell_1, v^*) = \pi_1(\ell_0^*, \ell_1, v^*)$. If $\ell_1 \in (\ell_0^* - (v_1^* - v_0^*), \ell_0^* + (v_1^* - v_0^*))$ then $\pi_1(\ell_0^*, \ell_1, v^*) = -C_1(\ell_1 - s_1) < 0 < \pi_1(\ell^*, v^*)$.

If $\ell_1 \leq \ell'_1 = \ell_0^* - (v_1^* - v_0^*)$ then $\pi_1(\ell_0^*, \ell_1, v^*) = v_1^* F(\ell_1) - C_1(\ell_1 - s_1) \leq v_1^* F(\ell'_1) - C_1(\ell'_1 - s_1)$. For each $n \in \mathbb{N}$, let $x_n = \max\{\ell_{0,k(n)} - (v_{1,k(n)} - v_{0,k(n)}), 0\}$. Since $0 \leq \ell_1 \leq \ell'_1$, and $(x_n)_{n\in\mathbb{N}} \to \max\{\ell'_1, 0\}$, we have $(x_n)_{n\in\mathbb{N}} \to \ell'_1$. Additionally, for each n, $\pi_1(\ell_{0,k(n)}, x_n, v_{k(n)}) = v_{1,k(n)} F(x_n) - C_1(x_n - s_1)$. Now choose any $n \in N$. $\ell_{k(n)} \in \Phi(v_{k(n)})$ implies $\pi_1(\ell_{k(n)}, v_{k(n)}) \geq \pi_1(\ell_{0,k(n)}, x_n, v_{k(n)})$, which means $u_1(\ell_{k(n)}, v_{k(n)}) \geq v_{1,k(n)} F(x_n) - C_1(x_n - s_1)$. Since $(\ell_{k(n)}, v_{k(n)})_{n\in\mathbb{N}} \to (\ell^*, v^*)$ and $(x_n)_{n\in\mathbb{N}} \to \ell'_1$, continuity of u_1, F, and C_1 implies, $u_1(\ell^*, v^*) \geq v_1^* F(\ell'_1) - C_1(\ell'_1 - s_1)$. And since $u_1(\ell^*, v^*) = \pi_1(\ell^*, v^*)$ and $v_1^* F(\ell'_1) - C_1(\ell'_1 - s_1) \geq \pi_1(\ell_0^*, \ell_1, v^*)$, this means $\pi_1(\ell^*, v^*) \geq \pi_1(\ell_0^*, \ell_1, v^*)$.

This establishes that $\ell^* \in \Phi(v^*)$.

Case (b): $0 = d_{j,n} < d_{i,n}$ **i.o.**

In particular, and WLOG, we will suppose that $0 = d_{1,n} < d_{0,n}$ i.o. Choose any $n \in \mathbb{N}$ such that $0 = d_{1,n} < d_{0,n}$. We must have $v_{0,n} > 0$; and since $s_0, s_1 \neq \frac{1}{2}$ (by Assumption), Proposition A.1(iv) implies $v_{1,n} = 0$. Also see that for $\ell_0 \in (s_1 - v_{0,n}, s_1 + v_{0,n})$, $\pi_0(\ell_0, s_1, v_n) < 0$. Moreover, $\pi_0(\cdot, s_1, v_n)$ is strictly decreasing on $[s_1 + v_{0,n}, 1]$. So $\ell_n \in \Phi(v_n)$ implies that either $\ell_{0,n} < s_1 - v_{0,n}$ or $\ell_{0,n} \in \{s_1 - v_{0,n}, s_1 + v_{0,n}\}$. Then, one of the following must be true:

b(i) $0 = d_{1,n} < d_{0,n}$, $0 = v_{1,n} < v_{0,n}$, and $\ell_{0,n} < s_1 - v_{0,n}$ i.o.

b(ii) $0 = d_{1,n} < d_{0,n}$, $0 = v_{1,n} < v_{0,n}$, and $\ell_{0,n} = s_1 - v_{0,n}$ i.o.

b(iii) $0 = d_{1,n} < d_{0,n}$, $0 = v_{1,n} < v_{0,n}$, and $\ell_{0,n} = s_1 + v_{0,n}$ i.o.

We consider each case in turn and we will show $\ell^* \in \Phi(v^*)$. Before proceeding, we point out that $\ell_{1,n} = s_1$ and $v_{1,n} = 0$ i.o. implies $\ell_1^* = s_1$ and $v_1^* = 0$. Since, $\ell_1^* = s_1$ is a dominant strategy for supplier 1 when $v_1 = v_1^* = 0$, to establish that $\ell^* \in \Phi(v^*)$ it suffices to show $\ell_0^* \in \arg\max_{\ell_0 \in [0,1]} \pi_0(\ell_0, s_1, v^*)$.

Subcase b(i): $\ell_{0,n} < s_1 - v_{0,n}$ **i.o.**

There is a subsequence, $(\ell_{k(n)}, v_{k(n)})_{n\in\mathbb{N}}$ of $(\ell_n, v_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $\ell_{1,k(n)} = s_1$, $0 = v_{1,k(n)} < v_{0,k(n)}$, $\ell_{0,k(n)} < s_1 - v_{0,k(n)}$, and $\ell_{0,k(n)} \neq s_0$.

Take any $n \in \mathbb{N}$; since $\ell_{0,k(n)} < s_1 - v_{0,k(n)}$, for all ℓ_0 in some neighborhood of $\ell_{0,k(n)}$, $\pi_0(\ell_0, \ell_{1,k(n)}, v_{k(n)}) \equiv u_0(\ell_0, \ell_{1,k(n)}, v_{k(n)})$. Since $u_0(\cdot, \ell_{1,k(n)}, v_{k(n)})$ is differentiable in this neighborhood, $\ell_{0,k(n)}$ must satisfy the first-order condition, $\frac{\partial u_0(\ell_{k(n)}, v_{k(n)})}{\partial \ell_0} = 0$. Thus, for all $n \in \mathbb{N}$, $\frac{\partial u_0(\ell_{k(n)}, v_{k(n)})}{\partial \ell_0} = 0$; by Lemma A.3 $\frac{\partial u_0(\ell^*, v^*)}{\partial \ell_0} = 0$. From this point, the proof that $\ell_0^* \in \arg\max_{\ell_0 \in [0,1]} \pi_0(\ell_0, \ell_1^*, v^*)$ follows identical arguments as were made in case (a).

Subcase b(ii): $\ell_{0,n} = s_1 - v_{0,n}$ **i.o.**

There is a subsequence, $(\ell_{k(n)}, v_{k(n)})_{n\in\mathbb{N}}$ of $(\ell_n, v_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $\ell_{1,k(n)} = s_1$, $0 = v_{1,k(n)} < v_{0,k(n)}$, $\ell_{0,k(n)} = s_1 - v_{0,k(n)}$, and $\ell_{0,k(n)} \neq s_0$. Then for each n, $\pi_0(\ell_{k(n)}, v_{k(n)}) = u_0(\ell_{k(n)}, v_{k(n)})$. Moreover, see that, $(\ell_{0,k(n)})_{n\in\mathbb{N}} \to s_1 - v_0^*$, which means $\ell_0^* = s_1 - v_0^*$ and $\pi_0(\ell^*, v^*) = u_0(\ell^*, v^*)$.

We will now show that $\pi_0(\ell^*, v^*) \geq \pi_0(\ell_0, s_1, v^*)$ for all $\ell_0 \in [0,1]$. Since $\pi_0(\ell_0, s_1, v^*) < 0$ for $\ell_0 \in (s_1 - v_0^*, s_1 + v_0^*)$ and $\pi_0(\cdot, s_1, v^*)$ is strictly decreasing on $[s_1 + v_0^*, 1]$, it suffices to restrict attention to $\ell_0 \in [0, s_1 - v_0^*) \cup \{s_1 + v_0^*\}$.

Let $\ell_0 \in [0, s_1 - v_0^*)$. Since $(v_{0,k(n)})_{n\in\mathbb{N}} \to v_0^*$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $\ell_0 \in [0, s_1 - v_{0,k(n)})$, and hence $\pi_0(\ell_0, s_1, v_{k(n)}) = u_0(\ell_0, s_1, v_{k(n)})$. Then for all $n > N$, $\ell_{k(n)} \in \Phi(v_{k(n)})$ implies $u_0(\ell_{k(n)}, v_{k(n)}) \geq u_0(\ell_0, s_1, v_{k(n)})$. Since $(\ell_{k(n)}, v_{k(n)}) \to (\ell^*, v^*)$, continuity of u_0 implies, $u_0(\ell^*, v^*) \geq u_0(\ell_0, s_1, v^*)$. And since, $\pi_0(\ell^*, v^*) = u_0(\ell^*, v^*)$ and $u_0(\ell_0, s_1, v^*) = \pi_0(\ell_0, s_1, v^*)$, we have $\pi_0(\ell^*, v^*) \geq \pi_0(\ell_0, s_1, v^*)$.

Next, let $\ell_0 = s_1 + v_0^*$. Then, $\pi_0(\ell_0, s_1, v^*) = v_0^*(1 - F(\ell_0)) - C_0(\ell_0 - s_0)$. For each $n \in \mathbb{N}$ let $x_n = \min\{s_1 + v_{0,k(n)}, 1\}$. Since $s_1 + v_0^* \leq 1$, $(x_n)_{n\in\mathbb{N}} \to s_1 + v_0^* = \ell_0$; moreover, $\pi_0(x_n, s_1, v_{k(n)}) = v_{0,k(n)}(1 - F(x_n)) - C_0(x_n - s_0)$. Choose any $n \in \mathbb{N}$. Since $\ell_{k(n)} \in \Phi(v_{k(n)})$, $\pi_0(\ell_{k(n)}, v_{k(n)}) \geq \pi_0(x_n, s_1, v_{k(n)})$, which means $u_0(\ell_{k(n)}, v_{k(n)}) \geq v_{0,k(n)}(1 - F(x_n)) - C_0(x_n - s_0)$. Since $(\ell_{k(n)}, v_{k(n)})_{n\in\mathbb{N}} \to (\ell^*, v^*)$ and $(x_n)_{n\in\mathbb{N}} \to \ell_0$, continuity of u_0, F, and C_0 implies, $u_0(\ell^*, v^*) \geq v_0^*(1 - F(\ell_0)) - C_0(\ell_0 - s_0)$, which means $\pi_0(\ell^*, v^*) \geq \pi_0(\ell_0, s_1, v^*)$. This completes the proof for subcase b(ii).

Subcase b(iii): $\ell_{0,n} = s_1 + v_{0,n}$ **i.o.**

There is a subsequence, $(\ell_{k(n)}, v_{k(n)})_{n\in\mathbb{N}}$ of $(\ell_n, v_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $\ell_{1,k(n)} = s_1$, $0 = v_{1,k(n)} < v_{0,k(n)}$, $\ell_{0,k(n)} = s_1 + v_{0,k(n)}$. For each $n \in \mathbb{N}$, we have $\pi_0(\ell_{k(n)}, v_{k(n)}) =$

$v_{0,k(n)}(1 - F(\ell_{0,k(n)})) - C_0(\ell_{0,k(n)} - s_0)$. Moreover, see that, $(\ell_{0,k(n)})_{n\in\mathbb{N}} \to s_1 + v_0^*$, which means $\ell_0^* = s_1 + v_0^*$ and $\pi_0(\ell^*, v^*) = v_0^*(1 - F(\ell_0^*)) - C_0(\ell_0^* - s_0)$.

We will now show that $\pi_0(\ell^*, v^*) \geq \pi_0(\ell_0, s_1, v^*)$ for all $\ell_0 \in [0, 1]$. First note that since $\pi_0(\ell_0, s_1, v^*) < 0$ for $\ell_0 \in (s_1 - v_0^*, s_1 + v_0^*)$ and $\pi_0(\cdot, s_1, v^*)$ is strictly decreasing on $[s_1 + v_0^*, 1]$, it suffices to show $\pi_0(\ell^*, v^*) \geq \pi_0(\ell_0, s_1, v^*)$ for all $\ell_0 \in [0, s_1 - v_0^*]$. As the result is trivial if $s_1 - v_0^* \leq 0$, consider the case $s_1 - v_0^* > 0$.

Let $\ell_0 \in [0, s_1 - v_0^*]$, and note that $\pi_0(\ell_0, s_1, v^*) = u_0(\ell_0, s_1, v^*)$. Define a sequence $(x_n)_{n\in\mathbb{N}}$ as follows: For each $n \in \mathbb{N}$, let $x_n = \ell_0$ if $\ell_0 < s_1 - v_{0,k(n)}$ and $x_n = \max\{s_1 - v_{0,k(n)}, 0\}$ otherwise. By construction, $(x_n)_{n\in\mathbb{N}} \to \ell_0$. Moreover, since $s_1 - v_0^* > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $s_1 - v_{0,k(n)} > 0$. Fix $n > N$. It holds that $x_n \leq s_1 - v_{0,k(n)}$, which means $\pi_0(x_n, s_1, v_{k(n)}) = u_0(x_n, s_1, v_{k(n)})$; and since $\ell_{k(n)} \in \Phi(v_{k(n)})$, we must have, $\pi_0(\ell_{k(n)}, v_{k(n)}) \geq \pi_0(x_n, s_1, v_{k(n)})$. Thus, for all $n > N$, $v_{0,k(n)}(1-F(\ell_{0,k(n)}))-C_0(\ell_{0,k(n)}-s_0) \geq u_0(x_n, s_1, v_{k(n)})$. Since $(\ell_{k(n)}, v_{k(n)})_{n\in\mathbb{N}} \to (\ell^*, v^*)$, $(x_n)_{n\in\mathbb{N}} \to \ell_0$, continuity of F, C_0, and u_0 implies, $v_0^*(1 - F(\ell_0^*)) - C_0(\ell_0^* - s_0) \geq u_0(\ell_0, s_1, v^*)$, which means $\pi_0(\ell^*, v^*) \geq \pi_0(\ell_0, \ell_1^*, v^*)$. This establishes that $\ell^* \in \Phi(v^*)$, which completes the proof of case (b).

Case (c): $d_{0,n} = d_{1,n} = 0$ i.o.

First note that since $\ell_{0,n} = s_0$ and $\ell_{1,n} = s_1$ i.o. and $(\ell_n)_{n\in\mathbb{N}} \to \ell^*$, it must be that $\ell_i^* = s_i$ for each i. If $v_0^* = v_1^* = 0$ then clearly $\ell^* \in \Phi(v^*)$. So in the remainder of this proof we consider the case where $v_i^* > 0$ for some i. In particular, and WLOG, we suppose $v_1^* > 0$.

Since $v_1^* > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $v_{1,n} > 0$. Since, in addition, $\ell_{0,n} = s_0$ and $\ell_{1,n} = s_1$ i.o., there is a subsequence, $(\ell_{k(n)}, v_{k(n)})_{n\in\mathbb{N}}$ of $(\ell_n, v_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $\ell_{0,k(n)} = s_0$, $\ell_{1,k(n)} = s_1$, and $v_{1,k(n)} > 0$. Before we show that $\ell^* \in \Phi(v^*)$, we first derive some useful properties of the sequence $(\ell_{k(n)}, v_{k(n)})_{n\in\mathbb{N}}$ and its limit, (ℓ^*, v^*). To begin, we show that we must have $v_{0,k(n)} = 0$ for each n.

By way of contradiction, suppose $v_{0,k(n)} > 0$ for some $n \in \mathbb{N}$. Since $(\ell_{k(n)}, v_{k(n)}) \in \hat{\mathcal{F}}_P$, we have $s_1 - s_0 \geq |v_{1,k(n)} - v_{0,k(n)}|$. If $s_1 - s_0 > |v_{1,k(n)} - v_{0,k(n)}|$, then for ℓ_0 in some neighborhood of s_0, $\pi_0(\ell_0, \ell_{1,k(n)}, v_{k(n)}) \equiv u_0(\ell_0, \ell_{1,k(n)}, v_{k(n)})$. Since $\frac{\partial u_0(s_0, \ell_{1,k(n)}, v_{k(n)})}{\partial \ell_0} = v_{0,k(n)} f(y_0(\ell_{k(n)}, v_{k(n)})) > 0$, there is some $\ell_0' > s_0$ sufficiently close to s_0 that $\pi_0(\ell_0', s_1, v_{k(n)}) > \pi_0(\ell_{k(n)}, v_{k(n)})$, which contradicts the hypothesis that $\ell_{k(n)} \in \Phi(v_{k(n)})$. So it must be that $s_1 - s_0 = |v_{1,k(n)} - v_{0,k(n)}|$. Since $s_0 \neq s_1$ (by assumption), we have that $v_{0,k(n)} \neq v_{1,k(n)}$; in particular, suppose WLOG, $0 < v_{0,k(n)} < v_{1,k(n)}$. Then $\pi_0(\ell_{k(n)}, v_{k(n)}) = v_{0,k(n)} F(s_1) < v_{0,k(n)} = \lim_{\ell_0 \downarrow s_0} \pi_0(\ell_0, \ell_{1,k(n)}, v_{k(n)})$, and so supplier zero has a profitable deviation to some $\ell_0' > s_0$ sufficiently close to s_0. Again, this contradicts the hypothesis that $\ell_{k(n)} \in \Phi(v_{k(n)})$.

Thus, it must be that $v_{0,k(n)} = 0$ for each $n \in \mathbb{N}$, and so, $v_0^* = 0$.

We now show that $s_1 - s_0 = v_{1,k(n)}$ for all $n \in \mathbb{N}$. To see this, choose any $n \in \mathbb{N}$ and, by way of contradiction, suppose $s_1 - s_0 \neq v_{1,k(n)}$. Since $(\ell_{k(n)}, v_{k(n)}) \in \hat{\mathcal{F}}_P$ it must be that $s_1 - s_0 > v_{1,k(n)}$. Then for ℓ_1 in some neighborhood of s_1, $\pi_1(\ell_{0,k(n)}, \ell_1, v_{k(n)}) \equiv u_1(\ell_{0,k(n)}, \ell_1, v_{k(n)})$. Since $\frac{\partial u_1(\ell_{0,k(n)}, s_1, v_{k(n)})}{\partial \ell_1} = -v_{1,k(n)} f(y_0(\ell_{k(n)}, v_{k(n)}) < 0$, there is some $\ell_1' < s_1$ sufficiently close to s_1 that $\pi_1(\ell_{0,k(n)}, \ell_1', v_{k(n)}) > \pi_1(\ell_{k(n)}, v_{k(n)})$, which contradicts the hypothesis that $\ell_{k(n)} \in \Phi(v_{k(n)})$. Therefore, we must have $s_1 - s_0 = v_{1,k(n)}$ for all $n \in \mathbb{N}$. This means $\pi_1(\ell_{k(n)}, v_{k(n)}) = u_1(\ell_{k(n)}, v_{k(n})$ for all $n \in \mathbb{N}$; moreover, $s_1 - s_0 = v_1^*$, and so $\pi_1(\ell^*, v^*) = u_1(\ell^*, v^*) = v_1^* (1 - F(s_1))$.

We now show that $\ell^* \in \Phi(v^*)$. Since $v_0^* = 0$, the approach s_0 is a dominant strategy for supplier zero and so, $\ell_0^* = \arg\max_{\ell_0} \pi_0(\ell_0, \ell_1^*, v^*)$. Then, to establish that $\ell^* \in \Phi(v^*)$, it suffices to show $s_1 \in \arg\max_{\ell_1} \pi_1(\ell_0^*, \ell_1, v^*)$.

Let $\ell_1 \in [0,1]$. We will show $\pi_1(\ell^*, v^*) \geq \pi_1(\ell_0^*, \ell_1, v^*)$. First suppose $\ell_1 > s_1$. Then, $\pi_1(\ell_0^*, \ell_1, v^*) = v_1^* (1 - F(y_0(\ell_0^*, \ell_1, v^*))) - C_1(\ell_1 - s_1) < v_1^*(1 - F(s_1)) = \pi_1(\ell^*, v^*)$. Next, suppose $\ell_1 \in (s_0 - v_1^*, s_0 + v_1^*)$. Then $\pi_1(\ell_0^*, \ell_1, v^*) = -C_1(\ell_1 - s_1) < 0 < \pi_1(\ell^*, v^*)$.

Last, suppose $\ell_1 \leq \ell_1' = s_0 - v_1^*$. Since $\pi_1(\ell_0^*, \cdot, v^*)$ is strictly increasing on $[0, \ell_1']$, we have $\pi_1(\ell_0^*, \ell_1, v^*) \leq \pi_1(\ell_0^*, \ell_1', v^*) = v_1^* F(\ell_1') - C_1(\ell_1' - s_1)$. For each $n \in \mathbb{N}$, let $x_n = \max\{s_0 - v_{1,k(n)}, 0\}$. Since $0 \leq s_0 - v_1^*$, we have $(x_n)_{n\in\mathbb{N}} \to s_0 - v_1^* = \ell_1'$. Moreover, for each $n \in \mathbb{N}$, $\pi_1(\ell_{0,k(n)}, x_n, v_{k(n)}) = v_{1,k(n)} F(x_n) - C_1(x_n - s_1)$. Then $\ell_{k(n)} \in \Phi(v_{k(n)})$ implies, $\pi_1(\ell_{k(n)}, v_{k(n)}) \geq \pi_1(\ell_{0,k(n)}, x_n, v_{k(n)})$, which means for all $n \in \mathbb{N}$, $u_1(\ell_{k(n)}, v_{k(n)}) \geq v_{1,k(n)} F(x_n) - C_1(x_n - s_1)$. Since $(\ell_{k(n)}, v_{k(n)})_{n\in\mathbb{N}} \to (\ell^*, v^*)$ and $(x_n)_{n\in\mathbb{N}} \to \ell_0'$, continuity of u_1, F, and C_1, implies $u_1(\ell^*, v^*) \geq v_1^* F(\ell_1') - C_1(\ell_1' - s_1)$. And since $u_1(\ell^*, v^*) = \pi_1(\ell^*, v^*)$ and $\pi_1(\ell_0^*, \ell_1, v^*) \leq v_1^* F(\ell_1') - C_1(\ell_1' - s_1)$, we have $\pi_1(\ell^*, v^*) \geq \pi_1(\ell_0^*, \ell_1, v^*)$.

This completes the proof of case (c) and establishes the lemma.

□

Proof of Lemma 6. Let $(\ell^*, v^*) \in \mathcal{A}_P$; we show that $v_i^* > 0$ if and only if $d_i^* > 0$. The fact that $d_i^* > 0$ implies $v_i^* > 0$ trivially follows from feasibility. So we show that $v_i^* > 0$ implies $d_i^* > 0$. Consider the case, $v_0^* > 0$; the case $v_1^* > 0$ follows analogous arguments. Proceeding by contradiction, suppose that $d_0^* = 0$.

Case 1: $d_0^* = d_1^* = 0$. It must be that $|v_1^* - v_0^*| \geq s_1 - s_0$; otherwise, for ℓ_0 in some neighborhood of s_0, $\pi_0(\ell_0, s_1) = u_0(\ell_0, s_1)$ and $\frac{\partial \pi_0(s_0, s_1)}{\partial \ell_0} = \frac{v_0^*}{2} f(y_0(s_0, s_1)) > 0$, which means supplier zero has a profitable deviation to some $\ell_0' > s_0$. Then, Lemma A.2 implies, $|v_1^* - v_0^*| = s_1 - s_0$.

Consider the case, $s_0 < s_1$. Let $v_0' = v_1' = 0$ and note that $(s_0, s_1) \in \Phi(v')$. Moreover,

$\pi_b(\ell^*, v^*) = q - L(\ell^*, v^*) < q - L_{FB}(\ell^*) = \pi_b(\ell^*, v')$, where the inequality holds since $v_0^* \neq v_1^*$ implies $L_{FB}(\ell^*) < L(\ell^*, v^*)$. The final equality holds since $v_0' = v_1'$ implies $L_{FB}(\ell^*) = L(\ell^*, v')$. This contradicts the hypothesis that $(\ell^*, v^*) \in \mathcal{A}_P$.

Next, suppose $s_0 = s_1 = s$. Following similar arguments as were made in the proof of Lemma A.2, there exists $v_0' > 0 = v_1'$ and $\ell' \in \Phi(v')$ such that $\pi_b(\ell', v') > \pi_b(\ell^*, v^*)$, contradicting the hypothesis that $(\ell^*, v^*) \in \mathcal{A}_P$.

Case 2: $d_1^* > 0 = d_0^*$. Proposition A.1(iv) implies $s_0 = \frac{1}{2} = \ell_1^*$ and $v_0^* = v_1^*$. Let $v_0' = v_1' = 0$ and $\ell' = (s_0, s_1)$. Then, $\ell' \in \Phi(v')$ and $\pi_b(\ell^*, v^*) = \mathbb{E}\left[Q_0\left(\frac{1}{2}, y\right)\right] - C_1(d_1^*) < \mathbb{E}\left[\max\left\{Q_0\left(\frac{1}{2}, y\right), Q_1(s_1, y)\right\}\right] = \pi_b(\ell', v')$, contradicting the hypothesis that $(\ell^*, v^*) \in \mathcal{A}_P$. This establishes the lemma. □

Lemma A.5. *If $s_0, 1 - s_1 < s^*$, then $|v_1^* - v_0^*| < s_1 - s_0$.*

Proof. Let $s_0, 1 - s_1 < s^*$, $(\ell^*, v^*) \in \mathcal{A}_P$ and let $d_i^* = |\ell_i^* - s_i|$. Proceed by contradiction and suppose $|v_1^* - v_0^*| \geq s_1 - s_0$. If $v_0^*, v_1^* > 0$, then Proposition A.1(v) implies, $d_0^* = d_1^* = 0$, contradicting Lemma 6. So, we must have $v_i^* > 0 = v_j^*$. WLOG, suppose $v_1^* > 0 = v_0^*$. Lemma 6 implies $d_1^* > 0 = d_0^*$, and so $\ell_0^* = s_0$. Then, using Proposition A.1(vi), and the fact that $s_0 < \frac{1}{2} < s_1$, it is easy to show that we must have $\ell_1^* = s_0 + v_1^* > s_1$, which means $\pi_b(\ell^*, v^*) = q - L(\ell^*, v^*)$.

Now let $\ell' = (s_0, s_1)$ and $v' = (0, 0)$, and see that $\pi_b(\ell', v') = q - \kappa_{FB}(\ell')$ and $\ell' \in \Phi(v')$. We will show that $\pi_b(\ell', v') > \pi_b(\ell^*, v^*)$; equivalently, $\kappa_{FB}(\ell') < L(\ell^*, v^*)$. First, since $v_0^* \neq v_1^*$ and $\ell_1^* \neq s_1$, $L(\ell^*, v^*) > \kappa_{FB}(\ell^*)$. Next, since $s_1 > 1 - s^*$, for all $\ell_1 \in [s_1, 1]$, $\frac{\partial \kappa_{FB}(s_0, \ell_1)}{\partial \ell_1} = 2F(\ell_1) - F\left(\frac{s_0 + \ell_1}{2}\right) - 1 > 2F(1 - s^*) - F\left(\frac{s^* + 1 - s^*}{2}\right) - 1 = 0$, where the inequality follows since and $\frac{\partial \kappa_{FB}(s_0, \ell_1)}{\partial \ell_1}$ is strictly increasing in ℓ_1 and strictly decreasing in s_0. It follows that $\kappa_{FB}(s_0, s_1) < \kappa_{FB}(s_0, \ell_1^*)$, and thus, $\kappa_{FB}(\ell') < L(\ell^*, v^*)$. Since $(\ell', v') \in \mathcal{F}_P$, this contradicts the optimality of (ℓ^*, v^*). □

Proof of Proposition 3: Let $s_0, 1 - s_1 < s^*$ be given and let $(\ell^*, v^*) \in \mathcal{A}_P$. We first show that $v_0^*, v_1^* > 0$. Proceed by contradiction and suppose $v_i^* = 0$ for some i; in particular and WLOG, suppose $v_0^* = 0$. Obviously, this means $\ell_0^* = s_0$. Lemmas 6 and A.5 together with Proposition A.1(vii) imply, $\ell_1^* \geq s_0 + v_1^*$ and $\frac{\partial u_1(\ell^*, v^*)}{\partial \ell_1} \leq 0$, which means (ℓ^*, v^*) is feasible in (AUX 2); moreover, $\pi_b(\ell^*, v^*) = q - L(\ell^*, v^*)$.

We now show that $\ell_1^* > s_0 + v_1^*$ and $v_1^* > 0$. Proceeding by contradiction, suppose that either $\ell_1^* = s_0 + v_1^*$ or $v_1^* = 0$. Let (ℓ_1^a, v_1^a) solve (AUX 2), and let $\ell^a = (s_0, \ell_1^a)$ and $v^a = (0, v_1^a)$. By Lemma B.6, $\ell^a \in \Phi(v^a)$; so $(\ell^a, v^a) \in \mathcal{F}_P$; moreover, $\ell_1^a > s_0 + v_1^a$, which means $\pi_b(\ell^a, v^a) = q - L(\ell^a, v^a)$. Since $(\ell^*, v^*) \in \mathcal{A}_P$, we have $L(\ell^*, v^*) \leq L(\ell^a, v^a)$. However, since (ℓ^*, v^*) is feasible in (AUX 2) and $\ell_1^* = v_1^* - s_0$ or $v_1^* = 0$, Lemma B.6 implies (ℓ^*, v^*) cannot solve (AUX 2); so, $L(\ell^a, v^a) < L(\ell^*, v^*)$. We have a contradiction, and therefore, $v_1^* > 0$ and $\ell_1^* > s_0 + v_1^*$.

Now since $\ell_1^* > s_0 + v_1^*$, in some neighborhood of ℓ_1^*, $\pi_1(\ell^*, v^*) = u_1(\ell^*, v^*)$. Optimality of ℓ_1^* implies $\frac{\partial u_1(\ell^*, v^*)}{\partial \ell_1^*} = 0$. This means, $(\ell^*, v^*) \in \mathcal{F}_{P'}$. Now let $(\ell', v') \in \mathcal{A}_{P'}$. Since $v_0^* = 0$, Lemma B.2 implies $(\ell^*, v^*) \notin \mathcal{A}_{P'}$, which means $L(\ell', v') < L(\ell^*, v^*)$. However, by Proposition B.1, $\ell' \in \Phi(v')$ and $\pi_b(\ell', v') = q - L(\ell', v')$. By definition of (ℓ^*, v^*), we must have $L(\ell^*, v^*) \leq L(\ell', v')$. This yields a contradiction; therefore, we must have $v_0^*, v_1^* > 0$. By Lemma 6, $d_0^*, d_1^* > 0$. And by Proposition A.1(iii), $s_0 < \ell_0^* < \ell_1^* < s_1$.

Next, let $s^* \leq s_0, 1 - s_1$ and $(\ell^*, v^*) \in \mathcal{A}_P$. We will show that $v_i^* = 0$ for some i.

To begin, we show that if $s_0 < s_1$ then for all ℓ such that $s_0 < \ell_0 < \ell_1 < s_1$, $\kappa_{FB}(\ell) > \kappa_{FB}(s_0, s_1)$. To see this, suppose $s_0 < s_1$ and let ℓ be given such that $s_0 < \ell_0 < \ell_1 < s_1$. Convexity of κ_{FB} implies, $\kappa_{FB}(\ell) - \kappa_{FB}(s_0, s_1) \geq (\ell_0 - s_0)\frac{\partial \kappa_{FB}(s_0, s_1)}{\partial \ell_0} + (\ell_1 - s_1)\frac{\partial \kappa_{FB}(s_0, s_1)}{\partial \ell_1}$. Then, see that $\frac{\partial \kappa_{FB}(s_0, s_1)}{\partial \ell_0} = 2F(\ell_0) - F\left(\frac{s_0 + s_1}{2}\right) > 2F(s^*) - F\left(\frac{s^* + 1 - s^*}{2}\right) = 0$. A similar argument reveals that $\frac{\partial \kappa_{FB}(s_0, s_1)}{\partial \ell_1} < 0$. Since $\ell_0 > s_0$ and $\ell_1 < s_1$, $(\ell_0 - s_0)\frac{\partial \kappa_{FB}(s_0, s_1)}{\partial \ell_0} + (\ell_1 - s_1)\frac{\partial \kappa_{FB}(s_0, s_1)}{\partial \ell_1} > 0$. Thus, $\kappa_{FB}(\ell) > \kappa_{FB}(s_0, s_1)$.

We now show $v_i^* = 0$ for some i. Proceeding by contradiction, suppose $v_0^*, v_1^* > 0$. Lemma 6 implies $d_0^*, d_1^* > 0$. And by Proposition A.1(iii), $s_0 < \ell_0^* < \ell_1^* < s_1$ and $\ell_1^* - \ell_0^* > |v_1^* - v_0^*|$. Now let $\ell' = (s_0, s_1)$ and $v_0' = v_1' = 0$. Then, $\pi_b(\ell^*, v^*) = q - L(\ell^*, v^*) < q - \kappa_{FB}(\ell^*)$, $\pi_b(\ell', v') = q - \kappa_{FB}(\ell')$. By the argument provided in the previous paragraph, $\kappa_{FB}(\ell^*) > \kappa_{FB}(\ell')$. Hence, $\pi_b(\ell^*, v^*) < \pi_b(\ell', v')$. But since $\ell' \in \Phi(v')$, this contradicts the definition of (ℓ^*, v^*).

□

Proof of Proposition 4: We divide the proof into 3 parts. In Part 0, we provide several bounds that will be used in the main part of the proof. In Part I, we show that if s_0 is sufficiently close to s_1 then $(\ell^*, v^*) \in \mathcal{A}_P$ implies $v_j^* = 0 < v_i^*$. In Part II, we show that $s_1 - s_0 < |\ell_i^* - s_i| = v_i^*$.

For the remainder of the proof, fix $s_1 \in (0, 1)$, and WLOG, suppose $s_1 \geq \frac{1}{2}$.

Part 0

For $x \in \left[0, \frac{s_1}{2}\right]$, let $\Psi_1(x)$ be defined, $\Psi_1(x) = \kappa_{FB}(s_1 - x, s_1) - \kappa(s_1 - 2x, s_1, 2x, 0) - C_0(x)$; i.e.,

$$\Psi_1(x) = \int_0^{s_1-\frac{x}{2}} |s_1-x-y|dF(y)+\int_{s_1-\frac{x}{2}}^1 |s_1-y|dF(y)-\int_0^{s_1-2x} (s_1-2x-y)dF(y)-\int_{s_1-2x}^1 |s_1-y|dF(y)-C_0(x).$$

Note that $\Psi_1(0) = 0$ and $\Psi_1'(0) = F(s_1) > 0$. By continuity, there exists $\epsilon_1 > 0$ such $x \in (0, \epsilon_1)$ implies $\Psi_1(x) > 0$.

Next, for $x \in \left[0, \frac{s_1}{2}\right]$, let $\Psi_2(x) = A(x) - \max\{0, 2x(1 - F(s_1 + 2x)) - C_0(3x)\}$, where $A(x) = 2xF(s_1 - 2x) - C_0(x)$. We show that $\Psi_2(x) > 0$ for x sufficiently small. To begin, $s_1 \geq \frac{1}{2}$ implies $F(s_1 - 2x) \geq 1 - F(s_1 + 2x)$. And since $C_0(x) \leq C_0(3x)$, holding strictly for $x > 0$, we have $A(x) > 2x(1 - F(s_1 + 2x)) - C_0(3x)$ for all $x > 0$. Moreover, see that $A(0) = 0$ and $A'(0) = 2F(s_1) > 0$. Thus, there exists $\epsilon_2 > 0$ such that $x \in (0, \epsilon_2)$ implies $A(x) > 0$. Hence, $\Psi_2(x) > 0$ for all $x \in (0, \epsilon_2)$.

Next, we show that there exists $\epsilon_3 > 0$ such that if $x \in (0, \epsilon_3)$, $\ell, \ell' \in [s_1 - x, s_1]^4$, $\ell_0 \leq \ell_0' \leq \ell_1' \leq \ell_1$, and $\ell_0 < \ell_1$ then $\kappa_{FB}(\ell) < \kappa_{FB}(\ell')$. First, since $2F(s_1) > F(s_1)$ by continuity of F, there exists $\epsilon' > 0$ such that $2F(s_1 - \epsilon') > F(s_1)$. Similarly, there exists $\epsilon'' > 0$ such that $2F(s_1) < 1 + F(s_1 - \epsilon'')$. Now let $\epsilon_3 = \min\{\epsilon', \epsilon''\}$. And let $x \in (0, \epsilon_3)$.

We now show that for all $\ell \in [s_1 - x, s_1]^2$ with $\ell_0 \leq \ell_1$, $\frac{\partial \kappa_{FB}(\ell)}{\partial \ell_0} > 0 > \frac{\kappa_{FB}(\ell)}{\partial \ell_1}$. Let $\ell \in [s_1 - x, s_1]$. Then $\frac{\partial \kappa_{FB}(\ell)}{\partial \ell_0} = 2F(\ell_0) - F(m(\ell)) \geq 2F(s_1 - x) - F(s_1) > 0$, where the final inequality holds since $x < \epsilon'$. A similar argument shows that $\frac{\partial \kappa(\ell)}{\partial \ell_1} < 0$. Finally, since $\frac{\partial \kappa_{FB}(\ell)}{\partial \ell_0} > 0 > \frac{\kappa_{FB}(\ell)}{\partial \ell_1}$ for all $\ell \in [s_1 - x, s_1]^2$, this implies $\kappa_{FB}(\ell) < \kappa(\ell')$ for all $\ell, \ell' \in [s_1 - x, s_1]^4$ with $\ell_0 \leq \ell_0' \leq \ell_1' \leq \ell_1$ and $\ell_0 < \ell_1$.

Next, let $\Psi_3(x) = F(s_1 - 2x) - xf(s_1 - 2x) - C_1'(2x) - 1 + F(s_1 - x)$, and suppose $s_1 > \frac{1}{2}$. We have $\Psi(0) = 2F(s_1) - 1 > 0$. Continuity implies that there exists $\epsilon_4 > 0$ such that $\Psi(x) > 0$ for all $x \in [0, \epsilon_4)$.

For the remainder of the proof, let $0 < \epsilon < \min\left\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \frac{s_1}{2}\right\}$, fix $s_0 \in (s_1 - \epsilon, s_1)$, and let $(\ell^*, v^*) \in \mathcal{A}_P(s_0, s_1)$. We also let $\delta = s_1 - s_0$. The case where $s_0 = s_1$ follows similar, but distinct, arguments and is omitted for brevity; it is available upon request.

Part I

In this section, we show that show that $v_i^* > 0 = v_j^*$.

To begin, we establish that $v_i^* > 0$ for some i. To do so, it suffcices to show that for some

$(\ell', v') \in \mathcal{F}_P$, $\pi_b(\ell', v') > \pi_b^{\text{def}}$, where π_b^{def} is the buyer's payoff at the default.

Now let $v_0' = 2\delta$, $v_1' = 0$, $\ell_0' = s_1 - 2\delta$, and $\ell_1' = s_1$. First we show $\pi_b(\ell', v') > \pi_b^{\text{def}}$. See that $\pi_b^{\text{def}} = q - \kappa_{FB}(s_0, s_1) = q - \kappa_{FB}(s_1 - \delta, s_1)$ and $\pi_b(\ell', v') = q - \kappa(s_1 - 2\delta, s_1, 2\delta, 0) - C_0(\delta)$. Hence, $\pi_b(\ell', v') - \pi_b^{def} = \Psi_1(\delta)$, where Ψ_1 is as defined in Part 0. And since $\delta \in (0, \epsilon_1)$, we have $\Psi_1(\delta) > 0$. Hence, $\pi_b(\ell', v') > \pi_b^{def}$.

Next, we show that $\ell' \in \mathcal{F}_P$. First, since $\delta < \epsilon < \frac{s_1}{2}$, clearly $\ell' \in (0,1)^2$. It remains to show that $\ell' \in \Phi(v')$. Since $v_1' = 0$, clearly $s_1 \in \arg\max_{\ell_1} \pi_1(\ell_0', \ell_1, v')$. Next, we show $\ell_0' \in B_0(\ell_1') = \arg\max_{\ell_0} \pi_0(\ell_0, \ell_1', v')$.

Since $v_1' = 0$ and $v_0' > \delta$, Proposition A.1(vi) implies $B_0(\ell_1') \subseteq \{s_0, s_1 - v_0, s_1 + v_0\} = \{s_0, \ell_0', s_1 + 2\delta\}$; thus, we need only show that $\pi_0(\ell', v') \geq \max\{\pi_0(s_0, \ell_1', v'), \pi_0(s_1 + 2\delta, \ell_1', v')\}$. See that, $\pi_0(\ell', v') = 2\delta F(s_1 - 2\delta) - C_0(\delta)$; $\pi_0(s_0, \ell_1', v') = 0$; and $\pi_0(s_1 + 2\delta, \ell_1', v') = 2\delta(1 - F(s_1 + 2\delta)) - C_0(3\delta)$. Hence, $\pi_0(\ell', v') - \max\{\pi_0(s_0, \ell_1', v'), \pi_0(s_1 + 2\delta, \ell_1', v')\} = \Psi_2(\delta)$, where Ψ_2 is as defined in Part 0. Since $\delta \in (0, \epsilon_2)$, $\Psi_2(\delta) > 0$. This establishes that $(\ell', v') \in \mathcal{F}_P$.

We have now established that $v_i^* > 0$ for some i. Next, we show that $v_j^* = 0$ for some j.

By way of contradiction, suppose $v_0^*, v_1^* > 0$. Lemma 6 implies $d_0^*, d_1^* > 0$. Then, by Proposition A.1(iii), $|v_1^* - v_0^*| < \ell_1^* - \ell_0^*$ and $s_0 < \ell_0^* < \ell_1^* < s_1$, which means $\pi_b(\ell^*, v^*) = q - L(\ell^*, v^*)$. See that $L(\ell^*, v^*) \geq \kappa_{FB}(\ell^*)$, $s_0 < \ell_0^* < \ell_1^* < s_1$ and $s_0, s_1, \ell_0^*, \ell_1^* \in [s_1 - \epsilon, s_1]$. Since $\epsilon < \epsilon_3$, as shown in Part 0, $\kappa_{FB}(\ell^*) > \kappa_{FB}(s_0, s_1)$. It follows that $\pi_b(\ell^*, v^*) < \pi_b^{\text{def}}$, which contradicts the optimality of (ℓ^*, v^*). This establishes that $v_j^* = 0$ for some j.

We have now established that $v_i^* > 0 = v_j^*$. This concludes the proof of Part I.

Part II

In this part of the proof, we show that $s_1 - s_0 < |\ell_i^* - s_i| = v_i^*$. To begin, we will show that $v_i^* > \delta$. We proceed by contradiction and suppose $v_i^* \leq \delta$.

First consider the case, $v_1^* = 0 < v_0^* \leq \delta$. Note that $\pi_0(\cdot, s_1)$ is strictly increasing on $[0, s_0)$, so optimality for supplier zero implies $\ell_0^* \notin [0, s_0)$. Following similar arguments to those used in the final paragraph of the proof of Part I, it can be shown that optimality for the buyer implies $\ell_0^* \notin (s_0, s_1)$; moreover, by Lemma 6, $\ell_0^* \neq s_0$. And for any $\ell_0 \in (s_1, s_1 + v_0^*)$, supplier zero wins with probability 0; thus, $\ell_0^* \notin (s_1, s_1 + v_0^*)$. So we must have, $\ell_0^* \in [s_1 + v_0^*, 1]$. On this interval, $\pi_0(\cdot, s_1)$ is strictly decreasing; therefore, $\ell_0^* = s_1 + v_0^*$. However, since $s_1 \geq \frac{1}{2}$, the approach, $s_1 - v_0^*$ gives supplier zero a (weakly) greater likelihood of victory at strictly lower cost than the approach $s_1 + v_0^*$; thus, $\pi_0(s_1 - v_0^*, s_1) > \pi_0(s_1 + v_0^*, s_1)$. This yields a contradiction and, therefore, we cannot have $0 = v_1^* < v_0^* \leq \delta$.

Next, consider the case, $v_0^* = 0 < v_1^* \leq \delta$. Following analogous arguments to those in the preceding paragraph, we must have $\ell_1^* \in [0, s_0 - v_1^*]$; but on this interval, $\pi_1(s_0, \cdot)$ is strictly increasing, and therefore, $\ell_1^* = s_0 - v_1^*$. Note that this also implies that we must have $s_0 > \frac{1}{2}$, for if $s_0 \leq \frac{1}{2}$ then the approach $s_0 + v_1^*$ gives supplier one a (weakly) greater likelihood of victory at strictly lower cost than the approach $s_0 - v_1^*$. In the remainder of this proof, we show that there is $(\ell', v') \in \mathcal{F}_P$ with $v_1' > v_1^*$ such that $\pi_b(\ell', v') > \pi_b(\ell^*, v^*)$.

As a first step, we show that there is some $\overline{\Delta}_1 \in (0, 1 - s_0 - v_1^*)$ such that for all $\Delta \in (0, \overline{\Delta}_1)$, $\pi_1(s_0, s_0 - v_1^* - \Delta, 0, v_1^* + \Delta) > \max\left\{0, \max_{\ell_1 \in [s_0 + v_1^* + \Delta, 1]} u_1(s_0, \ell_1, 0, v_1^* + \Delta)\right\}$. That is, if $\Delta \in (0, \overline{\Delta}_1)$ and supplier one's prize is $v_1 = v_1^* + \Delta$, then the approach $\ell_1 = s_0 - v_1$ gives supplier one a strictly positive payoff that is strictly higher than any approach in $[s_0 + v_1, 1]$.

For $\Delta \in [0, 1 - s_0 - v_1^*)$, Let $r_1(\Delta) = \arg\max_{\ell_1 \in [s_0 + v_1^* + \Delta, 1]} u_1(s_0, \ell_1, 0, v_1^* + \Delta)$; strict concavity of u_1 implies r_1 is a continuous single-valued function. Let $u_1^*(\Delta) = u_1(s_0, r_1(\Delta), 0, v_1^* + \Delta)$, and let $G(\Delta) = \pi_1(s_0, s_0 - v_1^* - \Delta, 0, v_1^* + \Delta) - \max\{0, u_1^*(\Delta)\}$:

$$\begin{aligned} G(\Delta) = {} & (v_1^* + \Delta) F(s_1 - \delta - v_1^* - \Delta) - C_1(\delta + v_1^* + \Delta) \\ & - \max\left\{0, (v_1^* + \Delta)\left[1 - F\left(\frac{r_1(\Delta) + s_1 - \delta + v_1^* + \Delta}{2}\right)\right] + C_1(s_1 - r_1(\Delta))\right\}. \end{aligned}$$

Note that $\ell^* \in \Phi(v^*)$ implies $G(0) \geq 0$. We next show that $G'(0) > 0$.

First, note that since $s_1 \in [s_0 + v_1^*, 1)$, $u_1^*(0) \geq u_1(s_0, s_1, v^*) > 0$. By continuity of u_1, there is some neighbhoord above zero such that for all Δ in this neighborhood, $u^*(\Delta) > 0$. Next, we bound $u^{*'}(0)$. Consider the case, $r_1(0) \in (s_0 + v_1^*, 1)$. By the Envelope Theorem:

$$\begin{aligned} u^{*'}(0) &= 1 - F\left(\frac{r_1(0) + s_1 - \delta + v_1^*}{2}\right) - \frac{v_1^*}{2} f\left(\frac{r_1(0) + s_1 - \delta + v_1^*}{2}\right) \\ &< 1 - F\left(\frac{r_1(0) + s_1 - \delta + v_1^*}{2}\right) \\ &< 1 - F(s_1 - \delta), \end{aligned}$$

Next, consider the case, $r_1(0) = s_0 + v_1^*$. It must be that $r_1(\Delta) = s_0 + v_1^* + \Delta$ for all $\Delta \in (0, 1 - s_0 - v_1^*)$, for if it is optimal for supplier one to choose the lower bound of the choice set when the prize is v_1^* then it is also optimal to do so when the prize is $v_1^* + \Delta > v_1^*$. Then,

$$u_1^*(\Delta) = (v_1^* + \Delta)[1 - F(s_1 - \delta + v_1^* + \Delta)] - C_1(\delta - v_1^* - \Delta),$$

and,

$$\begin{aligned}
u^{*'}(0) &= 1 - F(s_1 - \delta + v_1^*) - v_1^* f(s_1 - \delta + v_1^*) + C_1'(\delta - v_1^*) \\
&\leq 1 - F(s_1 - \delta + v_1^*) - \frac{v_1^*}{2} f(s_1 - \delta + v_1^*) \\
&< 1 - F(s_1 - \delta),
\end{aligned}$$

where the first inequality holds since $r_1(0) = s_0 + v_1^*$ implies $\frac{\partial u_1(s_0, \ell_0, v^*)}{\partial \ell_1}|_{\ell_1 = s_0 + v_1^*} = -\frac{v_1^*}{2} f(s_1 - \delta + v_1^*) + C_1'(\delta - v_1^*) \leq 0$. We have now established that $u^{*'}(0) < 1 - F(s_1 - \delta)$. We then have,

$$\begin{aligned}
G'(0) &= F(s_1 - \delta - v_1^*) - v_1^* f(s_1 - \delta - v_1^*) - C_1'(\delta + v_1^*) - u^{*'}(0) \\
&> F(s_1 - \delta - v_1^*) - v_1^* f(s_1 - \delta - v_1^*) - C_1'(\delta + v_1^*) - 1 + F(s_1 - \delta) \\
&> F(s_1 - 2\delta) - \delta f(s_1 - 2\delta) - C_1'(2\delta) - 1 + F(s_1 - \delta) \\
&= \Psi_3(\delta) > 0.
\end{aligned}$$

The first inequality holds since $u^{*'}(0) < 1 - F(s_1 - \delta)$. The second inequality holds since $v_1^* < \delta$ and the expression $F(s_1 - \delta - x) - x f(s_1 - \delta - x) - C_1'(\delta + x)$ is strictly decreasing in x. The function Ψ_3 on the last line is as defined in Part 0; $\Psi_3(\delta) > 0$ since $s_1 > \frac{1}{2}$ and $\delta < \epsilon_4$. Then since $G(0) \geq 0$ and $G'(0) > 0$, there exists $\overline{\Delta}_1 \in (0, 1 - s_0 - v_1^*)$ such that $G(\Delta) > 0$ for all $\Delta \in (0, \overline{\Delta}_1)$.

Next, we show that there exists $\overline{\Delta}_2 \in (0, s_0 - v_1^*)$ such that for all $\Delta \in (0, \overline{\Delta}_2)$, $\pi_b(s_0, s_0 - v_1^* - \Delta, 0, v_1^* + \Delta) > \pi_b(\ell^*, v^*)$. That is, for $\Delta \in (0, \overline{\Delta}_2)$, the buyer is strictly better off choosing $\ell = (s_0, s_0 - v_0^* - \Delta)$ and $v = (0, v_1^* + \Delta)$ than choosing (ℓ^*, v^*).

For $\Delta \in [0, s_0 - v_1^*]$, let $\hat{\pi}_b(\Delta) = \pi_b(s_0, s_0 - v_1^* - \Delta, 0, v_1^* + \Delta)$. See that $\hat{\pi}_b(\Delta) = q - \nu(\Delta)$, where

$$\nu(\Delta) = \int_0^{s_0 - v_1^* - \Delta} (s_0 - v_1^* - \Delta - y) dF(y) + \int_{s_0 - v_1^* - \Delta}^1 |s_0 - y| dF(y) + C_1(\delta + v_1^* + \Delta).$$

We now show $\hat{\pi}_b'(0) > 0$; equivalently, $\nu'(0) < 0$. See that

$$\begin{aligned}\nu'(0) &= -F(s_1 - \delta - v_1^*) + v_1^* f(s_1 - \delta - v_1^*) + C_1'(\delta + v_1^*) \\ &\leq -F(s_1 - 2\delta) + \delta f(s_1 - 2\delta) + C_1'(2\delta) \\ &= -\Psi_3(\delta) + F(s_1 - \delta) - 1 \\ &< 0\end{aligned}$$

The first inequality holds since the expression, $-F(s_1-\delta-x)+v_1^* f(s_1-\delta-x)+C_1'(\delta+x)$ is increasing in x and $v_1^* < \delta$. The final inequality holds since $\Psi_3(\delta) > 0$. This establishes that $\nu'(0) < 0$. Then since $\hat{\pi}_b(0) = \pi_b(\ell^*, v^*)$ and $\hat{\pi}'(0) > 0$, there exists $\overline{\Delta}_2 \in (0, s_0 - v_1^*)$ such that for all $\Delta \in (0, \overline{\Delta}_2)$, $\hat{\pi}_b(\Delta) > \pi_b(\ell^*, v^*)$; that is, $\pi_b(s_0, s_0 - v_1^* - \Delta, 0, v_1^* + \Delta) > \pi_b(\ell^*, v^*)$.

Now let $\Delta \in (0, \min\{\Delta_1, \Delta_2\})$, and let $\ell_1' = s_0 - v_0^* - \Delta$ and $v_1' = v_1^* + \Delta$. Let $\ell' = (s_0, \ell_1')$ and $v' = (0, v_1')$. We first show $\ell' \in \Phi(v')$. Clearly, $\ell_0' \in \arg\max_{\ell_0 \in [0,1]} \pi_0(\ell_0, \ell_1', v')$ so consider supplier one; we will show $\ell_1' \in \arg\max_{\ell_1 \in [0,1]} \pi_1(\ell_0', \ell_1, v')$. Let $\ell_1 \in [0, 1]$ be given. If $\ell_1 < s_0 - v_1'$ then since $\pi_1(\ell_0', \cdot, v')$ is strictly increasing on $[0, s_0 - v_1']$, $\pi_1(\ell_0', \ell_1, v') < \pi_1(\ell', v')$. If $\ell_1 \in (s_1 - v_1', s_0 + v_1')$ then supplier one wins with probabilty zero and so, $\pi_1(\ell_0', \ell_1, v') < 0 < \pi_1(\ell', v')$, where the final inequality holds since $\Delta < \overline{\Delta}_1$. Finally, if $\ell_1 \in (s_0 + v_1', 1]$ then $\pi_1(\ell_0', \ell_1, v') = u_1(\ell_0', \ell_1, v') \leq u^*(\Delta) < \pi_1(\ell', v')$, where the first inequality follows by definition of $u^*(\Delta)$ and the second since $\Delta < \overline{\Delta}_1$. This establishes that $\ell' \in \Phi(v')$ and, thus, $(\ell', v') \in \mathcal{F}_P$. And since $\Delta < \overline{\Delta}_2$, we have $\pi_b(\ell^*, v^*) < \pi_b(\ell', v')$, and this contradicts the definition of (ℓ^*, v^*). Therefore, we must have $0 = v_j^* < \delta < v_i^*$. Lemma 6 implies $\ell_i^* \neq s_i$ and Proposition A.1(vi) implies $|\ell_i^* - s_j| = v_i^*$. This completes the proof.

□

Proof of Proposition 5: We first prove part (i). Let $s_0 = \frac{1-\delta}{2}$ and $s_1 = \frac{1+\delta}{2}$; WLOG, suppose $C_0' < C_1'$. By Proposition 4 there is $\epsilon > 0$ such that for all $\delta \in (0, \epsilon)$, $(\ell^*, v^*) \in \mathcal{A}_P$, then, $v_j^* = 0 < v_i^*$ and $\delta < |\ell_i^* - s_j| = v_i^*$.

Fix $\delta \in (0, \epsilon)$ and let $(\ell^*, v^*) \in \mathcal{A}_P$. Proceed by contradiction and suppose $0 < v_1^*$; Proposition 4 implies $0 = v_0^* < \delta < v_1^* = |\ell_1^* - s_0|$. This means $\ell_1^* \in \{s_0 - v_1^*, s_0 + v_1^*\}$. And since $s_0 < \frac{1}{2} < s_1$, the approach, $s_0 + v_1^*$ gives supplier one a strictly greater probability of victory at strictly lower cost than the approach $s_0 - v_1^*$; thus, we must have $\ell_1^* = s_0 + v_1^*$. Now let $\ell_0' = s_1 - v_1^*$, $\ell_1' = s_1$, $v_0' = v_1^*$ and $v_1' = 0$. See that, $\pi_b(\ell^*, v^*) = q - L(\ell^*, v^*)$,

$\pi_b(\ell', v') = q - L(\ell', v')$ and

$$\begin{aligned}
L(\ell^*, v^*) &= \int_0^{s_0+v_1^*} |y - s_0| f(y) dy + \int_{s_0+v_1^*}^1 (y - s_0 - v_1^*) f(z) dz + C_1(s_0 + v_1^* - s_1) \\
&= \int_0^{s_1-v_1^*} (s_1 - v_1^* - z) f(z) dz + \int_{s_1-v_1^*}^1 |s_1 - z| f(z) dz + C_1(s_1 - s_0 - v_1^*) \\
&> \int_0^{s_1-v_1^*} (s_1 - v_1^* - z) f(z) dz + \int_{s_1-v_1^*}^1 |s_1 - z| f(z) dz + C_0(s_1 - s_0 - v_1^*) \\
&= L(\ell', v').
\end{aligned}$$

The second equality follows since $C(x) = C(-x)$ and through a change of variables in which we let $z = 1 - y$, $dz = -dy$ and then use the facts that $s_0 = 1 - s_1$ and $f(z) = f(1 - z)$. The strict inequality holds since $C_0' < C_1'$ implies $C_0(d) < C_1(d)$ for all $d > 0$. The string of inequalities shows that, $\pi_b(\ell^*, v^*) < \pi_b(\ell', v')$.

We now show that $\ell' \in \Phi(v')$. We focus on supplier zero. Let $B_0 = \arg\max_{\ell_0} \pi_0(\ell_0, \ell_1', v')$ denote supplier zero's best-reply when the prizes are v' and supplier one chooses $\ell_1' = s_1$. Since $\delta = s_1 - s_0 < v_1^*$, Proposition A.1 implies $B_0 \in \{s_0, s_1 - v_1^*, s_1 + v_1^*\}$. Now since $\ell^* \in \Phi(v^*)$, it must be that $\pi_1(\ell^*, v^*) \geq 0$. But using symmetry of the distribution/cost function and the facts that $s_0 = 1 - s_1$ and $C_0(d) < C_1(d)$, it is easily shown that $\pi_0(\ell', v') > \pi_1(\ell^*, v^*) \geq 0 = \pi_0(s_0, \ell_1', v')$, which means $B_0 \neq s_0$. Moreover, since $s_0 < \frac{1}{2} < s_1$, the approach $s_1 - v_1^*$ gives supplier zero a strictly greater probability of victory as strictly lower cost than the approach $s_1 + v_1^*$; hence $\pi_0(s_1 - v_1^*, \ell_1', v') > \pi_0(s_1 + v_1^*, \ell_1', v')$. This establishes that $B_0 = s_1 - v_1^* = \ell_0'$, and hence, $\ell' \in \Phi(v')$, so $(\ell', v') \in \mathcal{F}_P$. But since $L(\ell', v') < L(\ell^*, v^*)$, this contradicts the hypothesis that $(\ell^*, v^*) \in \mathcal{A}_P$.

We now prove Part (ii) for the case $\frac{1}{2} < s_1$; the case $s_1 \leq \frac{1}{2}$ is symmetric. So let $s_1 \in \left(\frac{1}{2}, 1\right)$. By Proposition 4 there exists $\epsilon_1 > 0$ such that $s_0 \in (s_1 - \epsilon_1, s_1)$ implies $0 = v_j^* < s_1 - s_0 < v_i^*$. Then let $\epsilon \in \left(0, \min\left\{s_1 - \frac{1}{2}, \epsilon_1\right\}\right)$ and let $s_0 \in (s_1 - \epsilon, s_1)$. See that $s_0 > \frac{1}{2}$ and $s_0 - \frac{1}{2} < s_1 - \frac{1}{2}$. We will show that $(\ell^*, v^*) \in \mathcal{A}_P$ implies $v_1^* = 0 < v_0^*$.

Let $(\ell^*, v^*) \in \mathcal{A}_P$. Since $s_1 - s_0 < \epsilon_1$, Proposition 4 implies that either, $v_1^* = 0 < s_1 - s_0 < v_0^*$ and $\ell_0^* \in \{s_1 - v_0^*, s_1 + v_0^*\}$ or $v_0^* = 0 < s_1 - s_0 < v_1^*$ and $\ell_1^* \in \{s_0 - v_1^*, s_0 + v_1^*\}$. Since $s_1 > \frac{1}{2}$, in the case where $v_0^* > 0$, it is straightforward to show that we must have $\ell_0^* = s_1 - v_1^*$, as this approach gives supplier zero a strictly greater probability of victory at strictly lower cost than the approach $s_1 + v_1^*$. So, it suffices to compare three scenarios. Below we provide a sketch of the comparison between the three scenarios. More explicit details are available upon request.

Scenario 1: $v_1 = 0 < v_0$ and $\ell_0 = s_1 - v_0$: For all (ℓ, v) such that $v_1 = 0 < v_0$, $\ell_1 = s_1$, and $\ell_0 = s_1 - v_0$, the buyer purchases from supplier zero whenever $y \leq \ell_0$, and purchases from supplier one when $y > \ell_0$. Consider the problem

$$\min_{\ell_0, v_0} \int_0^{\ell_0} (\ell_0 - y)dy + \int_{\ell_0}^{s_1} (s_1 - y)dy + \int_{s_1}^{1} (y - s_1)dy + \frac{c}{2}(\ell_0 - s_0)^2,$$

subject to the constraints that $\ell_0 = s_1 - v_0$ and $u_0(\ell, v) = v_0 F(\ell_0) - C_0(\ell_0 - s_0) \geq 0$. Taking FOCs and solving, it may be verified that the constraint $u_0 \geq 0$ is non-binding and that $\ell_0^* = \frac{cs_0 + s_1}{2+c} < s_0$.

Scenario 2: $v_0 = 0 < v_1$ and $s_0 + v_1 = \ell_1$: In this case, the buyer purchases from supplier one only when $y > \ell_1$. Consider the problem

$$\min_{\ell_1} \int_0^{s_0} (s_0 - y)dy + \int_{s_0}^{\ell_1} (y - s_0)dy + \int_{\ell_1}^{1} (y - \ell_1)dy + \frac{c}{2}(\ell_1 - s_1)^2$$

subject to the constraints that $\ell_1 = s_0 + v_1$ and $u_1(\ell, v) \geq \max\{0, u_1(s_0, s_0 - v_1, 0, v_1)\}$. For our purposes, it is sufficient to consider a relaxed problem in which we assume the constraint $u_1(\ell, v) \geq \max\{0, u_1(s_0, s_0 - v_1, 0, v_1)\}$ is non-binding. Solving this problem, we find that $\ell_1^* = \frac{1 + s_0 + cs_1}{2+c}$.

Scenario 3: $v_0 = 0 < v_1$ and $\ell_1 = s_0 - v_1$: In this case, the buyer purchases from supplier one only when $y < \ell_1$. Consider the problem

$$\min_{\ell_1} \int_0^{\ell_1} (\ell_1 - y)dy + \int_{\ell_1}^{s_0} (s_0 - y)dy + \int_{s_0}^{1} (y - s_0)dy + \frac{c}{2}(s_1 - \ell_1)^2$$

subject to the constraints that $\ell_1 = s_0 - v_1$ and $u_1(\ell, v) \geq \max\{0, u_1(s_0, s_0 + v_1, 0, v_1)\}$. As in scenario 2, for our purposes, it is sufficient to consider a relaxed problem in which we assume the constraint $u_1(\ell, v) \geq \max\{0, u_1(s_0, s_0 + v_1, 0, v_1)\}$ is non-binding. Solving this problem, we find that $\ell_1^* = \frac{s_0 + cs_1}{2+c}$.

Plugging in the solutions characterized above and comparing value functions, it is straightforward to verify that the buyer is best off in scenario 1.

□

Proofs for sections 4.3 – 4.4

***Proof of Lemma 7**:* Let $\mathcal{F}_{P'} = \left\{(\ell,v)\,|v_0,v_1 \geq 0,\ |v_0-v_1| \leq 1 \text{ and } \frac{\partial u_i(\ell,v)}{\partial \ell_i} = 0,\ \ i=0,1\right\}$ denote the feasible set in the problem (P') and let $\mathcal{A}_{P'} = \arg\min_{(\ell,v)\in\mathcal{F}_{P'}} L(\ell,v)$ denote the solution set. We wish to show that $\mathcal{A}_P = \mathcal{A}_{P'}$.

We first show that $\mathcal{A}_P \subset \mathcal{F}_{P'}$ and $\mathcal{A}_{P'} \subset \mathcal{F}_P$. Let $(\ell^*,v^*) \in \mathcal{A}_P$. Since $1-s_1, s_0 < s^*$, Proposition 3 implies $v_i^* > 0$ for each i. Then, Lemma 6 implies, $d_0^*, d_1^* > 0$. And Proposition A.1(iii) implies $\frac{\partial u_i(\ell^*,v^*)}{\partial \ell_i} = 0$ and $|v_1^* - v_0^*| < 1$, which means $(\ell^*,v^*) \in \mathcal{F}_{P'}$. Next, let $(\ell',v') \in \mathcal{A}_{P'}$. By proposition B.1, $\ell' \in \Phi(v')$, which means $(\ell',v') \in \mathcal{F}_P$. This establishes that $\mathcal{A}_P \subset \mathcal{F}_{P'}$ and $\mathcal{A}_{P'} \subset \mathcal{F}_P$.

We now show that $\mathcal{A}_P = A_{P'}$. Let $(\ell^*,v^*) \in \mathcal{A}_P$ and let $(\ell',v') \in \mathcal{A}_{P'}$. Proposition A.1(iii) then implies $\ell_1^* - \ell_0^* > |v_1^* - v_0^*|$, which means $\pi_b(\ell^*,v^*) = q - L(\ell^*,v^*)$. Moreover, Proposition B.1 implies $\ell_1' - \ell_0' > |v_1' - v_0'|$, which means $\pi_b(\ell',v') = q - L(\ell',v')$. Then, since $(\ell^*,v^*) \in \mathcal{F}_{P'}$, by definition of (ℓ',v'), $L(\ell',v') \leq L(\ell^*,v^*)$. And since $(\ell',v') \in \mathcal{F}_P$, we must have $L(\ell',v') \geq L(\ell^*,v^*)$. Thus, $L(\ell^*,v^*) = L(\ell',v')$, which implies $(\ell^*,v^*) \in \mathcal{A}_{P'}$ and $(\ell',v') \in \mathcal{A}_P$. □

***Proof of Proposition 6**.* We first show (i) $\iff$ (iii). Suppose there exists $v_0^{FB} = v_1^{FB}$ such that $\ell^{FB} \in \Phi(v^{FB})$. Since $v_0^{FB} = v_1^{FB}$, $\overline{Q}_b(\cdot, v^{FB}) = \overline{Q}_{FB}(\cdot)$; it follows that, $\pi_b(\ell^{FB}, v^{FB}) = S_{FB}(\ell^{FB})$. Next, let $(\ell^*,v^*) \in \mathcal{A}_P$ and suppose $\pi_b(\ell^*,v^*) = S_{FB}(\ell^{FB})$. Since $L(\ell,v) < L_{FB}(\ell)$ if $v_0 \neq v_1$, it must be that $v_0^* = v_1^*$. Moreover, by uniqueness of ℓ^{FB}, we must have $\ell^* = \ell^{FB}$. Then $(\ell^{FB}, v^*) \in \mathcal{F}_P$ implies $\ell^{FB} \in \Phi(v^*)$.

We now show (ii) $\implies$ (iii). Suppose $C_0'(\ell_0^{FB} - s_0) = C_1'(s_1 - \ell_1^{FB}) > 0$; the case where $C_0'(\ell_0^{FB} - s_0) = C_1'(s_1 - \ell_1^{FB}) = 0$ is straightforward and left to the reader. We will show that there exists $v_0^{FB} = v_1^{FB}$ such that $\ell^{FB} \in \Phi(v^{FB})$. We first show that $s_0, 1 - s_1 < s^*$. Adding (7) and (8) yields, $\Gamma(\ell^{FB}) = 0$, so (10) implies, $\ell_0^{FB} = 1 - \ell_1^{FB}$. (7) then implies $F(\ell_0^{FB}) < \frac{1}{4}$, which means $\ell_0^{FB} < s^*$. Moreover, note that since $C_0'(\ell_0^{FB} - s_0) = C_1'(s_1 - \ell_1^{FB}) > 0$, we have $\ell_0^{FB} > s_0$ and $\ell_1^{FB} < s_1$. It follows that $s_0, 1 - s_1 < s^*$. Next, let $v_0^{FB} = v_1^{FB} = \frac{2C_0'(\ell_0^{FB} - s_0)}{f(\frac{1}{2})}$. By construction, $\frac{\partial u_i(\ell^{FB}, v^{FB})}{\partial \ell_i} = 0$.

We now show $v_0^{FB} < 2\ell_1^{FB} - \ell_0^{FB} - 1$. First note that since $\ell_1^{FB} = 1 - \ell_0^{FB}$, $2\ell_1^{FB} - \ell_0^{FB} - 1 =$

$1 - 3\ell_0^{FB}$. (7) implies, $C_0'(\ell_0^{FB} - s_0) = \frac{1}{2} - 2F(\ell_0^{FB})$, Thus,

$$\begin{aligned} v_0^{FB} &= \frac{2}{f\left(\frac{1}{2}\right)} \int_{\ell_0^{FB}}^{\frac{1}{2}} f(x)dx - \frac{1}{f\left(\frac{1}{2}\right)} \int_0^{\ell_0^{FB}} 2f(x)dx \\ &< 2\left(\frac{1}{2} - \ell_0^{FB}\right) - \ell_0^{FB} \\ &= 2\ell_1^{FB} - \ell_0^{FB} - 1, \end{aligned}$$

where the inequality holds since $f(x) \le f\left(\frac{1}{2}\right)$ for all $x \in \left[\ell_0^{FB}, \frac{1}{2}\right]$ and since $2f(x) > f\left(\frac{1}{2}\right)$ for all $x \in \left[0, \ell_0^{FB}\right]$. The final equality holds since $1 - \ell_1^{FB} = \ell_0^{FB}$. Then by Proposition A.2, $\ell^{FB} \in \Phi(v^{FB})$.

Finally, we show (iii) $\implies$ (ii). Suppose there exists $v_0^{FB} = v_1^{FB} \ge 0$ such that $\ell^{FB} \in \Phi(v^{FB})$. Proposition 2 implies $\frac{\partial u_i(\ell^{FB}, v_0^*)}{\partial \ell_i} = 0$. Since $v_0^* = v_1^* \ge 0$, these first-order conditions imply $C_0'(\ell_0^{FB} - s_0) = C_1'(s_1 - \ell_1^{FB}) \ge 0$. □

Proof of Corollary 1. By Proposition 6, it suffices to show that $C_0'(\ell_0^{FB} - s_0) = C_1'(s_1 - \ell_1^{FB}) \ge 0$. First, the fact that $\ell_0^{FB} \ge s_0$ and $\ell_1^{FB} \le s_1$ follows by Proposition 1. Now proceeding by contradiction, suppose $C_0'(\ell_0^{FB} - s_0) \neq C_1'(s_1 - \ell_1^{FB})$; in particular, and WLOG, suppose $C_0'(\ell_0^{FB} - s_0) > C_1'(s_1 - \ell_1^{FB})$. Since $C_0 \equiv C_1$, this implies $\ell_0^{FB} + \ell_1^{FB} > s_0 + s_1 = 1$. Then, by (10), $\Gamma(\ell^{FB}) > 0$. Adding (7) and (8), we have $0 = \Gamma(\ell^{FB}) + C_0'(\ell_0 - s_0) - C_1'(s_1 - \ell_1^{FB})$, which yields a contradiction since $\Gamma(\ell^{FB}) > 0$ and, by assumption, $C_0'(\ell_0^{FB} - s_0) > C_1'(s_1 - \ell_1^{FB})$. □

Proof of Proposition 7. By Lemmas B.3 and B.4, it suffices to show that $C_i'\left(\left|\ell_i^{FB} - s_i\right|\right) < C_j'\left(|\ell_j - s_j|\right) \implies v_i^* < v_j^*$. By way of contradiction, suppose $C_0'(\ell_0^{FB} - s_0) < C_1'(s_1 - \ell_1^{FB})$, but $v_0^* \ge v_1^*$. Lemma B.4 implies $\ell_i^* \le \ell_i^{FB}$. Then, we have the following string of inequalities:

$$C_1'(s_1 - \ell_1^*) \ge C_1'(s_1 - \ell_1^{FB}) > C_0'(\ell_0^{FB} - s_0) \ge C_0'(\ell_0^* - s_0).$$

But since $v_0^* \ge v_1^*$ and $\frac{\partial u_i(\ell^*, v^*)}{\partial \ell_i} = 0$ for each i, this means $C_0'(\ell_0^* - s_0) \ge C_1'(s_1 - \ell_1^*)$, which yields a contradiction. Thus, we must have $v_0^* < v_1^*$. The proof for the case where $C_1'(s_1 - \ell_1^{FB}) < C_0'(\ell_0^{FB} - s_0)$ is analogous. □

Proof of Corollary 2. WLOG, suppose supplier zero is more flexible and conservative. That is, suppose $1 - s_1 \le s_0$ and $C_0' \le C_1'$, where at least one of the two inequalities is strict. We will show that $C_0'(\ell_0^{FB} - s_0) < C_1'(s_1 - \ell_1^{FB})$. Proceed by contradiction and suppose $C_0'(\ell_0^{FB} - s_0) \ge C_1'(s_1 - \ell_1^{FB})$. Since $C_0'(d) \le C_1'(d)$ for all $d \ge 0$, convexity of the cost functions implies, $\ell_0^{FB} + \ell_1^{FB} \geq s_0 + s_1 \geq 1$, where either the first (if $C_0' < C_1'$) or second (if $1 < s_0 + s_1$) inequality is strict. Then, (10) implies $\Gamma(\ell^{FB}) > 0$. Adding (7) and (8), $0 = \Gamma(\ell^{FB}) + C_0'(\ell_0^{FB} - s_0) - C_1'(s_1 - \ell_1^{FB}) > 0$, yielding a contradiction. □

Proof of Proposition 8. Suppose the buyer commits to purchase the highest quality innovation. Now, using the first-order conditions in Proposition 1, it is straightforward to show that when $s_0, 1-s_1 < s^*$, then, $\ell_0^{FB} > s_0$ and $\ell_1^{FB} < s_1$. Then, suppose the buyer offers the following prizes: $v_0^{FB} = \frac{2C_0'\left(\ell_0^{FB}-s_0\right)}{f(m(\ell^{FB}))}$ and $v_1^{FB} = \frac{2C_1'\left(s_1-\ell_1^{FB}\right)}{f(m(\ell^{FB}))}$. Given the buyer's purchasing rule, for all $\ell_0 \leq \ell_1$, the expected payoff to supplier zero is $\pi_0(\ell, v^{FB}) = F\left(m\left(\ell\right)\right) v_0^{FB} - C_0(\ell_0 - s_0)$ and the expected payoff to supplier one is $\pi_1(\ell, v^{FB}) = \left[1 - F\left(m\left(\ell\right)\right)\right] v_1^{FB} - C_1\left(s_1 - \ell_1\right)$. Using arguments similar to those given in the proof of Lemma 2, it is straightforward to confirm that there is a unique equilibrium in the stage-2 game between the suppliers, which is characterized by the following first-order conditions: $\frac{\partial \pi_0}{\partial \ell_0} = \frac{v_0^{FB}}{2} f\left(m\left(\ell\right)\right) - C_0'\left(\ell_0 - s_0\right) = 0$ and $\frac{\partial \pi_1}{\partial \ell_1} = \frac{-v_1^{FB}}{2} f\left(m\left(\ell\right)\right) + C_1'(s_1 - \ell_1) = 0$. By construction, ℓ^{FB} is the unique solution to this system. Then, if the buyer sets entry fees such that $t_i^{FB} = \pi_i\left(\ell^{FB}, v^{FB}\right)$, the buyer's stage-1 expected payoff is equal to equal to the first-best expected surplus. □

Proofs for Section 5

Proof of Proposition 9. Immediate from the text. □

Proof of Proposition 10. WLOG, suppose it is supplier zero who has a singular advantage. In particular, let us suppose she is more conservative so that $c_0 = c_1$, and $1 - s_1 < s_0 < s^*$; the steps for the case of a cost advantage are similar. Moreover, the statements in parts (i) and (ii) of the proposition follow almost immediately if there is a corner solution (i.e., the optimal prize is zero) in the anonymous FPC, so we initially assume the solution is interior, but we revisit this issue in the proof of part (iii).

We begin with part (i). Consider the anonymous auction (AAUC). Obviously, since the menus are unaffected by the constraint, equilibrium behavior in stages 2-4 is as characterized in Section 5.1: The suppliers choose the first-best approaches, and total surplus is equal to the first-best. In stage 1, the buyer's payoff is,

$$\pi_b^{AAUC} = \overline{Q}_{FB}(\ell^{FB}) - \mathbb{E}[\sigma_1(\ell^{FB}, y) + \sigma_0(\ell^{FB}, y)] + 2t$$

The buyer chooses t to maximize π_b^{AAUC} subject to individual rationality constraints, $\pi_i^{AAUC} = \overline{Q}_{FB}(\ell^{FB}) - \mathbb{E}\left[Q_j\left(\ell_j^{FB}, y\right)\right] - C_i(\ell_i^{FB} - s_i) \geq t$. Optimality dictates, $t = \min\left\{\pi_0^{AAUC}, \pi_1^{AAUC}\right\}$.

And since $\pi_i^{AAUC} > \pi_j^{AAUC}$ if i has a singular advantage, part (i) follows.

Now, for use later in this proof, we compute the buyer's value function and the total surplus. First, we compute the buyer's value function. Note that when supplier zero has a singular advantage, $\pi_0^{AAUC} > \pi_1^{AAUC}$; so, $t = \pi_1^{AAUC}$. And using the fact that $\mathbb{E}\left[\sigma_i\left(\ell, y\right)\right] = \overline{Q}_{FB}(\ell) - \mathbb{E}\left[Q_j\left(\ell_j, y\right)\right]$, as shown in the main body, we may write the buyer's stage-1 payoff as

$$\pi_b^{AAUC} = \overline{Q}_{FB}(\ell^{FB}) + \mathbb{E}\left[Q_1\left(\ell_1^{FB}, y\right)\right] - \mathbb{E}\left[Q_0\left(\ell^{FB}, y\right)\right] - 2C_1(\ell_1^{FB} - s_1).$$

Using Proposition 1, we can solve for the first-best approaches in the linear/quadratic model:

$$\ell_0^{FB} = \frac{2c^2 s_0 + 3cs_0 + cs_1 + 1}{2(c+1)(c+2)}$$

$$\ell_1^{FB} = \frac{2c^2 s_1 + cs_0 + 3cs_1 + 2c + 3}{2(c+1)(c+2)}.$$

Plugging in these first-best approaches, we find the buyer's value function:

$$\pi_b^{AAUC} = \frac{c\left(c\left[4q + s_0^2 + 2s_0\left(s_1 - 2\right) + s_1\left(8 - 7s_1\right) - 2\right] + 12q - 4s_0 - 8s_1^2 + 8s_1 - 2\right) + 8q - 1}{4(c+1)(c+2)}.$$

Total surplus is equal to the first-best and is given by

$$S^{AAUC} = S^{FB} = \frac{c^2\left[4q - 3s_0^2 + 2\left(s_0 + 2\right)s_1 - 3s_1^2 - 2\right] + 2c\left[6q - 2s_0^2 + s_0 + s_1\left(3 - 2s_1\right) - 2\right] + 8q - 1}{4(c+1)(c+2)}.$$

Next, we show part (ii). Consider the anonymous FPC (AFPC). Abusing our earlier notation, and now letting v and t be scalars (rather than prize and entry-fee profiles), The buyer's problem is,

$$\max_{(v,t)} \overline{Q}_{FB}(\ell) - v + 2t \text{ s.t. } \ell \in \Phi(v, v), v \geq 0.$$

Following similar logic as in the proof of Lemma 7, we may solve the buyer's problem by considering an auxiliary problem in which we replace the equilibrium constraint with the first-order conditions, $\frac{\partial u_i}{\partial \ell_i} = 0$, $i = 0, 1$. Moreover, as with the auction, the buyer optimally sets $t = \min\{u_0, u_1\}$, and when i has a singular advantage, $\frac{\partial u_i}{\partial \ell_i} = 0 \implies u_i \geq u_j$, holding strictly whenever $v > 0$. Assuming it is supplier zero who has a singular advantage, the Lagrangian associated with AFPC is

$$\mathcal{L} = \overline{Q}_{FB}(\ell) - v(2F(m(\ell)) - 1) - 2C_1(\ell_1 - s_1) + \lambda_0 \frac{\partial u_0(\ell, v)}{\partial \ell_0} + \lambda_1 \frac{\partial u_1(\ell, v)}{\partial \ell_1} + \mu_v v.$$

In the uniform/quadratic model, we can solve explicitly for the solution to this problem by taking the FOC. First, note that interiority requires that,

$$c < \bar{c} = \frac{2s_1 - 2s_0 - 1}{2(s_0 + s_1 - 1)}.$$

Then, at an interior solution,

$$v^{AFPC} = \frac{c\left[2s_1 - 2s_0 - 1 - 2c\left(s_0 + s_1 - 1\right)\right]}{2 + c},$$

and $\ell_0^{AFPC} = s_0 + \frac{v^{ac}}{2c}$ and $\ell_1^{AFPC} = s_1 - \frac{v^{ac}}{2c}$. Computing each ℓ_i^{AFPC} and comparing it with ℓ_i^{FB}, we find that $s_0 < \ell_0^{AFPC} < \ell_0^{FB}$ and $\ell_1^{FB} < \ell_1^{AFPC} < s_1$.[24] Total surplus in the AFPC is

$$S^{AFPC} = \frac{-4c^2(s_0 + s_1 - 1)^2 + c\left[4q - 3s_0^2 + 2s_0 s_1 + s_1\left(4 - 3s_1\right) - 2\right] + 8q - 2(s_0 + s_1)^2 + 4s_0 + 4s_1 - 3}{4(c + 2)}$$

and the buyer's value function is

$$\pi_b^{AFPC} = S^{AFPC} - \frac{c(1 - s_0 - s_1)\left[2c\left(s_0 + s_1 - 1\right) + 2s_0 - 2s_1 + 1\right]}{c + 2}.$$

We now compare the total surplus in the AFPC with the total surplus in the discriminatory FPC (DFPC). Using the first-order conditions for (P') found in Appendix B.1 we solve explicitly for the solution to the DFPC:

$$v_0^{DFPC} = \frac{c\left[1 - 4s_0 + c(s_1 - 3s_0 + 8cs_1 - 8cs_0 - 4c)\right]}{(2 + c)(4c^2 + c + 1)},$$

$$v_1^{DFPC} = \frac{c\left[s_1(4 + 3c + 8c^2) - c(2 + s_0 + 8cs_0 + 4c) - 3\right]}{(2 + c)(4c^2 + c + 1)},$$

and $\ell_0^{DFPC} = s_0 + \frac{v_0^{DFPC}}{2c}$ and $\ell_1^{*} = s_1 - \frac{v_1^{DFPC}}{2c}$. Then, computing total surplus in the DFPC and comparing it with total surplus in the AFPC, we find:

$$S^{DFCP} - S^{AFPC} = \frac{\left(c(4c + 1)\left(4c^2 + 1\right) + 2\right)\left(s_0 + s_1 - 1\right)^2}{4(c + 2)\left(4c^2 + c + 1\right)} > 0.$$

This establishes part (ii).

Finally, we compare the buyer's payoff in the AFPC with the AAUC. Assuming $c < \bar{c}$,

[24] Throughout this proof, we omit the (messy) algebra pertaining to the comparisons herein, but this can be provided upon request.

we have that,

$$\pi_b^{AFPC} - \pi_b^{AAUC} = \frac{(s_0 + s_1 - 1)\left(c\left(4c^2(s_0 + s_1 - 1) + 8cs_0 + 3s_0 - 5s_1 + 7\right) - 2(s_0 + s_1 - 1)\right)}{4(c+1)(c+2)} > 0.$$

When $c > \bar{c}$, the optimal prizes in the AFPC are zero, and the buyer's payoff is the default:

$$\pi_b^{AFPC} = \pi_b^{def} = \frac{1}{4}\left(4q - 3s_0^2 + 2s_0 s_1 + s_1(4 - 3s_1) - 2\right).$$

And for $c > \bar{c}$, it holds that $\pi_b^{AAUC} < \pi_b^{def} = \pi_b^{AFPC}$.

□

B Auxiliary problems used in proofs

As equilibrium outcomes in the stage-2 subgame need not vary continuously in the prize values, our proofs characterizing the optimal contest require making some discrete comparisons between the payoffs attained by the buyer in different equilibrium configurations. To facilitate such comparisons, it is useful to derive properties of the solutions to a few auxiliary problems.

B.1 Analysis of (P')

In this section, we assume that $s_0, 1 - s_1 < s^*$, and we analyze the problem (P'). We let $\mathcal{F}_{P'}$ and $\mathcal{A}_{P'}$ be as defined in the proof of Lemma 7. Let $\mathcal{L} = L(\ell, v) - \lambda_0 \frac{\partial u_0(\ell)}{\partial \ell_0} - \lambda_1 \frac{\partial u_1(\ell)}{\partial \ell_1} - \mu_0 v_0 - \mu_1 v_1 + \psi_0(v_0 - v_1 - 1) + \psi_1(v_1 - v_0 - 1)$ denote the Lagrangian associated with (P'). The FOCs and complementary slackness conditions are:

$$\frac{\partial \mathcal{L}}{\partial \ell_0} = 2F(\ell_0) - F(y_0) + \frac{1}{2}(v_1 - v_0) f(y_0) + C_0'(\ell_0 - s_0) - \lambda_0 \left[\frac{v_0}{4} f'(y_0) - C_0''(\ell_0 - s_0)\right] + \lambda_1 \frac{v_1}{4} f'(y_0) = 0 \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial \ell_1} = 2F(\ell_1) - F(y_0) - 1 + \frac{1}{2}(v_1 - v_0) f(y_0) + C_1'(\ell_1 - s_1) - \lambda_0 \frac{v_0}{4} f'(y_0) + \lambda_1 \left[\frac{v_1}{4} f'(y_0) + C_1''(\ell_1 - s_1)\right] = 0 \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial v_0} = \frac{1}{2}(v_0 - v_1)f(y_0) - \lambda_0\left[\frac{1}{2}f(y_0) - \frac{v_0}{4}f'(y_0)\right] - \lambda_1\frac{v_1}{4}f'(y_0) - \mu_0 + \psi_0 - \psi_1 = 0 \quad (14)$$

$$\frac{\partial \mathcal{L}}{\partial v_1} = \frac{1}{2}(v_1 - v_0)f(y_0) - \lambda_0\left[\frac{v_0}{4}f'(y_0)\right] + \lambda_1\left[\frac{1}{2}f(y_0) + \frac{v_1}{4}f'(y_0)\right] - \mu_1 - \psi_0 + \psi_1 = 0 \quad (15)$$

$$v_i\mu_i = 0, v_i \geq 0; \psi_i(v_i - v_j - 1) = 0, v_i - v_j \leq 1, \psi_i \geq 0; \frac{\partial u_i(\ell)}{\partial \ell_i} = 0; i = 0, 1 \quad (16)$$

Lemma B.1. *If $(\ell, v) \in \mathcal{A}_{P'}$, then, $y_0(\ell, v) \in [0, 1]$.*

Proof. Let $(\ell, v) \in \mathcal{F}_{P'}$ such that $y_0(\ell, v) \notin [0, 1]$; we will show that $(\ell, v) \notin \mathcal{A}_{P'}$. To begin, see that $f(y_0(\ell, v)) = 0$ and $\frac{\partial u_i(\ell,v)}{\partial \ell_i} = 0$ imply $\ell_i = s_i$. This necessarily means $v_0 \neq v_1$. Now let $v' = (0, 0)$ and note that $(\ell, v') \in \mathcal{F}_{P'}$ and $L(\ell, v) < L_{FB}(\ell) = L(\ell, v')$, where the inequality holds since $v_0 \neq v_1$. Thus, $(\ell, v) \notin \mathcal{A}_{P'}$. □

Lemma B.2. *If $(\ell, v) \in \mathcal{A}_{P'}$, then, $v_0, v_1 \in (0, 1)$, $\ell_0 > s_0$, and $\ell_1 < s_1$.*

Proof. It will first be convenient to establish the following claim:

Claim 1. *For all $\ell_0, \ell_1 \in [0, 1]$, $2F(\ell_0) - F(m(\ell)) + \frac{1}{2}f(m(\ell)) \geq 0 \geq 2F(\ell_1) - F(m(\ell)) - 1 - \frac{1}{2}f(m(\ell))$, holding with strict inequalities if $\ell_0, \ell_1 \in (0, 1)$.*

Proof of Claim 1: We prove that $2F(\ell_0) - F(m(\ell)) + \frac{1}{2}f(m(\ell)) \geq 0$. The fact that $2F(\ell_1) - F(m(\ell)) - 1 - \frac{1}{2}f(m(\ell)) \leq 0$ follows by symmetry of the distribution. First suppose $m(\ell) \leq \frac{1}{2}$. Then, $2F(\ell_0) - F(m(\ell)) + \frac{1}{2}f(m(\ell)) = 2F(\ell_0) + \int_0^m \left[\frac{1}{2m}f(m) - f(x)\right] dx \geq 2F(\ell_0) + f(m)\left(\frac{1}{2} - m(\ell)\right) \geq 0$, where the first inequality follows since $f(m) \geq f(x)$ for $x \in [0, m]$. Note that the final inequality is strict if $\ell_0 > 0$. Next, suppose $m(\ell) > \frac{1}{2}$. This means $\ell_0 > 1 - \ell_1$. Note, moreover, that $2F(\ell_0) - F(m(\ell)) + \frac{1}{2}f(m(\ell))$ is strictly increasing in ℓ_0. It follows that $2F(\ell_0) - F(m(\ell)) + \frac{1}{2}f(m(\ell)) > 2F(1 - \ell_1) - \frac{1}{2} + \frac{1}{2}f\left(\frac{1}{2}\right) \geq 0$, where the final inequality holds since $f\left(\frac{1}{2}\right) \geq 1$. This establishes the claim. □

We now show that $\psi_0 = \psi_1 = 0$. Proceeding by contradiction, let us suppose $\psi_1 > 0$. Then, $v_1 - v_0 = 1$, which implies $v_1 > 0$, $\mu_1 = 0$, and $\psi_0 = 0$. First, suppose $v_0 = 0$. Then, we have $v_1 = 1$. (15) yields: $\frac{1}{2}f(y_0) + \lambda_1\left[\frac{1}{2}f(y_0) + \frac{1}{4}f'(y_0)\right] + \psi_1 = 0$, which implies $\lambda_1 < 0$. Subtracting (15) from (13) reveals that $0 < 2F(\ell_1) - F(y_0) - 1 - \frac{1}{2}f(y_0)$. And since $F(y_0) + \frac{1}{2}f(y_0) > F(m) + \frac{1}{2}f(m)$, we have $0 < 2F(\ell_1) - F(m) - 1 - \frac{1}{2}f(m)$, contradicting Claim 1. Thus, if $\psi_1 > 0$ we must have $v_0 > 0$ and so, $\mu_0 = 0$.

Next, adding (14) and (15) reveals that $\lambda_0 = \lambda_1 = \lambda$. Equation (14) then yields: $\frac{1}{2}f(y_0) + \lambda\left[\frac{1}{2}f(y_0) + \frac{1}{4}f'(y_0)\right] + \psi_1 = 0$, which implies $\lambda < 0$. Adding (12) and (13) reveals that $0 < 2F(\ell_0) + 2F(\ell_1) - 2F(y_0) - 1 < \Gamma(\ell)$, where Γ is as defined in (9). Then, by (10), $\ell_0 + \ell_1 > 1$. But since $v_1 - v_0 = 1$, this implies $y_0(\ell, v) = \frac{\ell_0+\ell_1+v_1-v_0}{2} > 1$, contradicting Lemma B.1. This establishes that $\psi_1 = 0$; analogous arguments reveal $\psi_0 = 0$.

We now show that $v_0, v_1 > 0$. First, suppose $v_1 > 0 = v_0$. Then, $\mu_0 \geq 0 = \mu_1$. (15) implies $\lambda_1 < 0$. Adding (14) and (15) reveals that $\lambda_1 \geq \lambda_0$. (12) and (15) then imply $2F(\ell_0) - F(y_0) = \lambda_1 \frac{1}{2}f(y_0) - \lambda_0 C_0''(\ell_0 - s_0) \geq \lambda_0\left(\frac{1}{2}f(y_0) - C_0''(\ell_0 - s_0)\right) > 0$, where the first inequality holds since $\lambda_1 \geq \lambda_0$; the strict inequality holds since $\lambda_0 < 0$ and $C_0'' > \frac{1}{2}f$. Thus, we have that $2F(\ell_0) - F(y_0) > 0$. But see that $v_0 = 0$ and $\frac{\partial u_0}{\partial \ell_0} = 0$ implies $\ell_0 = s_0$. Moreover, $\frac{\partial u_1}{\partial \ell_1} = 0$ implies $\ell_1 > s_1 - \frac{v_1}{4}$; this means $\ell_1 + v_1 > s_1 > 1 - s^*$. It follows that, $2F(s_0) - F\left(\frac{s_0+\ell_1+v_1}{2}\right) < 2F(s^*) - F\left(\frac{1}{2}\right) = 0$, where the first inequality holds since $2F(s_0) - F\left(\frac{s_0+\ell_1+v_1}{2}\right)$ is strictly increasing in s_0 and decreasing in ℓ_1. The final equality holds by definition of s^*. We have a contradiction, and therefore, it cannot be the case that $v_1 > 0 = v_0$. Following analogous arguments, it can be shown that we cannot have $v_0 > 0 = v_1$.

Then, suppose $v_0 = v_1 = 0$. (14) implies $\lambda_0 \leq 0$. And noting that we must have $\ell_i = s_i$ for each i, (12) implies $2F(s_0) - F\left(\frac{s_0+s_1}{2}\right) \geq 0$. However, since $s_0, 1 - s_1 < s^*$, $2F(s_0) - F\left(\frac{s_0+s_1}{2}\right) < 2F(s^*) - F\left(\frac{1}{2}\right) = 0$, which yields a contradiction. This establishes that $v_0, v_1 > 0$.

Next, we show that $v_0, v_1 < 1$. WLOG, let us suppose $v_1 \geq v_0$; we show that $v_1 < 1$. Adding (14) and (15) reveals that $\lambda_0 = \lambda_1$; we drop the subscript and denote λ_i by λ. (14) then yields, $\frac{1}{2}(v_0 - v_1) = \lambda\left[\frac{1}{2}f(y_0) + \frac{v_1-v_0}{2}f'(y_0)\right]$. Since $v_1 - v_0 \leq 1$, Assumption 2(i) implies that the term in square brackets is positive, which means $\lambda \leq 0$. Adding (13) and (14) reveals $2F(\ell_1) - F(y_0) - 1 - \frac{v_1}{2}f(y_0) \geq 0$; it follows that, $\frac{v_1-1}{2}f(y_0) \leq 2F(\ell_1) - F(y_0) - 1 - \frac{1}{2}f(y_0)$. Note that $v_1 \geq v_0$ implies $F(y_0) + \frac{1}{2}f(y_0) \geq F(m) + \frac{1}{2}f(m)$; together with Claim 1 we have, $2F(\ell_1) - F(y_0) - 1 - \frac{1}{2}f(y_0) \leq 2F(\ell_1) - F(m) - 1 - \frac{1}{2}f(m) < 0$. Thus, $v_1 < 1$.

We now show that $\ell_0 > s_0$ and $\ell_1 < s_1$. Since $v_i > 0$ and, by Claim B.1, $y_0(\ell, v) \in [0, 1]$, $\frac{v_1}{2}f(y_0) > 0$. Then, $\frac{\partial u_i(\ell,v)}{\partial \ell_i} = 0$ implies $C_1'(\ell_1 - s_1) = -\frac{v_1}{2}f(y_0) < 0 < \frac{v_0}{2}f(y_0) = C_0'(\ell_0 - s_0)$, which means $\ell_1 < s_1$ and $\ell_0 > s_0$. This establishes the result. □

Now, by Lemma B.2 and its proof, if $(\ell, v) \in \mathcal{A}_{P'}$, then, $v_0, v_1 \in (0, 1)$, $\mu_i = \psi_i = 0$, and $\lambda_0 = \lambda_1 = \lambda$. After some manipulation, Equations (12)-(15) yield

$$2F(\ell_0) - F(y_0) + C_0'(\ell_0 - s_0) = \lambda\left[\frac{1}{2}f(y_0) - C_0''(\ell_0 - s_0)\right] \tag{17}$$

$$2F(\ell_1) - F(y_0) - 1 + C_1'(\ell_1 - s_1) = \lambda\left[\frac{1}{2}f(y_0) - C_1''(\ell_1 - s_1)\right] \tag{18}$$

$$\frac{1}{2}f(y_0)(v_0 - v_1) = \lambda\left[\frac{1}{2}f(y_0) + \frac{v_1 - v_0}{4}f'(y_0)\right] \tag{19}$$

We next establish the following lemma:

Lemma B.3. *Let* $(\ell, v) \in \mathcal{A}_{P'}$. *Then,* $sgn(v_1 - v_0) = sgn(\ell_0 + \ell_1 - 1)$.

Proof. Adding (17) and (18) and using the fact that $\frac{\partial u_i(\ell,v)}{\partial \ell_i} = 0$, we have that,

$$2F(\ell_0) + 2F(\ell_1) - 2F(y_0) - 1 = \frac{v_1 - v_0}{2}f(y_0) + \lambda[f(y_0) - C_0''(\ell_0 - s_0) - C_1''(\ell_1 - s_1).]$$

By (19), $\text{sgn}(v_0 - v_1) = \text{sgn}(\lambda)$. Since, $f(y_0) - C_0''(\ell_0 - s_0) - C_1''(\ell_1 - s_1) < 0$, it follows that $\text{sgn}(2F(\ell_0) + 2F(\ell_1) - 2F(y_0) - 1) = \text{sgn}(v_1 - v_0)$. If $v_1 > [=]v_0$, then, $0 < [=]2F(\ell_0) + 2F(\ell_1) - 2F(y_0) - 1 < [=]\mathit{\Gamma}(\ell)$, and by (10), $\ell_0 + \ell_1 - 1 > [=]0$. If $v_0 > v_1$, then, $0 > 2F(\ell_0) + 2F(\ell_1) - 2F(y_0) - 1 > \mathit{\Gamma}(\ell)$ and (10) implies $\ell_0 + \ell_1 - 1 < 0$. □

Proposition B.1. *If* $(\ell, v) \in \mathcal{A}_{P'}$, *then,* $\ell \in \mathit{\Phi}(v)$ *and* $\ell_1 - \ell_0 > |v_1 - v_0|$.

Proof. Let $(\ell, v) \in \mathcal{A}_{P'}$; WLOG, suppose $v_0 \leq v_1$.

We first show that $\ell \in \mathit{\Phi}(v)$. By Proposition A.2, it suffices to show that $v_1 \leq 2\ell_1 - \ell_0 - 1$. (19) implies $\lambda \leq 0$. Then, using the fact that $C_1'(\ell_1 - s_1) = -\frac{v_1}{2}f(y_0)$, (18) implies, $2F(\ell_1) - F(y_0) - 1 - \frac{v_1}{2}f(y_0) \geq 0$; then,

$$\begin{aligned} v_1 &\leq \frac{2}{f(y_0)}\int_{y_0}^{\ell_1} f(x)dx - \frac{2}{f(y_0)}\int_{\ell_1}^{1} f(x)dx \\ &\leq \frac{2}{f(y_0)}(\ell_1 - y_0)f(y_0) - \frac{2f(1)}{f(y_0)}(1 - \ell_1) \\ &< 2(\ell_1 - y_0) - (1 - \ell_1) \\ &\leq 2\ell_1 - \ell_0 - 1. \end{aligned}$$

The second inequality holds since $f(1) \leq f(x)$ for each $x \in [0, 1]$ and since $f(y_0) \geq f(x)$ for $x \in [y_0, \ell_1]$. To see this second point, recall that f is decreasing on $[\frac{1}{2}, 1]$. And since $v_0 \leq v_1$, Lemma B.3 implies $\ell_0 + \ell_1 \geq 1$, which means $y_0 \geq \frac{1}{2}$. The third inequality holds since $2f(1) > f(y_0)$, and the final inequality follows since $v_1 \geq v_0$. This establishes that $\ell \in \mathit{\Phi}(v)$.

Finally, see that $v_1 - v_0 < v_1 \leq 2\ell_1 - \ell_0 - 1 < \ell_1 - \ell_0$, where the strict inequalities holds since Lemma B.2 implies $v_0 > 0$ and $\ell_1 < 1$.

□

Lemma B.4. *Let $(\ell, v) \in \mathcal{A}_{P'}$. If $v_0 < v_1$, then, $\ell_i^{FB} < \ell_i$, $i = 0, 1$. If $v_0 = v_1$, then, $\ell_i = \ell_i^{FB}$, $i = 0, 1$. If $v_1 < v_0$, then, $\ell_i < \ell_i^{FB}$, $i = 0, 1$.*

Proof. Let $(\ell, v) \in \mathcal{A}_{P'}$. First note that if $v_0 = v_1$, then, $\lambda = 0$ and (17)-(18) correspond to the FOCs in (7)-(8), meaning that $\ell = \ell^{FB}$. We now show that $v_0 < v_1 \implies \ell_i^{FB} < \ell_i$. The proof for the case where $v_1 < v_0$ is analogous.

Proceeding by contradiction, first suppose that $\ell_1 \leq \ell_1^{FB}$ and $\ell_0^{FB} \leq \ell_0$. Since $v_0 < v_1 \implies \lambda < 0$ and $m < y_0$, (18) implies, $0 < 2F(\ell_1) - F(m) - 1 + C'(\ell_1 - s_1) \leq 2F(\ell_1^{FB}) - F\left(m(\ell^{FB})\right) - 1 + C'(\ell_1^{FB} - s_1) = 0$, where the second inequality holds since $\ell_1 \leq \ell_1^{FB}$ and $\ell_0^{FB} \leq \ell_0$; the final equality holds by (8). This yields a contradiction.

Next, suppose $\ell_1 \leq \ell_1^{FB}$ and $\ell_0 < \ell_0^{FB}$. Adding (17) and (18) and using the fact that $\lambda < 0$ and $m < y_0$, we have, $0 < \Gamma(\ell) + C_0'(\ell_0 - s_0) + C_1'(\ell_1 - s_1) < \Gamma(\ell^{FB}) + C_0'(\ell_0^{FB} - s_0) + C_1'(\ell_1^{FB} - s_1) = 0$, where the second inequality holds since $\Gamma(\cdot)$ and C_i are strictly increasing and $\ell_1 \leq \ell_1^{FB}$ and $\ell_0 < \ell_0^{FB}$. The final equality holds by adding (7) and (8). We have a contradiction and, this establishes that $\ell_1 > \ell_1^{FB}$.

Finally, suppose $\ell_1^{FB} < \ell_1$ and $\ell_0 \leq \ell_0^{FB}$. Since $\lambda < 0$ and $m < y_0$, (17) implies, $0 < 2F(\ell_0) - F(m) + C_0'(\ell_0 - s_0) < 2F(\ell_0^{FB}) - F(m(\ell^{FB}) + C_0'(\ell_0^{FB} - s_0) = 0$, where the second inequality holds since $\ell_0^{FB} \leq \ell_0$ and $\ell_1^{FB} < \ell_1$ and the final equality holds by (7). We have a contradiction, and this establishes that $\ell_1 > \ell_1^{FB}$ and $\ell_0 > \ell_0^{FB}$. □

Proposition B.2. *Suppose $1 - s_1 < s_0 < s^*$ and $C_0 = C_1 = C$. If $(\ell, v) \in \mathcal{A}_{P'}$, then, $v_1 > v_0$, $\ell_0 > 1 - \ell_1$, $\ell_0 > \ell_0^{FB}$, and $\ell_1 > \ell_1^{FB}$.*

Proof. We first show that $v_1 > v_0$. Proceed by contradiction and suppose $v_0 \geq v_1$. By Lemma B.3, $\ell_0 + \ell_1 \leq 1$. But since $\frac{\partial u_1}{\partial \ell_1} = \frac{\partial u_0}{\partial \ell_0} = 0$ and $C_0 \equiv C_1 \equiv C$, we have $C'(\ell_0 - s_0) \geq C'(s_1 - \ell_1)$, which means $\ell_0 - s_0 \geq s_1 - \ell_1$; equivalently, $\ell_0 + \ell_1 \geq s_0 + s_1 > 1$. We have a contradiction, and therefore, we must have $v_0 > v_1$; and by Lemma B.3, $\ell_0 + \ell_1 > 1$.

□

B.2 Auxiliary Problem 2

Fix $\ell_0 = s_0$, $v_0 = 0$, and consider the following problem:

$$\min_{\ell_1, v_1} L(s_0, \ell_1, 0, v_1) \text{ s.t. } \frac{\partial u_1(\ell)}{\partial \ell_1} \leq 0, 0 \leq v_1 \leq \ell_1 - s_0 \tag{AUX 2}$$

Before characterizing the solution to this problem, we state and prove the following lemma:

Lemma B.5. *If $s_0, 1 - s_1 < s^*$, $v_j = 0 < v_i$, $\frac{\partial u_i(\ell_i^*, s_j)}{\partial \ell_i} = 0$ and $v_i < |\ell_i^* - s_j|$, then, $(\ell_i^*, s_j) \in \Phi(v)$.*

Proof. Suppose WLOG, $v_1 = 0 < v_0$. Since supplier one has a dominant strategy to play s_1, we focus on supplier zero. Let ℓ_0^* satisfy, $\frac{\partial u_0(\ell_0^*, s_1)}{\partial \ell_0} = 0$ and suppose $v_0 < |\ell_0^* - s_1|$. We will show that $\ell_0^* \in \arg\max_{\ell_0} \pi_0(\ell_0, s_1)$.

It is straightforward to verify that $v_0 < |\ell_0^* - s_1|$ and $\frac{\partial u_0(\ell_0^*, s_1)}{\partial \ell_0} = 0$ imply $\ell_0^* \in (s_0, s_1)$. Then, since $\ell_0^* < s_1$ we have that $v_0 < s_1 - \ell_0^*$. It follows that for all $\ell_0 \in [0, s_1 - v_0]$, $\pi_0(\ell_0, s_1) = u_0(\ell_0, s_1)$. By concavity of u_0 in ℓ_0, $u_0(\ell_0^*, s_1) = \max_{\ell_0 \in [0, s_1 - v_0]} \pi_0(\ell_0, s_1)$. Next, note that for $\ell_0 \in (s_1 - v_0, s_1 + v_0)$, $\pi_0(\ell_0, s_1) = -C_0(\ell_0 - s_0) < 0 < u_0(\ell_0^*, s_1)$. Finally, for $\ell_0 \in [s_1 + v_0, 1]$,

$$\begin{aligned}
\pi_0(\ell_0, s_1) &= v_0 \left(1 - F\left(\frac{\ell_0 + s_1 + v_0}{2}\right)\right) - C_0(\ell_0 - s_0) \\
&< v_0 F(s_1 - v_0) - C_0(s_1 - v_0 - s_0) \\
&= u_0(s_1 - v_0, s_1) \\
&< u_0(\ell_0^*, s_1) \\
&= \pi_0(\ell_0^*, s_1)
\end{aligned}$$

The first inequality holds since $\ell_0 \geq s_1 + v_0$ and $s_1 \geq \frac{1}{2}$ implies $1 - F\left(\frac{\ell_0 + s_1 + v_0}{2}\right) = F\left(1 - \frac{\ell_0 + s_1 + v_0}{2}\right) \leq F(s_1 - v_0)$ and $C_0(s_1 - v_0 - s_0) < C_0(\ell_0 - s_0)$. The strict inequality holds by definition of ℓ_0^*. We have now established that $\ell_0^* \in \arg\max_{\ell_0} \pi_0(\ell_0, s_1)$; therefore, $(\ell_0^*, s_1) \in \Phi(v)$. □

The next result characterizes the solution to (AUX 2).

Lemma B.6. *Suppose $s_0, 1 - s_1 < s^*$. If (ℓ_1^a, v_1^a) solves (AUX 2), then, $0 < v_1^a < \ell_1^a - s_0$ and $\frac{\partial u_1(s_0, \ell_1^a)}{\partial \ell_1} = 0$; moreover, $(s_0, \ell_1^a) \in \Phi(0, v_1^a)$.*

Proof. The Lagrangian associated with (AUX 2) is, $\mathcal{L} = L(\ell, v) - \lambda \left[\frac{v_1}{2} f(y_0) + C_1'(\ell_1 - s)\right] + \mu \left[s_0 + v_1 - \ell_1\right] - \mu_v v_1$. The first-order conditions yield,

$$\frac{\partial \mathcal{L}}{\partial \ell_1} = \frac{1}{2}v_1 f(y_0) + C_1'(\ell_1 - s_1) + 2F(\ell_1) - F(y_0) - 1 - \lambda\left[\frac{v_1}{4}f'(y_0) + C_1''(\ell_1 - s_1)\right] - \mu = 0 \quad (20)$$

$$\frac{\partial \mathcal{L}}{\partial v_1} = \frac{1}{2}v_1 f(y_0) - \lambda\left[\frac{1}{2}f(y_0) + \frac{v_1}{4}f'(y_0)\right] + \mu - \mu_v = 0 \quad (21)$$

$$\lambda\frac{\partial u_1(\ell)}{\partial \ell_1} = 0, \lambda \geq 0; \mu\left[s_0 + v_1 - \ell_1\right] = 0, \mu \geq 0; \mu_v v = 0, \mu \geq 0 \quad (22)$$

We will show that at the optimum $\lambda > 0 = \mu = \mu_v$. Let us first suppose $\lambda = 0$ and $v_1 > 0$. Then, by (22), $\mu_v = 0$. But then, (21) implies $\frac{v_1}{2}f(y_0) + \mu_1 = 0$; but since $v_1 > 0$ and $\mu_1 \geq 0$, we have an immediate contradiction. So, if $\lambda = 0$, we must have $v_1 = 0$. (21) then implies $\mu = \mu_v$. Then, note that $v_1 = 0$ and $\frac{\partial u_1}{\partial \ell_1} \leq 0$ implies $C_1'(\ell_1 - s_1) \geq 0$, which means $\ell_1 \geq s_1 > s_0 = s_0 + v_1$. By (22), we must have $\mu = 0$. (20) then yields, $0 \leq C_1'(\ell_1 - s_1) = 1 + F(y_0) - 2F(\ell_1)$. But see that,

$$1 + F(y_0) - 2F(\ell_1) = 1 + F\left(\frac{s_0 + \ell_1}{2}\right) - 2F(\ell_1) < 1 + F\left(\frac{s^* + 1 - s^*}{2}\right) - 2F(1 - s^*) = 0.$$

The inequality follows since $s_0 < s^*$ and $\ell_1 \geq s_1 > 1 - s^*$ and since $F\left(\frac{s_0+\ell_1}{2}\right) - 2F(\ell_1)$ is strictly increasing in s_0 and strictly decreasing in ℓ_1. The final equality holds by definition of s^*. We have a contradiction; therefore, we must have $\lambda > 0$, which means $\frac{v_1}{2}f(y_0) = C_1'(\ell_1 - s_1)$. Then, (20) yields, $F(\ell_1) - F(y_0) - [1 - F(\ell_1)] = \lambda\left[\frac{v_1}{4}f'(y_0) + C_1''(\ell_1 - s_1)\right] + \mu > 0$, where the inequality holds since $\lambda\left[\frac{v_1}{4}f'(y_0) + C_1''(\ell_1 - s_1)\right] > 0$ and $\mu \geq 0$. Since $F(\ell_1) \leq 1$, we must have $F(\ell_1) > F(y_0)$, which means $\ell_1 > s_0 + v_1$; by (22), $\mu = 0$. Finally, (21) implies $\frac{v_1}{2}f(y_0) = \lambda\left[\frac{1}{2}f(y_0) + \frac{v_1}{4}f'(y_0)\right] + \mu_v > 0$, which means $v_1 > 0$ and $\mu_v = 0$. The fact that $(s_0, \ell_1^a) \in \Phi(0, v_1^a)$ now follows by Lemma B.5. □