The infinitesimal rigidity of symmetric bar-joint frameworks with non-free joints

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Abstract

The infinitesimal rigidity of symmetric (bar-joint) frameworks has been studied extensively for over two decades. The area splits into the research of forced and incidental symmetric rigidity. Whereas forced symmetric rigidity only considers infinitesimal motions which maintain the symmetry of the framework, incidental symmetric rigidity allows infinitesimal motions that break symmetry.

In both settings and for various symmetry groups, combinatorial characterisations have been obtained for 'symmetry-generic' frameworks, i.e. frameworks which are as generic as possible subject to being symmetric. Assuming that the symmetry group acts freely on the joints, forced symmetric infinitesimally rigid frameworks have been characterised for all cyclic groups, and for dihedral groups $\mathcal{C}_{kv}$, where $k \geq 3$ is odd. With the same free-action assumption, incidentally symmetric infinitesimally rigid frameworks have been characterised for cyclic groups of order 2,4,6, and of odd order less than 1000.

A limitation of these results is the assumption that the symmetry group acts freely on the joints of the framework. In this thesis, we fill this mathematical gap. This is also motivated by problems in applied areas such as structural engineering or formation control, where symmetric frameworks are frequently studied, and frameworks may contain joints fixed by the point group (e.g. joints on the symmetry line of a reflection-symmetric framework, or in the centre of rotation of a rotationally-symmetric framework).

We consider plane frameworks which are symmetric with respect to cyclic groups or dihedral groups. We provide necessary conditions for incidentally infinitesimally rigid frameworks for all cyclic groups, and for the dihedral group of order 4; we also show that such conditions are sufficient for cyclic groups of order 2,4,6, or of odd order less than 1000. For cyclic groups of even order, we present counterexamples to show that the expected sparsity count is necessary, but not sufficient. We also give necessary conditions for the forced infinitesimal rigidity of frameworks that are symmetric with respect to dihedral groups of arbitrary finite order.

In order to do so, we introduce a generalisation of tools commonly used in the study of symmetric frameworks, known as 'orbit matrices' and 'gain graphs'. Orbit matrices are symmetry-adapted rigidity matrices, whose underlying combinatorial structures are gain graphs, directed multigraphs whose edges are labelled with group elements. A generalisation of gain graphs, and hence of orbit rigidity matrices, is needed when working with joints which are fixed by the symmetry group. A further generalisation is required if some joints are neither free nor fixed by the symmetry group (when working, say, with dihedral groups). We introduce such a generalisation, and show how some of the properties of usual gain graphs hold for this new definition, whilst others do not. This generalisation of gain graph is expected to be useful in future research, for the combinatorial characterisation of infinitesimally rigid dihedral-symmetric frameworks.

Acknowledgements

First, I would like to express my gratitude towards my supervisor, Bernd Schulze for his invaluable advice, continuous support, and patience during my PhD. His expertise in the research area, as well as his enthusiasm for the topic, were always reflected in the long and useful mathematical discussions, during which he provided valuable insight. Discussing mathematics with him has truly been a pleasure. His years of experience in the world of academia, and the opportunities he has presented me with, were also crucial for my growth both as a mathematician and as a researcher. Moreover, Bernd Schulze was always available when I needed help. I am very grateful to have had him as my supervisor.

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My family has also played a key role in my academic development, all of which have given me support throughout my whole studies, and are always there for me.

Declaration

I hereby declare that the work presented in this thesis is, to the best of my knowledge and belief, original and my own work. The material has not been submitted, either in whole or in part, for a degree at this, or any other university.

This thesis contains research carried out jointly with my supervisor, Dr. Bernd Schulze: Some of the work that appears in Chapters 3, 4 and 5, and mostly all of the work in Chapters 6 and 7, can be found in [47] and [48]. I declare that I contributed fully to all aspects of this research and the writing of these papers.

This thesis does not exceed the maximum permitted word length of 80,000 words including appendices and footnotes, but excluding the bibliography.

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Chapter 1

Introduction

1.1 Rigidity theory

Rigidity theory is a research area which uses combinatorial and geometric tools to analyse whether a given structure is flexible or rigid in a given space. Rigid and flexible structures have received significant interest from mathematicians for centuries. 'Rigidity theory', though it was not always recognised as such, was studied as far back as the eighteenth century, when L. Euler conjectured that *"a closed spatial figure allows no change, as long as it is not ripped apart"* [18].

Contributions from the nineteenth century include A.-L. Cauchy's theorem on the uniqueness of convex triangulated polyhedra whose edge lengths are fixed [7] and R. Bricard's construction of the first family of flexible polyhedra, the *Bricard octahedra* [6]. Work continued through the twentieth century. One noteworthy contributor is A.D. Alexandrov, with his work on the unique realisability of convex polyhedra [2] (see also [10]). Following J.C. Maxwell's necessary counting conditions for structures to be 'infinitesimally rigid' [41], in 1927, H. Pollaczek-Geiringer proved one of the central results in rigidity theory, which characterises rigid frameworks in the Euclidean plane, under genericity assumptions [46]. The same result was proved independently in 1970 by G. Laman [38]. It is now referred to as the *Geiringer-Laman theorem.*

The term rigidity started being used in the 1970's, when a series of strong papers in the topic were published, including G. Laman's characterisation. In 1975, H. Gluck refined Euler's rigidity conjecture [22]. In 1977, R. Connelly provided a counterexample, the *Connelly sphere*, to show that Euler's conjecture was incorrect. In 1978 and 1979, L. Asimow and B. Roth published a pair of papers in which they showed that, under genericity assumptions, rigidity can be treated as a graph theoretic property, and in which they presented a counterexample to show that the Geiringer-Laman Theorem does not hold for dimensions $d \geq 3$, the famous *double banana* [3, 4].

Since frameworks are a good mathematical model for physical structures, engineers have also played a key role in the developments of rigidity theory. One of the most notable contributors is L. Henneberg, who developed engineering techniques for generating rigid structures [26], the *Henneberg moves.* The Henneberg moves were converted into mathematical methods in 1985 by T.-S. Tay and W.J. Whiteley [69].

Over the last 50 years, rigidity theory has undergone significant mathematical development, and it has received interest from various areas of applied sciences. Now, geometric rigidity is a strong research area, both from a mathematical and an application-driven prospective: as a mathematical theory, it increasingly connects to other areas, including geometry, homotopy theory, tropical geometry, algebra, group theory, and others. For instance, the combinatorial characterisations of rigid graphs usually induce count matroids, which may be of interest in matroid theory; Results in geometric rigidity also have applications to a variety of sciences, ranging from mechanical and structural engineering and robotics to biophysics (flexibility of molecules), material sciences, formation control and Computer Aided Design (CAD).

A strong bridge between pure mathematical rigidity and real life applications is given by graphic statics, a geometric theory originating in structural engineering that provides visual information about the relation between forces and forms of a structure, and which often adopts results from rigidity theory. A classical result is the *Maxwell-Cremona correspondence*, linking self-stressed frameworks with dual (force) diagrams and polyhedral liftings [33, 40]. The recent paper [55] by B. Schulze and C. Millar provides a way of using the combinatorial results on rigid symmetric frameworks to study the forces of a structure.

In fact, *symmetric* structures have received significant interest for over two decades, as symmetry occurs naturally in many application areas. (See, for instance, [20] and [76] for a description on how symmetric rigidity can be used in CAD and formation control, respectively.) Exploiting symmetry is beneficial in engineering, and structure design should take into consideration *fully-symmetric* and *anti-symmetric* loads, i.e. loads which, respectively, maintain and break the symmetry of a structure [62]. Therefore, the research of symmetric rigidity splits in the study of *forced* and *incidental* symmetric rigidity. In both cases, the framework starts in a symmetric position. However, in the former, the symmetry of the framework must be maintained throughout its motions, whereas in the latter the framework can move in unrestricted ways [12, 60].

Since the study of symmetric frameworks is strongly motivated by its applications, some of the first strong contributors to the theory were engineers: in 2000, R. Kangwai and S. Guest made the breakthrough observation that the rigidity matrix of a symmetric frameworks block-diagonalises in a way that each block corresponds to exactly one irreducible representation of the symmetry group [31]. This was followed by a series of further observations on symmetric frameworks, again in an engineering setting (see. e.g., [19] and [30]).

These conclusions started being translated into mathematical results in 2010, when B. Schulze gave a rigorous proof to show that the observation given in [31] does in fact hold mathematically [49]. The same result was proved by J.C. Owen and S.C. Power, using different techniques [45]. One outcome of the result is the introduction of the *orbit rigidity matrix*, a tool used to study the forced rigidity properties of a symmetric structure [61]. However, the block-diagonalisation of the rigidity matrix is much stronger, as it also allows for a rigorous mathematical study of incidentally rigid symmetric frameworks: the problem of characterising the infinitesimal rigidity of a symmetric framework reduces into sub-problems, one for each irreducible representation of the symmetry group, which can be solved separately (see e.g. [50]).

Though symmetric frameworks are often non-generic, we can still make symmetry adapted genericity assumptions and, under these assumptions, rigidity (both forced and incidental) can be treated as graph theoretic properties. Moreover, assuming 'symmetry-genericity', forced symmetric *infinitesimal* rigidity and forced symmetric *continuous* rigidity coincide [54]. There are now many papers which provide rigidity characterisations for symmetry-generic frameworks for both forced [5, 16, 29, 39, 39, 42, 52, 53, 57] and incidental [9, 13, 27, 36, 37, 43, 56] rigidity, in various settings. The conditions are often given in terms of sparsity counts on group-labelled quotient graphs, which induce new types of count matroids [28, 64], and the sufficiency proofs use Henneberg-type inductive constructions.

So far, most of the results in symmetric rigidity make the assumption that the point group acts freely on the joints of the symmetric framework. Joints that are fixed by non-trivial symmetries make the theory substantially more difficult, but it is important to understand this for a variety of applications, as such framework configurations naturally appear in areas such as structural engineering or formation control. Notably, positioning joints on a line, such as a reflection line, can be beneficial in engineering structures, e.g. to avoid torsion.

In this thesis, we drop this free action requirement, in order to fill a gap in the existing knowledge. This requires a generalisation of algebraic (*phase-symmetric orbit rigidity matrices*) and combinatorial (*gain graphs*) objects which are central tools in the study of symmetric rigidity. The main results of the thesis are combinatorial characterisations of incidentally infinitesimally rigid symmetry-generic plane frameworks for which the symmetry group is either the reflection group $\mathcal{C}_s$ or a rotation group $\mathcal{C}_k$, where $k \leq 7$ or $7 < k < 1000$ is odd. The counts that we provide can be checked in polynomial time by combinatorial algorithms. We also provide necessary conditions for the incidental infinitesimal rigidity of '$\mathcal{C}_k$-generic' plane frameworks for all $k \geq 8$ and counterexamples to show that such conditions are not sufficient. For these groups, we provide a combinatorial characterisation of forced infinitesimally rigid symmetry-generic plane frameworks. Moreover, we explore frameworks which are symmetric with respect to a dihedral group $\mathcal{C}_{kv}$ (where $k \geq 2$): when $k = 2$, we provide necessary conditions for the incidental infinitesimal rigidity of symmetry-generic plane frameworks; for all $k \geq 3$, we provide necessary conditions for the forced infinitesimal rigidity of symmetry-generic plane frameworks.

1.2 Outline of the thesis

In this thesis, we consider symmetric bar-joint frameworks in $\mathbb{R}^2$, whose joints may be free, fixed, or neither. We assume throughout the thesis that the frameworks satisfy some genericity condition, and we restrict the study to finite frameworks. Therefore, the symmetry groups we consider are the reflection group $\mathcal{C}_s$, rotation groups $\mathcal{C}_k$, and dihedral groups $\mathcal{C}_{kv}$, where $k \geq 2$ is finite.

In Chapter 2 we present some important concepts in graph theory and in rigidity theory, while reviewing some key results and setting the terminology which will be used throughout the thesis. We introduce three distinct but connected notions of rigidity: continuous rigidity, infinitesimal rigidity and generic rigidity. We also formalise the notions of symmetric framework and symmetric rigidity. We conclude the chapter by setting the symmetry notation which will be used for the rest of the thesis.

In Chapter 3 we present the notion of gain graph, a widely used tool in the study of symmetric frameworks whose joints are free under the symmetry group action. In order to use this tool in the setting where the symmetry group need not act freely on the joints of the framework, we generalise the notion of gain graph in two steps: first to consider cyclic groups; then to consider all groups. We also generalise some key results concerning gain graphs.

In Chapter 4 we examine the rigidity matrix, the main tool in the study of infinitesimal rigidity. We present the main result in [49], which asserts that the rigidity matrix of a symmetric framework may be block-diagonalised. We then utilise this result to construct 'phase-symmetric orbit rigidity matrices', which will be used in later chapters to find a combinatorial characterisation of infinitesimally rigid symmetry-generic frameworks.

In Chapter 5 we provide necessary conditions for the infinitesimal rigidity of frameworks whose symmetry group is either a cyclic group or a dihedral group of order 4. We also provide necessary conditions for the forced infinitesimal rigidity of frameworks which are symmetric with respect to a dihedral group of arbitrary but finite order. A summarised version of the arguments in Chapter 5 for cyclic groups can also be found in [48].

In Chapters 6 and 7 we show that the necessary conditions given in Chapter 5 for infinitesimally rigid frameworks which are symmetric with respect to a cyclic group are also sufficient, provided the group has order 2,4,6, or odd order less than 1000. We give explanation on the restrictions on the group order. Namely, we provide counterexamples to show that such conditions are not sufficient for cyclic groups of even order $k \geq 8$. Chapters 6 and 7 are based on [48] and [47], respectively.

In Chapter 8 we explore ways in which the research of symmetric frameworks with joints fixed by the symmetry action can be further developed. Namely, we expand on the difficulties which arise when we consider certain cyclic and dihedral groups, and we offer possible solutions. We also briefly present some ways in which the techniques presented in this thesis can be adopted in different settings.

Chapter 2

Graphs, frameworks and symmetry

2.1 Graphs: undirected, directed and symmetric

We start by recalling some graph theoretic concepts. The main purpose of this section is to set the notation which will be used throughout the thesis, as well as to remind the expert reader of concepts in graph theory, which are crucial for a good understanding of the thesis.

2.1.1 Basic concepts and notation

We start by recalling the notions of undirected and directed graphs. Both play a key role in the thesis.

Definition 2.1.1. An *undirected graph* is a pair $G = (V, E)$, where V is a finite set of objects called *vertices*, and E is a multiset of unordered pairs of vertices of G called *edges*. Given two vertices $u, v \in V$, an edge $e \in E$ between u and v is denoted uv.

A *directed graph* is a pair $G = (V, E)$, where V is a finite set of objects called *vertices*, and E is a multiset of ordered pairs of vertices of G called *edges*. Given two vertices $u, v \in V$, an edge $e \in E$ from u to v is denoted (u, v).

Given $e = uv \in E$ (if G is undirected) or $e = (u, v) \in E$ (if G is directed), we say u, v are the *end-vertices* of e, that u, v are *adjacent*, and that u and e are *incident*, as are v and e.

When clear from the context, we use the term graph for both a directed and undirected graph. We denote the *vertex set* of a graph G by $V(G)$, and its *edge set* by $E(G)$. All notions given for the rest of the section are defined for both directed and undirected graphs.

Definition 2.1.2. Given a graph G, we define:

(i) A *loop* at a vertex $u \in V(G)$ to be an edge whose end-vertices are both u.

(ii) Two *parallel edges* to be edges $e, f \in E(G)$ which share the same end-vertices.

If $E(G)$ contains no loops and no parallel edges, then we say G is *simple*. Otherwise, we say G is a *multigraph*. Note, when G is simple, $E(G)$ is a set of pairs of vertices.

Definition 2.1.3. Let G be a graph with a vertex u. A *neighbour* of u is a vertex v which is adjacent to u. The *neighbourhood* of u is the set of all neighbours of u, and is denoted $N_G(u)$.

Definition 2.1.4. Let G be a graph with a vertex u. The *degree* of u is the number of edges incident to u, and is denoted by $\deg_G(u)$. If e is a loop at u, then e contributes two to the degree of u.

Definition 2.1.5. A *bipartite graph* (or *bigraph*) is a graph G whose vertex set may be partitioned into two sets U and V, called *partite sets*, such that for every edge $e \in E(G)$, one end-vertex of e lies in U and the other lies in V.

Definition 2.1.6. Let G be a graph, and $U \subseteq V(G)$. The *graph induced by U* is the graph whose vertex set is U and whose edge set is the set of all edges in $E(G)$ that have both end vertices in U. We denote such a graph by $[U]_G$.

Definition 2.1.7. Let G be a graph. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G, we write $H \subseteq G$. We say a subgraph H of G is *proper* if $E(H) \subsetneq E(G)$, in which case we write $H \subsetneq G$. A subgraph H of G is called *spanning* if $V(H) = V(G)$.

Let G be a graph. A simple way of obtaining subgraphs of G is by recursively removing vertices and/or edges from G.

(i) For some $e \in E(G)$, we use $G - e$ to denote the graph obtained from G by removing e. For some $F \subseteq E(G)$, we use $G - F$ to denote the graph obtained from G by removing all edges in F.

(ii) For some $u \in V(G)$, we use $G - u$ to denote the graph obtained from G by removing u, together with all edges incident to u. For some $U \subseteq V(G)$, we use $G - U$ to denote the graph obtained from G by removing all vertices in U, together with all edges incident to the vertices in U.

In a similar way, one can obtain graphs by recursively adding vertices and/or edges to a graph. Let G be a graph and $H \subsetneq G$.

(i) For some $e \in E(G) \setminus E(H)$ such that both end vertices of e lie in $V(H)$, we use $H + e$ to denote the graph obtained from H by adding e. For some $F \subseteq E(G) \setminus E(H)$ such that both end vertices of each edge in F lie in $V(H)$, we use $H + F$ to denote the graph obtained from G by adding all edges in F.

(ii) For some $u \in V(G) \setminus V(H)$ (if it exists), we use $H +_G u$ to denote the graph obtained from H by adding u, together with all edges in G which are incident to u and some $v \in V(H)$, as well as all loops at u in G. If there is some non-empty $U \subseteq V(G) \setminus V(H)$, we use $H +_G U$ to denote the graph obtained from H by adding all vertices in U, together with all edges in G incident to two vertices in $U \cup V(H)$, as well as all loops in G adjacent to the vertices in U. When G is clear from the context, we write $H + u$ and $H + U$ for $H +_G u$ and $H +_G U$, respectively.

Another very simple way of building graphs is by taking the union and the intersection of two different graphs.

Definition 2.1.8. Let G_1 and G_2 be two graphs. The *union* $G_1 \cup G_2$ of G_1 and G_2 is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and whose edge set is the set $E(G_1) \cup E(G_2)$. The *intersection* $G_1 \cap G_2$ of G_1 and G_2 is the graph whose vertex set is $V(G_1) \cap V(G_2)$ and whose edge set is $E(G_1) \cap E(G_2)$.

Definition 2.1.9. Let G be graph and $u, v \in V(G)$. A $u - v$ *walk* W in G is a finite sequence $e_1, \dots, e_k$ of edges of G for which there is a sequence of vertices $u = u_1, \dots, u_{k+1} = v$ such that, for each $1 \leq i \leq k$, e_i has end-vertices u_i and u_{i+1}. The vertices $u_2, \dots, u_k$ are called the *internal vertices of* W. We say W is a $u - v$ *path* if $v_1, \dots, v_k$ are all distinct. A *closed walk* is a $u - v$ walk with $u = v$, and a *cycle* is a closed path.

Definition 2.1.10. We say a graph G is *connected* if, for all $u \neq v \in V(G)$, there is a $u - v$ path in G. Otherwise, G is *disconnected.* Given a connected graph G, we say a subset $U \subseteq V(G)$ is a *separating set of* G if $G - U$ is disconnected. If a singleton U is a separating set of G, we call its unique element a *separating vertex of* G. Given $k \geq 2$, we say a connected graph G with $|V(G)| > k$ is k*-connected* if it has no separating set of size k. For $k \geq 2$, a k*-connected component* (respectively, a *connected component*) of a graph G with $|V(G)| > k$ is a k-connected subgraph (respectively, a connected subgraph) H of G such that $H = H'$ for all k-connected graphs (respectively, all connected graphs) H' with $H \subseteq H' \subseteq G$.

Definition 2.1.11. A graph with no cycles is called a *forest*, and a connected forest is called a *tree.* A spanning subgraph of G is called a *spanning forest* of G it if is a forest, and it is called a *spanning tree* of G if it is a tree.

2.1.2 Symmetric graphs

In this section, we define symmetric graphs. For the purpose of this thesis, symmetric graphs are simple, undirected graphs.

Definition 2.1.12. An *automorphism* of an undirected graph G is a permutation π of $V(G)$ such that $uv \in E(G)$ if and only if $\pi(u)\pi(v) \in E(G)$. The automorphisms of G form a group under composition, called the *automorphism group* of G, which we denote $\mathrm{Aut}(G)$.

Definition 2.1.13. Let Γ be an (abstract) group and G be a simple undirected graph. If there is a homomorphism $\theta : \Gamma \to \mathrm{Aut}(G)$, we say G is Γ-*symmetric* with respect to the *group action* θ.

Definition 2.1.14. Let Γ be a group, G be a Γ-symmetric graph with respect to an action θ, and $u \in V(G), e \in E(G)$. The *(vertex)* Γ-*orbit* of u with respect to θ is the set $\theta(\Gamma)u = \{\theta(\gamma)(u) : \gamma \in \Gamma\}$, and the *(edge)* Γ-*orbit* of e with respect to θ is the set $\theta(\Gamma)e = \{\theta(\gamma)(e) : \gamma \in \Gamma\}$. The sets of vertex Γ-orbits and edge Γ-orbits of G with respect to θ are denoted by $\theta(\Gamma)V(G)$ and $\theta(\Gamma)E(G)$, respectively.

Example 2.1.15. Let G be the graph in Figure 2.1(a). The automorphism group of G is $\mathrm{Aut}(G) = \langle\gamma\rangle$, where $\gamma = (123)(456)(789)$. See Figure 2.1(a,b,c) for all automorphisms in $\mathrm{Aut}(G)$ applied to G. So, G is $\mathbb{Z}_3$-symmetric with respect to the isomorphism $\theta : \mathbb{Z}_3 \to \mathrm{Aut}(G)$ which maps the generator 1 of $\mathbb{Z}_3$ to γ.

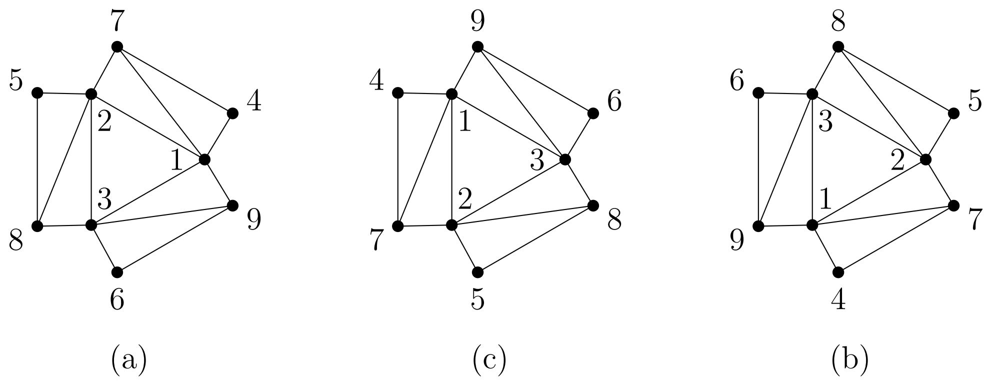


Figure 2.1: A $\mathbb{Z}_3$-symmetric graph G. (a,b,c) are, respectively, the graphs obtained from G by applying the automorphisms $(1)(2)\ldots(9), (123)(456)(789)$ and $(132)(465)(798)$.

Definition 2.1.16. Let Γ be a group and G be a Γ-symmetric graph with respect to a group action θ. We say an element $\gamma \in \Gamma$ *fixes a vertex* $u \in V(G)$ *with respect to* θ if $\theta(\gamma)(u) = u$, and it *fixes a subset* U *of* $V(G)$ *with respect to* θ if, for all $u \in U$, it fixes u with respect to θ. The *stabiliser of* $u \in V(G)$ *with respect to* Γ *and* θ is the set $S_{\Gamma,\theta}(u) = \{\gamma \in \Gamma : \gamma \text{ fixes } u \text{ with respect to } \theta\}$. A vertex $u \in V(G)$ is:

(i) *fixed by (or under)* Γ *with respect to* θ if $S_{\Gamma,\theta}(u) = \Gamma$. Equivalently, we can say Γ *fixes* u *with respect to* θ. The set of all vertices of G which are fixed by Γ with respect to θ is denoted by $V_{|\Gamma|}(G)_{\Gamma,\theta}$.

(ii) *free under* Γ *with respect to* θ if $S_{\Gamma,\theta}(u) = \{\mathrm{id}_\Gamma\}$. Equivalently, we can say Γ *acts freely on* u *with respect to* θ. The set of all vertices of G which are free under Γ with respect to θ is denoted by $V_1(G)_{\Gamma,\theta}$.

(iii) *semi-free under* Γ *with respect to* θ if it is neither free nor fixed under Γ with respect to θ. For $2 \leq i \leq |\Gamma| - 1$, the set of all vertices of G whose stabiliser with respect to Γ and θ has size i is denoted by $V_i(G)_{\Gamma,\theta}$. Notice that the set of semi-free vertices of G under Γ with respect to θ is $V_2(G)_{\Gamma,\theta} \dot\cup \dots \dot\cup V_{|\Gamma|-1}(G)_{\Gamma,\theta}$, where $A \dot\cup B$ denotes the disjoint union of two sets A and B.

When the group Γ and the action θ are clear from the context, we say a vertex is fixed/free/semi-free, and we write $V_1(G), V_2(G), \dots V_{|\Gamma|}(G)$.

Definition 2.1.17. Let Γ be a group and G be a Γ-symmetric graph with respect to a group action θ. We say an element $\gamma \in \Gamma$ *fixes an edge* $e \in E(G)$ *with respect to* θ if $\theta(\gamma)(e) = e$, and it *fixes a subset* F *of* $E(G)$ *with respect to* θ if, for all $e \in F$, it fixes e with respect to θ. The *stabiliser of* $e \in E(G)$ *with respect to* Γ *and* θ is the set $S_{\Gamma,\theta}(e) = \{\gamma \in \Gamma : \gamma \text{ fixes } e \text{ with respect to } \theta\}$. An edge $e \in E(G)$ is:

(i) *fixed by (or under)* Γ *with respect to* θ if $S_{\Gamma,\theta}(e) = \Gamma$. Equivalently, we can say Γ *fixes* e *with respect to* θ. The set of all edges of G which are fixed by Γ with respect to θ is denoted by $E_{|\Gamma|}(G)_{\Gamma,\theta}$.

(ii) *free under* Γ *with respect to* θ if $S_{\Gamma,\theta}(e) = \{\mathrm{id}_\Gamma\}$. Equivalently, we can say Γ *acts freely on* e *with respect to* θ. The set of all edges of G which are free under Γ with respect to θ is denoted by $E_1(G)_{\Gamma,\theta}$.

(iii) *semi-free under* Γ *with respect to* θ if it is neither free nor fixed under Γ with respect to θ. For $2 \leq i \leq |\Gamma| - 1$, the set of all edges of G whose stabiliser with respect to Γ and θ has size i is denoted by $E_i(G)_{\Gamma,\theta}$. The set of semi-free edges of G under Γ with respect to θ is $E_2(G)_{\Gamma,\theta} \dot\cup \ldots \dot\cup E_{|\Gamma|-1}(G)_{\Gamma,\theta}$.

When the group Γ and the action θ are clear from the context, we say an edge is fixed/free/semi-free, and we write $E_1(G), E_2(G), \ldots E_{|\Gamma|}(G)$. For the purposes of this thesis, we assume that whenever Γ is cyclic $V(G) = V_1(G) \dot\cup V_{|\Gamma|}(G)$ and that $E(G) = E_1(G) \dot\cup E(G)$. (See Subsection 2.3.1 for an explanation.)

Example 2.1.18. Let W_5 be the wheel graph on 5 vertices, i.e. the graph which connects a single vertex to all vertices of a cycle on 4 vertices, as shown in Figure 2.2. Let $\mathbb{D}_8$ denote the dihedral group $\langle s, r : s^2 = r^4 = (sr)^2 = \mathrm{id}\rangle$. With the same notation as in Figure 2.2, W_5 is $\mathbb{D}_8$-symmetric with respect to the action $\theta : \mathbb{D}_8 \to \mathrm{Aut}(W_5)$ which maps r to (1234) and s to (13). Note that $1, 2, 3, 4$ are all semi-free under $\mathbb{D}_8$ with respect to θ, and 0 is fixed by $\mathbb{D}_8$ with respect to θ. Note also that the edge 12 is semi-free under $\mathbb{D}_8$ with respect to θ, since it is fixed by $(12)(34)$, but not by $(14)(23)$.

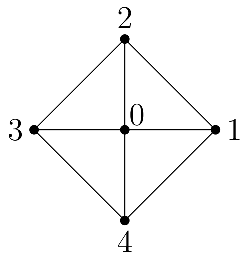


Figure 2.2: The $\mathbb{D}_8$-symmetric graph W_5.

We conclude this section by recalling the notion of quotient graph. This thesis uses G when referring to the quotient of a graph, and it uses $\tilde{G}$ when referring to the 'lifting' of a graph.

Definition 2.1.19. Let Γ be a group and $\tilde{G} = (\tilde{V}, \tilde{E})$ be a Γ-symmetric graph with respect to a group action θ. The Γ-*quotient graph of* $\tilde{G}$ *with respect to* θ is an (undirected) multigraph $G = (V, E)$. The vertex set V is the set of vertex orbits of $\tilde{G}$ with respect to Γ and θ, and the edge set E is the set of edge orbits of $\tilde{G}$ with respect to Γ and θ. G is indeed a graph, since each edge orbits correspond to a pair of vertex orbits. The graph $\tilde{G}$ is the Γ-*lifting of* G *with respect to the action* θ.

Example 2.1.20. Let K_4 be the complete graph on the 4 vertices $\tilde{V} = \{\tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}\}$, as shown in Figure 2.3(a). View K_4 as a $\mathbb{Z}_2$-symmetric graph with respect to the action $\theta_1 : \mathbb{Z}_2 \to \mathrm{Aut}(G)$ which maps the non-identity element 1 of $\mathbb{Z}_2$ to $(\tilde{1}\tilde{2})(\tilde{3}\tilde{4})$. The vertex orbits of $\tilde{G}$ with respect to $\mathbb{Z}_2$ and θ_1 are $1 := \{\tilde{1}, \tilde{2}\}$ and $4 := \{\tilde{3}, \tilde{4}\}$, and the edge orbits of $\tilde{G}$ with respect to $\mathbb{Z}_2$ and θ_1 are $e_{12} = \{\tilde{1}\tilde{2}\}, e_{13} = \{\tilde{1}\tilde{3}, \tilde{2}\tilde{4}\}$, $e_{14} = \{\tilde{1}\tilde{4}, \tilde{2}\tilde{3}\}$ and $e_{34} = \{\tilde{3}\tilde{4}\}$. So, the $\mathbb{Z}_2$-quotient graph of $\tilde{G}$ with respect to θ_1 has vertex set $\{1, 4\}$ and edge set $\{e_{12}, e_{13}, e_{14}, e_{34}\}$ (see Figure 2.3(b)). Now, view K_4 as a $\mathbb{Z}_4$-symmetric graph with respect to the action $\theta_2 : \mathbb{Z}_4 \to \mathrm{Aut}(G)$ which maps 1 to $(\tilde{1}\tilde{2}\tilde{3}\tilde{4})$. The only vertex orbit of $\tilde{G}$ with respect to $\mathbb{Z}_4$ and θ_2 is $\tilde{V}$ itself. The edge orbits of $\tilde{G}$ with respect to $\mathbb{Z}_4$ and θ_2 are $f_{12} = \{\tilde{1}\tilde{2}, \tilde{2}\tilde{3}, \tilde{3}\tilde{4}, \tilde{4}\tilde{1}\}$ and $f_{13} = \{\tilde{1}\tilde{3}, \tilde{2}\tilde{4}\}$. So, the $\mathbb{Z}_4$-quotient graph of $\tilde{G}$ with respect to θ_2 has vertex set $\{\tilde{V}\}$ and edge set $\{f_{12}, f_{13}\}$ (see Figure 2.3(c)).

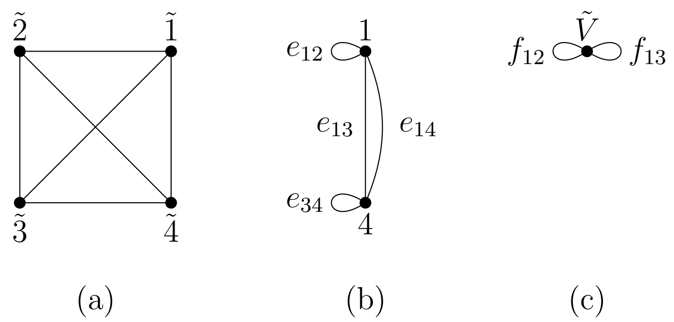


Figure 2.3: Two different quotient graphs of K_4.

Though Example 2.1.18 shows that the action θ cannot be neglected, it is sometimes clear from the context. In such cases we abbreviate $\theta(\gamma)(v), \theta(\gamma)(e)$ to $\gamma v, \gamma e$, respectively, and Γ-symmetric graphs with respect to θ are referred to as Γ-symmetric graphs. Notice that the partitioning $V(\tilde{G}) = V_1(\tilde{G}) \dot{\cup} \ldots \dot{\cup} V_{|\Gamma|}(\tilde{G})$ induces a partition of $V(G)$ into the sets $V_i(G) = \{\Gamma v \in V(G) : |\Gamma| = i\}$ for all $1 \leq i \leq k$.

2.2 Rigidity theory

In this section, we introduce some base concepts in rigidity theory [12, 23, 60]. Throughout this section, we fix a dimension d. We will later fix d to be 2.

2.2.1 Basic concepts in rigidity theory

One of the main objects of interest in geometric rigidity is the bar-joint framework, which is comprised of flexible joints connected by straight bars of fixed length.

Definition 2.2.1. A *(bar-joint) framework* in $\mathbb{R}^d$ (equivalently, *d-framework*) is a pair (G, p) where G is a simple undirected graph and $p : V(G) \to \mathbb{R}^d$ is a map. Alternatively, (G, p) is known as a *d-realisation* of the *underlying graph* G. The map p is known as a *d-configuration* of G.

When the dimension d is clear, d-framework, d-realisation and d-configuration may be abbreviated to framework, realisation and configuration, respectively. By ordering the vertices of G so that $V(G) = \{1, \ldots, n\}$, we may identify p with the column vector in $\mathbb{R}^{dn}$ that has the vector $p(k)$ on the entries $d(k-1)+i$ for $1 \leq i \leq d$ and $1 \leq k \leq n$.

We call the realisation $p(u)$ of a vertex $u \in V(G)$ a *joint* of (G, p). Realising the vertices of G implicitly realises its edges. The realisation of an edge $e = uv \in E$ is the straight line segment between $p(u)$ and $p(v)$, and so its length is defined to be $||p(u) - p(v)||$, where $|| \cdot ||$ denotes the Euclidean norm. We allow an abuse of notation, and use $p(e)$ to denote the realisation of e. We call the realisation $p(e)$ of an edge $e \in E(G)$ a *bar* of (G, p). We sometimes abbreviate the notation of a joint from $p(u)$ to p_u, and the notation of a bar from $p(e)$ to p_e.

The main interest in rigidity theory is to decide whether a given framework is rigid or flexible. We introduce three distinct, but related definitions of rigidity.

2.2.2 Continuous rigidity

We start by presenting the definition of (continuous) rigidity. A continuous motion of a framework is a smooth displacement of its joints, as shown in Figure 2.4.

Definition 2.2.2. Let $G = (V, E)$ be a graph with vertex set $V = \{1, \ldots, n\}$, and let (G, p) be a d-realisation of G. A *continuous motion* of (G, p) is a collection $\{P_i\}_{i=1}^n$ of n continuous maps $P_i : [0, 1] \to \mathbb{R}^d$, each corresponding to a vertex of G, such that

(i) $P_i(0) = p_i$ for all $1 \leq i \leq n$;

(ii) $P_i(t)$ is differentiable on the interval $[0, 1]$ for all $1 \leq i \leq n$;

(iii) $||P_i(t) - P_j(t)||^2 = ||p_i - p_j||^2$ for all $t \in [0, 1]$ and all edges $ij \in E$.

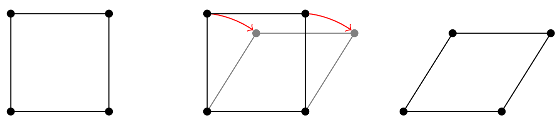


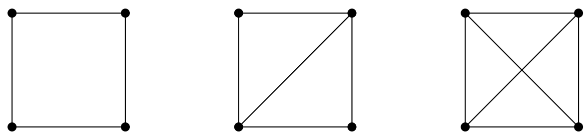
Figure 2.4: Continuous motion of a 2-framework.

Definition 2.2.3. Let (G, p) be a framework with $V(G) = \{1, \ldots, n\}$, and $\{P_i\}_{i=1}^n$ be a continuous motion of (G, p). We say $\{P_i\}_{i=1}^n$ is *trivial* (or *rigid*) if, for all $t \in [0, 1]$ and all pairs of vertices $i, j \in V$,

$$||P_i(t) - P_j(t)||^2 = ||p_i - p_j||^2.$$

Definition 2.2.4. We say (G, p) is *continuously rigid* if all of its continuous motions are trivial. Otherwise, we say (G, p) is *continuously flexible*. We say (G, p) is *minimally continuously rigid* if it is continuously rigid, and there is some $e \in E$ such that $(G - e, p)$ is continuously flexible.

Figure 2.5 shows three examples of 2-frameworks: (a) is continuously flexible, (b) is minimally continuously rigid, and (c) is (not minimally) continuously rigid. As 3-frameworks, (b) and (c) are, respectively, continuously flexible and minimally continuously rigid.



(a) (b) (c)

Figure 2.5: Three different examples of 2-frameworks.

The trivial continuous motions of a framework (G, p) are exactly the motions composed of rotations and translations of (G, p).

2.2.3 Infinitesimal rigidity

Determining whether a framework is continuously rigid is NP-hard, as it requires solving a system of quadratic equations [1]. We simplify the theory by linearising it: instead of studying continuous motions, we consider velocity vectors assigned to the joints of a framework. Infinitesimal rigidity implies continuous rigidity (see, e.g., the frameworks in Figure 2.6), and it is strongly connected to static rigidity, a useful theory in engineering.

Definition 2.2.5. An *infinitesimal motion* of a d-framework (G, p) is a function $m : V(G) \to \mathbb{R}^d$ such that for all $uv \in E(G)$,

$$(p_u - p_v)^T (m_u - m_v) = 0, \tag{2.1}$$

where m_u and m_v denote $m(u)$ and $m(v)$, respectively.

Let $V(G) = \{1, \dots, n\}$. If (G, p) has a continuous motion $\{P_i\}_{i=1}^n$, then one can check that the map $m : V \to \mathbb{R}^d$ defined by letting $m(i) = P_i'(0)$ is an infinitesimal motion. We say an infinitesimal motion m of (G, p) is *trivial* (or *rigid*) if there is a trivial continuous motion $\{P_i\}_{i=1}^n$ such that $m(i) = P_i'(0)$. Otherwise, we say m is an *infinitesimal flex*. The following definition is equivalent.

Definition 2.2.6. An infinitesimal motion m of a framework (G, p) is *trivial* (or, *rigid*) if there is a skew-symmetric matrix $M \in M_d(\mathbb{R})$ and a d-dimensional vector t such that $m(u) = Mp_u + t$ for all $u \in V(G)$. Otherwise, m is called an *infinitesimal flex*.

Definition 2.2.7. We say (G, p) is *infinitesimally rigid* if all of its infinitesimal motions are trivial. Otherwise, we say (G, p) is *infinitesimally flexible*. We say (G, p) is *minimally infinitesimally rigid* (equivalently, *isostatic*) if it is infinitesimally rigid, and there is some $e \in E(G)$ such that $(G - e, p)$ is infinitesimally flexible.

Since all differentiable continuous motions can be differentiated into infinitesimal motions, and since the spaces of trivial continuous motions and of trivial infinitesimal motions of a framework have the same dimension, we expect all infinitesimally rigid frameworks to also be continuously rigid. This is, in fact, true. Though the idea seems intuitive, the proof is quite elaborate. [[11], Theorem 2.54] has three different proofs of the following statement.

Theorem 2.2.8. Let (G, p) be a d-framework. If (G, p) is infinitesimally rigid, then it is continuously rigid.

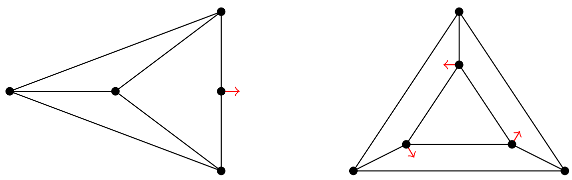


Figure 2.6: 2-frameworks which are continuously rigid, but not infinitesimally rigid.

As we will see in Subsection 2.2.4, under certain genericity conditions, continuous and infinitesimal rigidity coincide. However, this is not generally true. Figure 2.6

shows two examples of frameworks which are continuously rigid, but infinitesimally flexible.

We conclude the section by introducing the main tool used in the study of infinitesimal rigidity, the rigidity matrix.

Definition 2.2.9. Let (G,p) be a d-framework (G,p) with $V(G)=\{1,\ldots,n\}$. The *rigidity matrix* $R(G,p)$ of (G,p) is the $|E(G)|\times(dn)$ matrix

$$\begin{array}{cc} & \begin{array}{ccccccccccc} 1 & & & i & & & & j & & & n \end{array} \\ R(G,p) = & \begin{pmatrix} & & & & & \vdots & & & & & \\ 0 & \ldots & 0 & [p_i-p_j]^T & 0 & \ldots & 0 & [p_j-p_i]^T & 0 & \ldots & 0 \\ & & & & & \vdots & & & & & \end{pmatrix} (ij\in E(G)), \end{array}$$

where the d-dimensional row vector $[p_i-p_j]^T$ appears in the d columns corresponding to i, the d-dimensional row vector $[p_j-p_i]^T$ appears in the columns corresponding to j, and the other entries in the row corresponding to the edge ij are zero.

By ordering the vertices of G, we may identify an infinitesimal motion m of (G,p) with a vector in $\mathbb{R}^{dn}$ that has the vector $m(k)$ on the entries $d(k-1)+i$ for $1\le i\le d$ and for $1\le k\le n$. By Equation 2.1, the right kernel of $R(G,p)$ is the space of infinitesimal motions of (G,p). By Definition 2.2.6, it follows that the rigidity matrix of a framework (G,p) (such that $p(V)$ spans $\mathbb{R}^d$) has nullity at least $\binom{d+1}{2}$, and (G,p) is infinitesimally rigid if and only if nullity $R(G,p)=\binom{d+1}{2}$ or equivalently, rank $R(G,p)=d|V(G)|-\binom{d+1}{2}$. (For a formal proof, see [[3], Section 3].)

The rigidity matrix is also strongly connected to an important concept in rigidity theory, known as equilibrium stress.

Definition 2.2.10. An *equilibrium stress* of a framework (G,p) is a map $\omega: E(G)\to\mathbb{R}$ such that for all $u\in V(G)$, it satisfies

$$\sum_{v:uv\in E(G)}\omega(uv)(p_u-p_v)=0.$$

Notice that ω is an equilibrium stress if and only if $R(G,p)^T\omega = 0$. So a framework (G,p) has a non-zero equilibrium stress if and only if there is a non-trivial row dependency in $R(G,p)$.

2.2.4 Generic rigidity

Definition 2.2.11. Let G be a graph. A d-configuration p of G is said to be *generic* if $p(V(G))$ is algebraically independent over $\mathbb{Q}$. If p is generic, we say the framework (G,p) is *generic*.

Continuous and infinitesimal rigidity coincide for all generic frameworks. Moreover, if G has an infinitesimally rigid generic d-realisation, then all generic d-realisations of G are infinitesimally rigid [59].

Theorem 2.2.12 ([3], Theorem 1)**.** Let (G,p) be a generic d-framework. Then, (G,p) is infinitesimally rigid if and only if it is continuously rigid.

Lemma 2.2.13 ([3], Lemma 1)**.** Let G be a graph and (G,p) be a generic d-realisation of G. If (G,p) is continuously rigid, then (G,q) is continuously rigid for all generic d-configurations q of G. If (G,p) is infinitesimally rigid, then (G,q) is infinitesimally rigid for all generic d-configurations q of G.

In particular, Lemma 2.2.13 implies that rigidity becomes a property of the underlying graph if we only consider generic configurations.

Definition 2.2.14. A graph G is said to be *(generically) d-rigid* if (G,p) is infinitesimally rigid for some (equivalently, for all) generic d-realisation (G,p) of G. Otherwise, we say G is *(generically) d-flexible*. We say G is *minimally (generically) d-rigid* if it is generically d-rigid, and there is an edge $e \in E(G)$ such that $G - e$ is generically d-flexible.

Intuitively, a graph on n vertices is generically d-rigid if it has sufficient edges, which are 'adequately' distributed. Formally, this idea is translated in the notion of sparsity count.

Definition 2.2.15. Let m, l be non-negative integers with $m \leq l$. A graph G is (m, l)-*sparse* if, for all subgraphs H of G with $E(H) \neq \emptyset$ and $m|V(H)| \geq l$, $|E(H)| \leq m|V(H)| - l$. The graph G is said to be (m, l)-*tight* if it is (m, l)-sparse and $|E(G)| = m|V(G)| - l$.

Recall that a framework (G, p) (where $p(V)$ spans $\mathbb{R}^d$) is infinitesimally d-rigid if and only if the rank of its rigidity matrix is $d|V(G)| - \binom{d+1}{2}$ [40]. This implies that any generically d-rigid graph has a $\left(d, \binom{d+1}{2}\right)$-tight spanning subgraph. It was first shown by H. Pollaczek-Geiringer [46], and later by G. Laman [38], that, in the case where $d = 2$, such sparsity conditions are not only necessary, but also sufficient, to determine the generic rigidity of G.

Theorem 2.2.16 (Geiringer-Laman Theorem)**.** A graph G is minimally generically 2-rigid if and only if it is $(2, 3)$-tight.

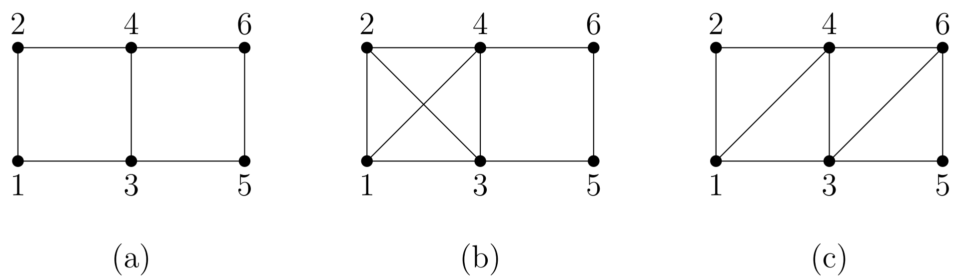


Figure 2.7: Three graphs. (a) is $(2, 3)$-sparse, but not $(2, 3)$-tight, (b) is not $(2, 3)$-sparse, and (c) is $(2, 3)$-tight.

Consider the graphs in Figure 2.7. Graphs (a,b) do not have a $(2, 3)$-tight spanning subgraph. Any generic 2-realisation of both graphs is continuously flexible, and hence infinitesimally flexible: in both cases, pinning the joints corresponding to the vertices 1,2,3,4, and sliding the joints corresponding to 5,6 upwards gives a continuous motion. Graph (c) is minimally generically 2-rigid, and is $(2, 3)$-tight.

Significant efforts have been made to obtain analogous results to the Geiringer-Laman Theorem for $d \geq 3$. An expected result would be that a graph G is minimally generically d-rigid if and only if it is $\left(d, \binom{d+1}{2}\right)$-tight. However, though these sparsity conditions are necessary, there are counterexamples to show they are not sufficient for generic d-rigidity. The most commonly used example is the 'double banana', a $(3, 6)$-tight graph which is not generically 3-rigid (see Figure 2.8). In this example, given a d-configuration p of G, the line through $p(u)$ and $p(v)$ acts as a pivot, around which either 'banana' may rotate [12, 23]. Many similar examples may be constructed in all dimensions [67].

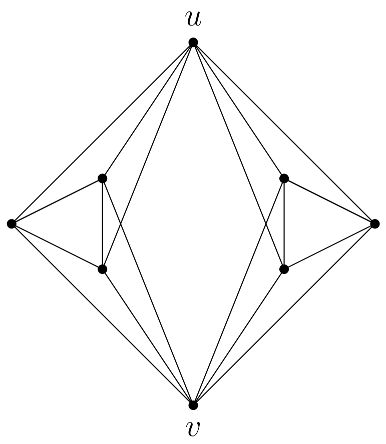


Figure 2.8: The double banana.

This thesis examines symmetric frameworks which are non-generic by definition. However, we consider frameworks which are as generic as possible subject to given symmetry constraints. Both the Geiringer-Laman Theorem itself and the general idea of its proof play a key role in the proofs of the main results of the thesis.

2.3 Symmetry in rigidity

In this section we introduce the main object of interest of this thesis, the symmetric framework. We give a formal definition of symmetric framework and we establish some fundamental concepts. Throughout the rest of the thesis, we assume that all configurations of a graph are injective.

2.3.1 Symmetric frameworks

Definition 2.3.1. A *symmetry group* on $\mathbb{R}^d$ is a subgroup of the orthogonal group $O(\mathbb{R}^d)$.

Definition 2.3.2. Let $\mathcal{G}$ be symmetry group on $\mathbb{R}^d$. We say a point $x \in \mathbb{R}^d$ is *fixed by an element* $g \in \mathcal{G}$ if $gx = x$. The *symmetry element corresponding to* g is the linear subspace F_g of $\mathbb{R}^d$ which consists of all points $x \in \mathbb{R}^d$ fixed by g. We define the space

$$U(x) = \bigcap_{g \in \mathcal{G} : gx = x} F_g.$$

Definition 2.3.3. Let Γ be a group, G be a Γ-symmetric graph and $\tau : \Gamma \to O(\mathbb{R}^d)$ be an injective homomorphism. A framework (G, p) is $\tau(\Gamma)$*-symmetric* if, for all $\gamma \in \Gamma, v \in V(G)$, we have $\tau(\gamma)p(v) = p(\gamma v)$.

Notice that the image $\tau(\Gamma)$ of Γ under τ, when equipped with composition, forms a *symmetry group* on $\mathbb{R}^d$. Recall that an element of $\tau(\Gamma)$ may also be seen as a $d \times d$ orthogonal matrix in $O(\mathbb{R}^d)$. When clear, we interchange the notions of $\tau(\gamma) \in \tau(\Gamma)$ as a map and as a matrix.

Definition 2.3.4. Let Γ be a group, G be a Γ-symmetric graph and $\tau : \Gamma \to O(\mathbb{R}^d)$ be an injective homomorphism. Let (G, p) be a $\tau(\Gamma)$-symmetric framework, and p_u be a joint of (G, p). We say p_u is *fixed by an element* $\tau(\gamma) \in \tau(\Gamma)$ if, as a point in $\mathbb{R}^d$, p_u is fixed by $\tau(\gamma)$. We say that p_u is:

(i) *fixed by* (or *under*) $\tau(\Gamma)$ if it is fixed by all elements of $\tau(\Gamma)$.

(i) *free under* $\tau(\Gamma)$ if the only element in $\tau(\Gamma)$ which fixes p_u is $\mathrm{id}_{\tau(\Gamma)}$.

(iii) *semi-free under* $\tau(\Gamma)$ if it is neither free nor fixed under $\tau(\Gamma)$.

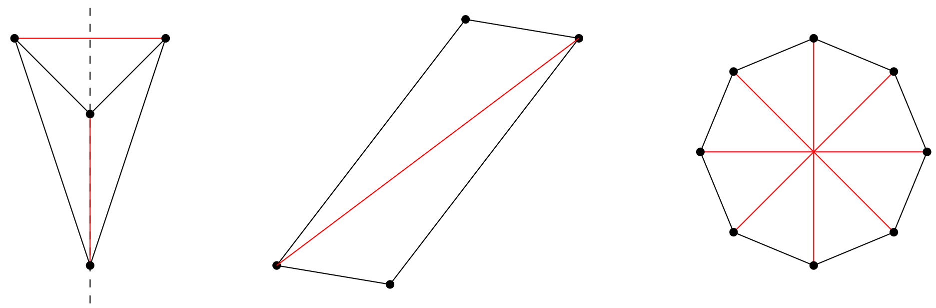
Definition 2.3.5. Let Γ be a group, G be a Γ-symmetric graph and $\tau : \Gamma \to O(\mathbb{R}^d)$ be an injective homomorphism. Let (G, p) be a $\tau(\Gamma)$-symmetric framework, and p_e be a bar of (G, p). We say p_e is *fixed by an element* $\tau(\gamma) \in \tau(\Gamma)$ if $\tau(\gamma)p_e = p_e$. We say that p_e is:

(i) *fixed by* (or *under*) $\tau(\Gamma)$ if it is fixed by all elements of $\tau(\Gamma)$.

(i) *free under* $\tau(\Gamma)$ if the only element in $\tau(\Gamma)$ which fixes p_e is $\mathrm{id}_{\tau(\Gamma)}$.

(iii) *semi-free under* $\tau(\Gamma)$ if it is neither free nor fixed under $\tau(\Gamma)$.

Figure 2.9 shows three examples of $\tau(\Gamma)$-symmetric frameworks which have non-free bars. The $\mathcal{C}_s$-symmetric framework in (a) contains two fixed bars coloured in red. The $\mathcal{C}_2$-symmetric framework in (b) contains a fixed bar coloured in red. The $\mathcal{C}_8$-symmetric framework in (c) contains four semi-free bars coloured in red. For all even $k \geq 4$, there is a $\mathcal{C}_k$-symmetric framework analogous to the framework in (c), which contains $k/2$ semi-free edges.



(a) (b) (c)

Figure 2.9: A $\mathcal{C}_s$-symmetric framework, a $\mathcal{C}_2$-symmetric framework, and a $\mathcal{C}_8$-symmetric framework, all showing non-free bars in red.

Let Γ be a group, G be a Γ-symmetric graph and $\tau : \Gamma \to O(\mathbb{R}^d)$ be an injective homomorphism. Consider a $\tau(\Gamma)$-symmetric realisation (G, p) of G. If a vertex $v \in V(G)$ (respectively, an edge $e \in E(G)$) is not free, then neither is the joint $p(v)$ (respectively, the bar $p(e)$). Moreover, if v (respectively, e) is fixed, then so is $p(v)$ (respectively, $p(e)$). Since we assume throughout the thesis that p is injective, the converse is also true: each fixed vertex corresponds to a fixed joint; each semi-free vertex corresponds to a semi-free joint; each free vertex corresponds to a free joint.

In this thesis, we will derive Geiringer-Laman type results for some symmetric frameworks, and hence we need a symmetry-adapted notion of genericity.

Definition 2.3.6. Let (G, p) be a $\tau(\Gamma)$-symmetric framework for some group Γ and some injective homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$. The d-configuration p is $\tau(\Gamma)$-*generic* if $\operatorname{rank} R(G, p) \geq \operatorname{rank} R(G, q)$ for all $\tau(\Gamma)$-symmetric realisations (G, q) of G. The framework (G, p) is $\tau(\Gamma)$-*generic* if p is $\tau(\Gamma)$-generic.

The set of all $\tau(\Gamma)$-generic configurations of G is a dense, open subset of the set of $\tau(\Gamma)$-symmetric configurations of G.

2.3.2 Forced and incidental symmetric rigidity

In this thesis, we consider all infinitesimal motions of a symmetric framework, including those which break the symmetry of the framework. This area of research is commonly known as incidental rigidity in the rigidity community.

A number of combinatorial characterisations of symmetry-generic infinitesimally rigid frameworks have been obtained. Notably, [56] characterises plane $\tau(\Gamma)$-generic infinitesimally rigid frameworks with no fixed joints, where $\tau(\Gamma)$ is the reflection group, the half-turn group, or the three-fold rotation group. In his Master Thesis, R. Ikeshita (in collaboration with S. Tanigawa) extends these results to the case where $\tau(\Gamma)$ is any cyclic group of odd order less than 1000, with a free group action on the joints [27].

Alternatively, one could restrict the study of a symmetric framework to consider solely its "fully-symmetric" motions, i.e. those motions which maintain the symmetry of the framework. Figure 2.10 shows a fully-symmetric motion of a plane $\mathcal{C}_s$-symmetric framework, where $\mathcal{C}_s$ is the cyclic group of order 2 generated by the reflection along the y-axis. (We will look at this symmetry group more in detail in Section 2.4.) Formally, we give the following definition.

Definition 2.3.7. Let (G, p) be a $\tau(\Gamma)$-symmetric framework, for some group Γ and some injective homomorphism $\tau : \Gamma \to \mathbb{R}^d$. We say an infinitesimal motion m of (G, p) is $\tau(\Gamma)$-*fully symmetric* if, for all $\gamma \in \Gamma$ and all $u \in V(G)$, $m(\gamma u) = \tau(\gamma) m(u)$.

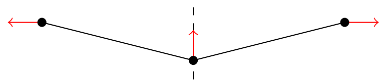


Figure 2.10: Example of a $\mathcal{C}_s$-fully symmetric motion of a $\mathcal{C}_s$-symmetric framework.

The study of fully-symmetric infinitesimal motions is commonly referred to as forced symmetric rigidity [12, 60, 61].

Definition 2.3.8. Let (G, p) be a $\tau(\Gamma)$-symmetric framework for some group Γ and some injective homomorphism $\tau : \Gamma \to \mathbb{R}^d$. We say (G, p) is $\tau(\Gamma)$-*fully symmetrically infinitesimally rigid* (or $\tau(\Gamma)$-*forced rigid*) if all of its $\tau(\Gamma)$-symmetric infinitesimal motions are trivial. Otherwise, we say (G, p) is $\tau(\Gamma)$-*fully symmetrically infinitesimally flexible* (or $\tau(\Gamma)$-*forced flexible*). We say (G, p) is *minimally* $\tau(\Gamma)$-*fully symmetrically infinitesimally rigid* (or *minimally* $\tau(\Gamma)$-*forced rigid*, $\tau(\Gamma)$-*forced isostatic*) if it is $\tau(\Gamma)$-forced rigid and there is an edge $e \in E(G)$ such that $(G - e, p)$ is $\tau(\Gamma)$-forced flexible.

A powerful tool in the area of forced symmetric rigidity is the orbit rigidity matrix (see Section 4.1), which was used to combinatorially characterise various classes of frameworks. In [29], there is a combinatorial characterisation of forced symmetrically rigid frameworks, where the symmetry group is either a cyclic group or a dihedral groups $\mathcal{C}_{kv}$, where k is odd, and it acts freely on the joints of the framework.

2.4 Symmetry groups in the plane

We work on the plane, and we only consider finite groups. Hence, we work with reflection, rotation and dihedral groups. We establish the notation which will be used throughout the thesis, in order to avoid any confusion.

Let $k \geq 2$ be an integer. For notational convenience, we often identify the additive group $\mathbb{Z}_k$ with the multiplicative group $\Gamma = \langle \gamma \rangle$ of order k via the isomorphism defined by $1 \mapsto \gamma$.

2.4.1 Reflection symmetry group

Throughout the thesis, we use the notation $\mathcal{C}_s := \{\mathrm{id}, \sigma\}$, where σ is a reflection whose mirror line passes through the origin. By conjugating by a rotation, we may always assume that the mirror line is the y-axis. Notice that the underlying graph of a $\mathcal{C}_s$-symmetric framework is $\mathbb{Z}_2$-symmetric.

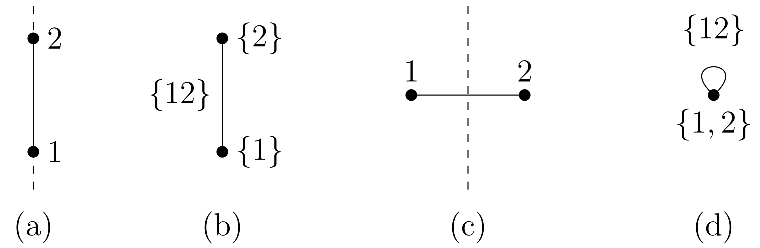


Figure 2.11: Fixed bars of a $\mathcal{C}_s$-symmetric framework, and the $\mathbb{Z}_2$-quotient graphs of their underlying $\mathbb{Z}_2$-symmetric graphs.

Let (G, p) be a $\mathcal{C}_s$-symmetric framework and let $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(G)$ be an injective homomorphism. Choose a joint p_v of (G, p). If p_v lies on the symmetry line $x = 0$, then p_v is fixed by $\mathcal{C}_s$. Otherwise, p_v is free under $\mathcal{C}_s$.

Now, choose a bar p_e of (G, p). If p_e lies on the symmetry line $x = 0$, then $p(e)$ is fixed by $\mathcal{C}_s$ (see Figure 2.11(a)). Moreover, if p_e is normal to the symmetry line $x = 0$ on its mid-point, then p_e is fixed by $\mathcal{C}_s$ (see Figure 2.11(c)). In every other instance, p_e is free under $\mathcal{C}_s$.

Let (G, p) be a $\mathcal{C}_s$-symmetric framework, and m be a fully-symmetric infinitesimal motion of (G, p). We investigate the effect of m on an arbitrary vertex $v \in V(G)$. Recall that the underlying graph G is $\mathbb{Z}_2$-symmetric. We identify $\mathbb{Z}_2$ with a multiplicative cyclic group $\Gamma = \langle \gamma \rangle$ or order 2. By definition, for all $v \in V(G)$

$$m(\gamma v) = \tau(\gamma) m(v) = \mathrm{diag}(-1, 1) m(v).$$

So, for each vertex orbit $\Gamma v = \{v, \gamma v\} \in \Gamma V(G)$, the vectors that m assigns to $v, \gamma v$ share the same y-coordinate, and their x-coordinates share the same size, but differ in sign. In particular, if p_v lies on the symmetry line, then $\gamma v = v$ and so

$m(v) = \mathrm{diag}(-1,1)m(v)$. This implies that $m(v)$ has zero x-coordinate. As one would expect, this means that all fully-symmetric infinitesimal motions of a $\mathcal{C}_s$-symmetric framework do not move the fixed vertices from the symmetry line.

2.4.2 Rotation symmetry groups

Let $k \geq 2$ be an integer. Throughout the thesis, we use the notation $\mathcal{C}_k := \langle C_k \rangle$, where C_k is a counterclockwise rotation about the origin by $2\pi/k$. Notice that the underlying graph of a $\mathcal{C}_k$-symmetric framework is $\mathbb{Z}_k$-symmetric.

Let (G,p) be a $\mathcal{C}_k$-symmetric framework and let $\theta : \mathbb{Z}_k \to \mathrm{Aut}(G)$ be an injective homomorphism. Choose a joint p_v of (G,p). If p_v lies at the origin, then p_v is fixed by $\mathcal{C}_k$. Otherwise, p_v is free under $\mathcal{C}_k$. Since p is injective, v is either fixed or free under $\mathbb{Z}_k$, and $V(G) = V_1(G) \dot\cup V_k(G)$. By injectivity, we may also assume that $|V_k(G)| \leq 1$.

Now, choose a bar p_e of (G,p). If k is even and the mid-point of p_e lies at the origin, then p_e is fixed by the subgroup $\mathcal{C}_2$ of $\mathcal{C}_k$, though it is not fixed by $\mathcal{C}_k$ if $k \geq 4$ (see Figure 2.12(a,c)). Otherwise, p_e is free under $\mathcal{C}_k$. So $E(G) = E_1(G) \dot\cup E_2(G)$. In particular, if $k \geq 3$ then (G,p) has no fixed bars, and if k is odd then all bars of (G,p) are free.

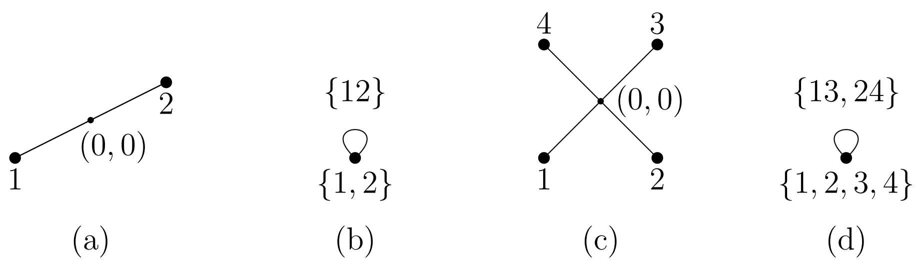


Figure 2.12: (a,b) are, respectively, the fixed bar of a $\mathcal{C}_2$-symmetric framework and the $\mathbb{Z}_2$-quotient graph of its underlying $\mathbb{Z}_2$-symmetric graph. (c,d) are, respectively, the semi-free bars of a $\mathcal{C}_4$-symmetric framework and the $\mathbb{Z}_4$-quotient graph of its underlying $\mathbb{Z}_4$-symmetric graph.

Let (G,p) be a $\mathcal{C}_k$-symmetric framework, and m be a fully-symmetric infinitesimal motion of (G, p). Similarly as with the reflection group, we investigate the effect of m on an arbitrary vertex $v \in V(G)$. The graph G is $\mathbb{Z}_k$-symmetric. We identify $\mathbb{Z}_k$ with a multiplicative cyclic group $\Gamma = \langle \gamma \rangle$ or order k, by the isomorphism which maps the 1 in $\mathbb{Z}_k$ to γ. By definition, for all $v \in V(G)$ and all $0 \leq j \leq k-1$, we have

$$m(\gamma^j v) = \tau(\gamma^j) m(v) = \begin{pmatrix} \cos(2\pi j/k) & -\sin(2\pi j/k) \\ \sin(2\pi j/k) & \cos(2\pi j/k) \end{pmatrix} m(v).$$

So, for each vertex orbit $\Gamma v = \{\gamma^j v : 0 \leq j \leq k-1\} \in \Gamma V(G)$, the infinitesimal motion m assigns a vector t to v and, for $1 \leq i \leq k-1$, it assigns the same vector to $\gamma^j v$, but rotated anti-clockwise by the angle $2\pi j/k$. In particular, if p_v lies at the origin, then $\gamma v = v$ and so

$$m(v) = \begin{pmatrix} \cos(2\pi j/k) & -\sin(2\pi j/k) \\ \sin(2\pi j/k) & \cos(2\pi j/k) \end{pmatrix} m(v).$$

Since $k \geq 2$, this implies that $m(v) = 0$. So, all fully-symmetric infinitesimal motions of a $\mathcal{C}_k$-symmetric framework do not have any effect on the joints at the origin.

2.4.3 Dihedral symmetry groups

Let $k \geq 2$ be an integer. Throughout the thesis, we use the notation $\mathcal{C}_{kv} := \langle \sigma, C_k \rangle$, where σ and C_k are defined as in Subsections 2.4.1 and 2.4.2, respectively. Notice that the underlying graph of a $\mathcal{C}_{kv}$-symmetric framework is $\mathbb{D}_{2k}$-symmetric, where $\mathbb{D}_{2k}$ is the Dihedral group $\big\langle s, r : s^2 = r^k = (sr)^2 = \mathrm{id} \big\rangle$.

Let (G, p) be a $\mathcal{C}_{kv}$-symmetric framework and let $\theta : \mathbb{D}_{2k} \to \mathrm{Aut}(G)$ be an injective homomorphism. Let l be the y-axis. Choose a joint p_v of (G, p). If p_v lies on a line obtained from l by applying a counterclockwise rotation around the origin by $2\pi j/k$ for some $0 \leq j \leq k-1$, then p_v is fixed by a reflection group. Otherwise, p_v is free under $\mathcal{C}_{kv}$. If p_v lies at the origin, then p_v is fixed by $\mathcal{C}_{kv}$. Since p is injective, it follows that $V(G) = V_1(G) \,\dot\cup\, V_2(G) \,\dot\cup\, V_{2k}(G)$.

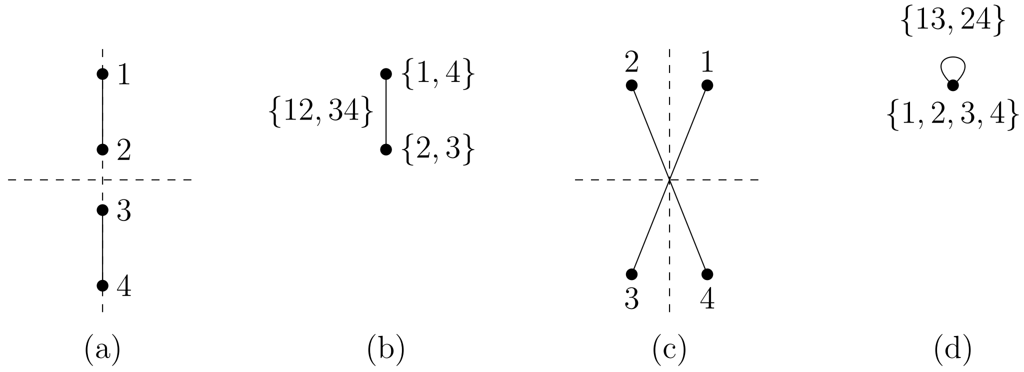


Figure 2.13: Semi-free bars of a $\mathcal{C}_{2v}$-symmetric framework, and the $\mathbb{D}_4$-quotient graphs of their underlying $\mathbb{D}_4$-symmetric graphs.

Now, choose a bar p_e of (G, p). If p_e lies on the anti-clockwise rotation of l around the origin by $2\pi j/k$ for some $0 \leq j \leq k-1$, then p_e is fixed by a reflection group (see Figure 2.13(a)). Moreover, if k is even and the mid-point of p_e lies at the origin, then p_e is fixed by the subgroup $\mathcal{C}_2$ of $\mathcal{C}_{kv}$ (see Figure 2.13(c)). In particular, if k is even, p_e lies on the anti-clockwise rotation of l around the origin by $2\pi j/k$ for some $0 \leq j \leq k-1$, and its mid-point lies at the origin, then p_e is fixed by the subgroup $\mathcal{C}_{2v}$ of $\mathcal{C}_{kv}$. In all other cases p_e is free under $\mathcal{C}_{kv}$. So $E(G) = E_1(G) \dot\cup E_2(G) \dot\cup E_4(G)$.

Let (G, p) be a $\mathcal{C}_{kv}$-symmetric framework, and m be a fully-symmetric infinitesimal motion of (G, p). We investigate the effect of m on a vertex $v \in V(G)$. Recall that $\mathbb{D}_{2k}$ is the Dihedral group $\langle s, r : s^2 = r^k = (sr)^2 = \mathrm{id} \rangle$. G is $\mathbb{D}_{2k}$-symmetric. By definition, for all $v \in V(G)$ and all $0 \leq j \leq k-1$, we have

$$m(r^j v) = \tau(r^j) m(v) = \begin{pmatrix} \cos(2\pi j/k) & -\sin(2\pi j/k) \\ \sin(2\pi j/k) & \cos(2\pi j/k) \end{pmatrix} m(v)$$

and

$$m(sr^j v) = \tau(s)\tau(r^j) m(v) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(2\pi j/k) & -\sin(2\pi j/k) \\ \sin(2\pi j/k) & \cos(2\pi j/k) \end{pmatrix} m(v).$$

So, for each vertex orbit $\Gamma v = \{r^j v, sr^j v : 0 \leq j \leq k-1\} \in \Gamma V(G)$, the infinitesimal motion m assigns a vector t to v and, for $1 \leq i \leq k-1$, it assigns the same vector to

$r^j v$, but rotated anti-clockwise by the angle $2\pi j/k$, and it assigns the same vector to $sr^j v$, but rotated anti-clockwise by the angle $2\pi j/k$ and then reflected along the y-axis. In particular, if p_v lies at the origin, then $rv = v$ and so

$$m(v) = \begin{pmatrix} \cos(2\pi j/k) & -\sin(2\pi j/k) \\ \sin(2\pi j/k) & \cos(2\pi j/k) \end{pmatrix} m(v).$$

Since $k \geq 2$, this implies that $m(v) = 0$. If, for some $0 \leq j \leq k-1$, p_v lies on the line l_j obtained by rotating l anti-clockwise around the origin by an angle $2\pi j/k$, then

$$m(v) = \tau(s)\tau(r^j)m(v) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(2\pi j/k) & -\sin(2\pi j/k) \\ \sin(2\pi j/k) & \cos(2\pi j/k) \end{pmatrix} m(v).$$

So, m assigns a vector t to v which is parallel to the symmetry line l_j.

Chapter 3

Generalised gain graphs

Let Γ be a non-trivial abstract group, $\tau : \Gamma \to O(\mathbb{R}^2)$ be an injective homomorphism and $(\tilde{G}, \tilde{p})$ be a $\tau(\Gamma)$-symmetric realisation of some Γ-symmetric graph $\tilde{G} = (\tilde{V}, \tilde{E})$. Consider a joint $\tilde{p}(u)$ of $(\tilde{G}, \tilde{p})$. By Definition 2.3.3, we know the position $\tilde{p}(\gamma u)$ of γu for all $\gamma \in \Gamma$. Similarly, given a bar $\tilde{p}(e)$ of $(\tilde{G}, \tilde{p})$, we may deduce $\tilde{p}(\gamma e)$ for all $\gamma \in \Gamma$. This suggests that a significant amount of information contained in $(\tilde{G}, \tilde{p})$ is redundant. A convenient way to discard this information is through the use of a combinatorial tool known as gain graph.

A Γ-gain graph is a directed multigraph whose edges are labelled with elements of the group Γ. This notion was first introduced and developed by J. Gross under the title of 'voltage graphs' as a tool to study the embeddings of graphs in a surface [24]. Gain graphs were substantially developed by T. Zaslavsky, due to his interest in biased graphs, special cases of gain graphs [75]. Whereas J. Gross was interested in how gain graphs could aid topological studies, T. Zaslavsky took a matroidal approach and developed the study of balanced gain graphs.

There is a bijective correspondence between Γ-symmetric graphs whose vertex set is free under Γ and Γ-gain graphs. Therefore, over time, gain graphs became an important tool in the study of infinitesimally rigid symmetric frameworks, with the assumption that the symmetry group acts freely on the joints of a given framework (see, e.g., [5, 8, 15, 27, 28, 34, 36, 39, 42, 57, 56, 60, 64]). As a result, the theory of gain graphs as a rigidity tool has developed greatly.

Currently, many papers on forced and incidental symmetric rigidity treat the gain graph as a central combinatorial object. A large amount of results in the area characterise infinitesimally rigid symmetric frameworks based on a sparsity count applied to gain graphs, which is analogous to (but more complex than) that given in Definition 2.2.15.

If we allow Γ-symmetric graphs to have vertices which are not free under Γ, then the bijective correspondence between Γ-gain graphs and Γ-symmetric graphs fails. However, by generalising the notion of Γ-gain graph, we are able to re-establish such a correspondence. Moreover, our generalised definition of Γ-gain graph can be simplified in the case where each vertex is either free or fixed under Γ (for instance, if Γ is a cyclic group). This allows for a better understanding of the gain graph structure.

We start this chapter by introducing the usual notion of gain graph, as well as some linked notions, and by presenting some important results (see Section 3.1). We then generalise the notion of gain graph, surrounding notions and crucial results, in order to study Γ-symmetric graphs whose vertex sets are not necessarily free under the group Γ. Since such concepts simplify when Γ is cyclic, we start by considering cyclic groups in Section 3.2. In Section 3.3 we consider all groups Γ.

3.1 The usual notion of gain graph

Throughout this section we will assume that, for any group Γ and any Γ-symmetric graph $\tilde{G} = (\tilde{V}, \tilde{E})$, $\tilde{v}$ is free under Γ for all $\tilde{v} \in \tilde{V}$.

Definition 3.1.1. Let Γ be an abstract group. A Γ-*gain graph* is a pair (G, ψ), where G is a directed multigraph and $\psi : E(G) \to \Gamma$ is a map such that:

1. If $e, f \in E(G)$ are parallel and have the same orientation, then $\psi(e) \neq \psi(f)$. If they are parallel and have the opposite orientation, then $\psi(e) \neq \psi(f)^{-1}$.

2. If $e \in E(G)$ is a loop, then $\psi(e) \neq \mathrm{id}$.

The map ψ is known as *gain map* or *labelling* of G (equivalently, of (G, ψ)). We say an edge $e \in E(G)$ has *gain* or *label* $\psi(e)$.

Given a group Γ, let (G, ψ) be a Γ-gain graph and suppose an edge $e = (u, v)$ in $E(G)$ has gain $\psi(e) = g$. We may redirect e and change its label to g^{-1}.

Let $\tilde{G}$ be a Γ-symmetric graph for some group Γ, and let G be the Γ-quotient of $\tilde{G}$, with some fixed vertex orbit representatives. We may direct the edges of G and label them with elements of Γ in order to obtain a Γ-gain graph. The process described in [56] gives a unique way of doing so, up to redirecting edges and relabelling them with the inverse of their original gain. (A generalised version of this process will be described in Section 3.2, and again in Section 3.3.) This establishes a bijective correspondence between Γ-symmetric graphs and Γ-gain graphs, up to the choice of the edge orientations and of the vertex orbit representatives. The choice of a vertex orbit representative $u^\star$ affects the labels of the edges incident to the vertex $u := \Gamma u^\star \in V(G)$. However, it is important to note that, once the orbit representative $u^\star$ is chosen, the process in [56] gives a unique labelling of each edge incident to u. (This is not the case for our generalised version of the process.) In Subsections 3.2.1 and 3.3.1 we will see that the choice of vertex orbit representatives is closely related to well-known gain graph operations known as switchings.

We call the Γ-gain graph (G, ψ) obtained from $\tilde{G}$ by applying the process described in [56] the Γ-*gain graph of* $\tilde{G}$ and, with a slight abuse of notation, we call $\tilde{G}$ the Γ-*lifting of* (G, ψ). If two Γ-gain graphs $(G_1, \psi_1), (G_2, \psi_2)$ are obtained from the same Γ-symmetric graph, we say (G_1, ψ_1) and (G_2, ψ_2) are *equivalent*.

Notice a connected graph with non-empty edge set E can be fully described by E. Since the gain map ψ is defined on the edges of a gain graph, this allows us to define the gain of a walk and of a connected graph with non-empty edge set.

Definition 3.1.2. Let Γ be a group and (G, ψ) be a Γ-gain graph. Let W be a walk in G of the form $W = e_1, \dots, e_t$, where e_i has end-vertices $v_i, v_{i+1} \in V(G)$ for all $1 \leq i \leq t$. We say the *gain of* W *under* ψ is $\psi(W) = \prod_{i=1}^t \psi(e_i)^{\text{sign}(e_i)}$, where $\text{sign}(e_i) = 1$ if e_i is directed from v_i to v_{i+1}, and $\text{sign}(e_i) = -1$ otherwise.

Given a connected subgraph H of G with $E(H) \neq \emptyset$ and a vertex $v \in V(H)$, the *gain of H under ψ with base vertex v* (equivalently, the *gain of $E(H)$ under ψ with base vertex v*) is the group generated by

$$\{\psi(W) : W \text{ is a closed walk in } H \text{ starting at } v\}.$$

We denote such a group by $\langle E(H)\rangle_{v,\psi}$ (or $\langle H\rangle_{v,\psi}$).

The following useful results allow us to drop v, ψ from the notation of $\langle H\rangle_{v,\psi}$ whenever the gain graph (G, ψ) is clear from the context and Γ is abelian.

Proposition 3.1.3 ([29], Proposition 2.1)**.** Let Γ be a group and (G, ψ) be a Γ-gain graph. Given a connected subgraph H of G with $E(H) \neq \emptyset$ and some $u, v \in V(H)$, $\langle H\rangle_{u,\psi}$ and $\langle H\rangle_{v,\psi}$ are conjugate.

Proposition 3.1.4 ([29], Proposition 2.2)**.** Let Γ be a group and $(G, \psi), (G, \psi')$ be equivalent Γ-gain graphs. Given a connected subgraph H of G with $E(H) \neq \emptyset$ and some $v \in V(H)$, $\langle H\rangle_{v,\psi}$ and $\langle H\rangle_{v,\psi'}$ are conjugate.

An important sub-class of gain graphs is the class of balanced gain graphs, which was studied extensively by T. Zaslavsky [75].

Definition 3.1.5. Let Γ be a group and (G, ψ) be a Γ-gain graph. We say a connected subgraph H of G (equivalently, $E(H), (H, \psi|_{E(H)})$) is *balanced under ψ* if every closed walk in H has gain id under ψ. Otherwise, we say H (equivalently, $E(H), (H, \psi|_{E(H)})$) is *unbalanced* under ψ. We say a (not necessarily connected) subgraph H of G is *balanced* if all of its connected components are balanced. Otherwise, we say it is *unbalanced.*

As mentioned in the introductory paragraph of this chapter, $\tau(\Gamma)$-generic infinitesimally rigid frameworks may be combinatorially characterised by a sparsity count applied to the Γ-gain graphs of the underlying Γ-symmetric graphs. However, the notions of sparsity counts for gain graphs are slightly more refined than those given in Definition 2.2.15, as they take into account balanced subgraphs, as well as other types of subgraphs in certain versions.

Definition 3.1.6. Let Γ be a group, let (G, ψ) be a Γ-gain graph, and let $0 \leq l \leq 3$ be an integer. We say (G, ψ) is $(2, 3, l)$-*gain sparse* if

- for all $H \subseteq G$ with $E(H) \neq \emptyset$, H is $(2, l)$-sparse.
- for all balanced $H \subseteq G$ with non-empty edge set, H is $(2, 3)$-sparse.

We say (G, ψ) is $(2, 3, l)$-*gain tight* if it is $(2, 3, l)$-gain sparse and $(2, l)$-tight.

Let $k \geq 2$ be an integer and $\tilde{G}$ be a $\mathbb{Z}_k$-symmetric graph. For an injective homomorphism $\tau : \mathbb{Z}_k \to O(\mathbb{R}^2)$, let $(\tilde{G}, \tilde{p})$ be a $\tau(\mathbb{Z}_k)$-generic realisation of $\tilde{G}$. In [39], J. Malestein and L. Theran show that $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic if and only if the $\mathbb{Z}_k$-gain graph (G, ψ) of $\tilde{G}$ is $(2, 3, 1)$-gain tight. This was done independently for the case of rotational-symmetry in [29] by T. Jordán, V. Kaszanitsky and S. Tanigawa. In [56], B. Schulze and S. Tanigawa extend these results to consider incidental infinitesimal rigidity, in the case where $k = 2, 3$.

Theorem 3.1.7 ([56], Theorem 6.4)**.** Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_s$-generic framework, and let (G, ψ) be the $\mathbb{Z}_2$-gain graph of $\tilde{G}$. $(\tilde{G}, \tilde{p})$ is infinitesimally rigid if and only if (G, ψ) has a $(2, 3, 1)$-gain tight spanning subgraph and a $(2, 3, 2)$-gain tight spanning subgraph.

Theorem 3.1.8 ([56], Theorem 6.9)**.** Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_2$-generic framework, and let (G, ψ) be the $\mathbb{Z}_2$-gain graph of $\tilde{G}$. $(\tilde{G}, \tilde{p})$ is infinitesimally rigid if and only if (G, ψ) has a $(2, 3, 1)$-gain tight spanning subgraph and a $(2, 3, 2)$-gain tight spanning subgraph.

Theorem 3.1.9 ([56], Theorem 6.11)**.** Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_3$-generic framework, and let (G, ψ) be the $\mathbb{Z}_3$-gain graph of $\tilde{G}$. $(\tilde{G}, \tilde{p})$ is infinitesimally rigid if and only if (G, ψ) has a $(2, 3, 1)$-gain tight spanning subgraph.

If $k \geq 4$, then (G, ψ) must have a $(2, 3, 0)$-gain tight spanning subgraph and a $(2, 3, 1)$-tight spanning subgraph in order for $(\tilde{G}, \tilde{p})$ to be infinitesimally rigid. However, such conditions are not sufficient, and we need a notion of sparsity which is more refined than that given in Definition 3.1.6. In [27], R. Ikeshita and S. Tanigawa combinatorially characterise $\mathcal{C}_k$-generic infinitesimally rigid frameworks, under the assumption that $4 < k < 1000$ is odd. These results are extended by K. Clinch and S. Tanigawa in [8] to consider the cases where $k = 4, 6$. In Section 3.2 we will give the refined count described in [8] and [27], with a generalisation which allows a joint to lie at the centre of rotation.

In [29], T. Jordán, V. Kaszanitzky and S. Tanigawa combinatorially characterise forced infinitesimally rigid $\mathcal{C}_{kv}$-generic frameworks, where $k \geq 3$ is odd. The sparsity conditions described in [29] are also more refined than that given in Definition 3.1.6. We will describe such conditions in Section 3.3, with a generalisation which allows joints to lie on the reflection lines of $\mathcal{C}_{kv}$ (including the origin).

3.2 Gain graphs for cyclic groups

Throughout this section we let Γ be a cyclic group of finite order. Recall that, in this setting, for all Γ-symmetric graphs $\tilde{G}$, we assume that $V(\tilde{G}) = V_1(\tilde{G}) \dot\cup V_{|\Gamma|}(\tilde{G})$.

Let $\tilde{G}$ be a Γ-symmetric graph, and $G = (V, E)$ be the quotient of $\tilde{G}$ with respect to Γ. We orient the edges of G and assign them a group label. We do so in the following way.

For each vertex orbit $v := \Gamma v^\star$ we fix a representative vertex $v^\star$. We also fix an orientation on the edges of the quotient graph G. For each directed edge $e = (u, v)$ in the directed quotient graph, let $u = \Gamma u^\star$ and $v = \Gamma v^\star$ have vertex orbit representatives $u^\star$ and $v^\star$ respectively. We assign the following labelling (or "gain") to e:

- If $u^\star, v^\star$ are free under Γ, then there exists a unique $\gamma \in \Gamma$ such that $u^\star v^\star_\gamma \in uv$, where $v^\star_\gamma$ denotes $\gamma v^\star$. We let γ be the *gain on e*.

- If at least one of $u^\star, v^\star$ is fixed under Γ, say $u^\star$ is fixed under Γ, then uv is the set $\{u^\star v^\star_\gamma |\, \gamma \in \Gamma, v^\star_\gamma := \gamma v^\star\}$. We define the *gain on e* to be any $\gamma \in \Gamma$.

In each of the cases above, we could re-direct e from v to u and re-label it with the group inverse of the original label chosen. The process gives rise to a class of group-labelled, directed multigraphs, which are a generalisation of the Γ-gain graphs given in Section 3.1.

Definition 3.2.1. Let Γ be a cyclic group. A Γ-*gain graph* is a pair (G, ψ), where G is a directed multigraph and $\psi : E(G) \to \Gamma$ is a map that assigns a label to each edge such that, for some partition $V(G) = V_1(G) \dot{\cup} V_{|\Gamma|}(G)$, where no vertex in $V_{|\Gamma|}(G)$ has a loop or is incident to parallel edges, the following conditions are satisfied:

1. If $e, f \in E(G)$ are parallel and have the same orientation, then $\psi(e) \neq \psi(f)$. If they are parallel and have opposite orientations, then $\psi(e) \neq \psi(f)^{-1}$.
2. If $e \in E(G)$ is a loop, then $\psi(e) \neq \mathrm{id}$.

We call ψ the *gain map* of (G, ψ). For each $e \in E(G)$, we call $\psi(e)$ the *gain* or *label* of e. The elements of $V_1(G)$ and $V_{|\Gamma|}(G)$ are called the *free* and *fixed vertices of* (G, ψ), respectively.

If $V(G) = V_1(G)$, then Definitions 3.1.1 and 3.2.1 coincide. Therefore, given a cyclic group Γ, we will henceforth use the terminology Γ-gain graph and the notation (G, ψ) to refer to the combinatorial object given in Definition 3.2.10.

When drawing a Γ-gain graph (G, ψ) it is important to distinguish between the fixed and free vertices of (G, ψ). We will be doing so by representing the elements of $V_1(G)$ and $V_{|\Gamma|}(G)$ by circles and squares, respectively. In Figure 3.1 we consider a cyclic group $\Gamma = \langle \gamma \rangle \simeq \mathbb{Z}_6$ of order 6, and we give an example of a Γ-symmetric graph $\tilde{G}$ with a Γ-gain graph obtained from $\tilde{G}$ by applying the process described above.

For a cyclic group $\Gamma = \langle \gamma \rangle$ of finite order k, let $\tilde{G}$ be a Γ-symmetric graph and (G, ψ) be a Γ-gain graph obtained from $\tilde{G}$ by applying the process described above. Take an edge $e = (u, v) \in E(G)$, where $u^\star, v^\star$ are the vertex representatives of

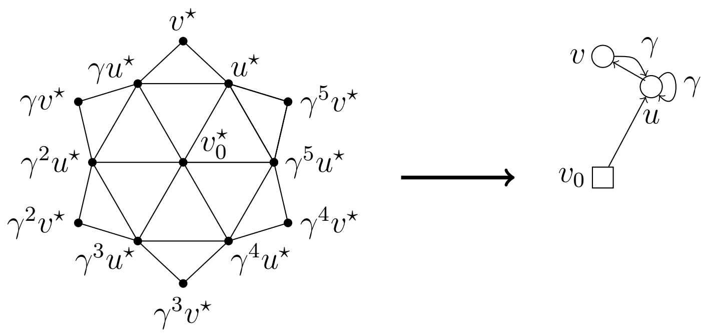


Figure 3.1: A Γ-symmetric graph $\tilde{G}$ and a Γ-gain graph constructed from $\tilde{G}$, where $\Gamma = \langle\gamma\rangle$ is a cyclic group of order 6. Here, the unlabelled edges have gain id.

u, v, respectively. Suppose that $u^\star v^\star_{\psi(e)}$ (where $v^\star_{\psi(e)} = \psi(e)v^\star$) is not free under Γ. Recall from Subsections 2.4.1 and 2.4.2 that, since we are interested in geometric realisations of $\tilde{G}$, we may always assume that one of the following holds:

(i) $\Gamma = \{\mathrm{id}, \gamma\}$, and $u^\star v^\star_{\psi(e)}$ is fixed (see Figure 3.2(a,b)).

(ii) $k \geq 4$ is even, $u^\star v^\star_{\psi(e)}$ is semi-free, and $u^\star, v^\star_{\psi(e)}$ are free (see Figure 3.2(c)).

Note that, in both cases, $\psi(e) = \gamma^{k/2}$.

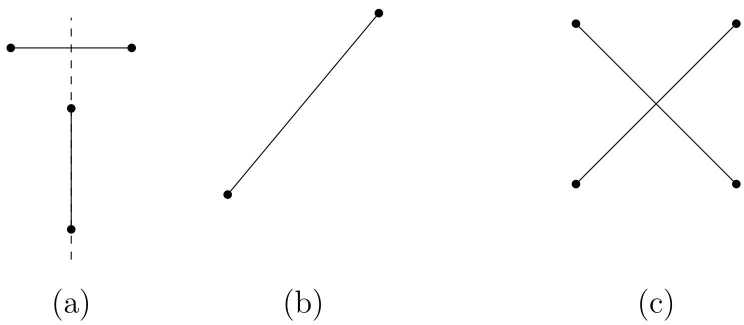


Figure 3.2: Non-free edges of Γ-symmetric graph, where Γ is a cyclic group. (a) and (b) show fixed bars of $\mathbb{Z}_2$-symmetric graphs, and (c) shows semi-free edges of a $\mathbb{Z}_4$-symmetric graph.

Therefore, we will assume throughout the rest of the thesis, that one of the following holds: $k = 2$ and $u^\star, v^\star$ are both fixed by Γ; k is even (possibly $k = 2$),

$u^\star = v^\star$ and $\psi(e) = \gamma^{k/2}$. We define $E_2(G) := \{e = (u,v) \in E(G) : u, v \in V_k(G)$ or $u = v, \psi(e) = \gamma^{k/2}\}$ and $E_1(G) := E(G) \setminus E_2(G)$. An edge $e \in E(G)$ is said to be *free* if $e \in E_1(G)$. Otherwise, we say it is *fixed* when $k = 2$, and we say it is *semi-free* when $k \geq 4$ is even.

Conversely, given a Γ-gain graph (G, ψ), we may construct a Γ-symmetric graph $\tilde{G}$. We do so in the following way. For each $v \in V_{|\Gamma|}(G)$, $V_{|\Gamma|}(\tilde{G})$ contains v, and for each $v \in V_1(G)$, $V_1(\tilde{G})$ contains the vertices in $\{v_\gamma : \gamma \in \Gamma\}$. Given an edge $e = (u, v) \in E(G)$, $E(\tilde{G})$ contains the following edges:

- If $u, v \in V_{|\Gamma|}(G)$, then $E(\tilde{G})$ contains uv.
- If $u \in V_{|\Gamma|}(G), v \in V_1(G)$, then $E(\tilde{G})$ contains the edges uv_γ for all $\gamma \in \Gamma$.
- If $u, v \in V_1(G)$, $E(\tilde{G})$ contains $u_\gamma v_{\gamma\psi(e)}$ for all $\gamma \in \Gamma$.

The graph obtained by applying this process is simple and it is unique up to isomorphism. Moreover, given any Γ-symmetric graph $\tilde{G}$, if we construct a Γ-gain graph (G, ψ) from $\tilde{G}$ using the process described at the beginning of the section, and we then construct a Γ-symmetric graph $\tilde{H}$ from (G, ψ) by applying the process just described, then $\tilde{G} \simeq \tilde{H}$. Hence, with a slight abuse of terminology, we call (G, ψ) the *(quotient)* Γ*-gain graph* of $\tilde{G}$ and we call $\tilde{G} \simeq \tilde{H}$ the Γ*-covering graph* (or Γ*-lifting*) of (G, ψ). Figure 3.3 gives an example of the construction applied to a Γ-gain graph, where $\Gamma = \langle\gamma\rangle$ is a cyclic group of order 2.

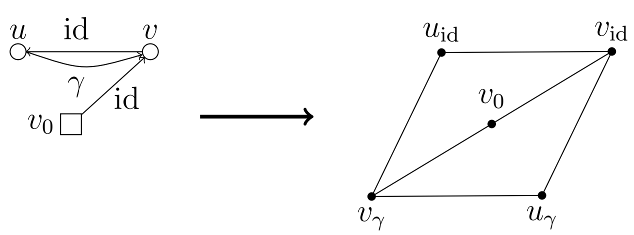


Figure 3.3: A Γ-gain graph and its lifting, where $\Gamma = \{\text{id}, \gamma\}$.

3.2.1 Switchings

Let Γ be a cyclic group, $\tilde{G}$ be a Γ-symmetric graph and (G, ψ) be the quotient Γ-gain graph of $\tilde{G}$. Consider $v \in V(G)$, and let $v^\star \in V(\tilde{G})$ be the vertex orbit representative of v chosen when constructing (G, ψ). Given any $\gamma \in \Gamma$, we could have instead chosen $\gamma v^\star$ as a vertex orbit representative of v when constructing (G, ψ). Changing the choice of vertex orbit representative from $v^\star$ to $\gamma v^\star$ when constructing (G, ψ) is equivalent to applying an operation called switching at v induced by γ. We give a formal definition of switching at a vertex.

Definition 3.2.2. For a cyclic group Γ, let (G, ψ) be a Γ-gain graph. Let $v \in V(G)$. A *(type I) switching at v induced by* $\gamma \in \Gamma$ is an operation which generates a new gain map ψ' by letting

$$\psi'(e) = \begin{cases} \gamma\psi(e)\gamma^{-1} & \text{if } e \text{ is a loop incident to } v \\ \gamma\psi(e) & \text{if } e \text{ is a non-loop edge directed from } v \\ \psi(e)\gamma^{-1} & \text{if } e \text{ is a non-loop edge directed to } v \\ \psi(e) & \text{otherwise} \end{cases}$$

for all $e \in E(G)$.

Consider again the Γ-symmetric graph $\tilde{G}$ and its quotient Γ-gain graph (G, ψ), and fix the vertex orbit representatives chosen when constructing (G, ψ). Suppose that there is an edge $e = (u, v) \in E(G)$ such that $u \in V_{|\Gamma|}(G)$, and let $g := \psi(e)$. For all $\gamma \in \Gamma$ we could have instead chosen $\psi(e)$ to be γ when constructing (G, ψ). Changing the choice of $\psi(e)$ when constructing (G, ψ) is equivalent to applying an operation called switching at e induced by the element γg^{-1}. We now give a formal definition of switching at an edge.

Definition 3.2.3. For a cyclic group Γ, let (G, ψ) be a Γ-gain graph. Let $e = (u, v)$ be an edge in $E(G)$ such that $u \in V_{|\Gamma|}(G)$. A *(type II) switching at e induced by the element* $\gamma \in \Gamma$ is an operation which generates a new gain map ψ' by letting $\psi'(e) = \gamma\psi(e)$ and $\psi'(f) = \psi(f)$ for all edges $f \neq e$ of G.

Definition 3.2.4. Given a cyclic group Γ and a Γ-gain graph (G, ψ), we define a *switching of* (G, ψ) to be any type I switching at a vertex $v \in V(G)$ or type II switching at an edge $e \in E(G)$.

Notice that applying a series of switchings to a Γ-gain graph defines an equivalence relation. We give a formal definition of equivalent Γ-gain graphs.

Definition 3.2.5. For a cyclic group Γ, let $(G, \psi_1), (G, \psi_2)$ be two Γ-gain graphs. We say ψ_1 and ψ_2 are *equivalent/type I equivalent/type II equivalent* if one can be obtained from the other by applying a sequence of switchings/type I switchings/type II switchings. We say (G, ψ_1) and (G, ψ_2) are *equivalent/type I equivalent/type II equivalent* if ψ_1 and ψ_2 are equivalent/type I equivalent/type II equivalent.

Let Γ be a finite cyclic group. Take a tree T in a Γ-gain graph (G, ψ), and choose a root v of $E(T)$. Let $e_1, \ldots, e_t \in T$ be the edges incident to v in T, and assume that each such edge is directed from v. For $1 \leq i \leq t$, let $e_i = (v, v_i)$ and $g_i = \psi(e_i)$. Then, for each $1 \leq i \leq t$, we may apply a switching at v_i induced by g_i in order to obtain a Γ-gain graph (G, ψ') type I equivalent to (G, ψ) such that $\psi'(e_i) = \text{id}$ for all $1 \leq i \leq t$. We may then apply similar switching operations to all neighbours of v_i in T for $1 \leq i \leq t$, so that all edges incident to a vertex in $\{v, v_1, \ldots, v_t\}$ have identity gain. Applying this process recursively, we obtain a gain graph type I equivalent to (G, ψ) which assigns the identity gain to all edges in T. Taking a forest F in (G, ψ), we may apply this process to each connected component of F. This was the argument used in [29] to prove the following result.

Proposition 3.2.6 ([29], Proposition 2.2)**.** Let Γ be a cyclic group and (G, ψ) be a Γ-gain graph with $V(G) = V_1(G)$. For any forest F in G, there is a Γ-gain graph (G, ψ') type I equivalent to (G, ψ) such that $\psi'(e) = \text{id}$ for all $e \in E(F)$.

3.2.2 Balanced subgraphs

Recall the notion of balancedness given in Definition 3.1.5. In this section, we give a slight generalisation of this definition, and we explain the intuition behind the concept of balanced gain graph. We start by generalising Definition 3.1.2.

Definition 3.2.7. Let Γ be a cyclic group and (G,ψ) be a Γ-gain graph. Let W be a walk in G of the form $W = e_1,\dots,e_t$, where e_i has end-vertices $v_i, v_{i+1} \in V(G)$ for all $1 \le i \le t$. We say the *gain of* W *under* ψ is $\psi(W) = \prod_{i=1}^{t}\psi(e_i)^{\mathrm{sign}(e_i)}$, where $\mathrm{sign}(e_i) = 1$ if e_i is directed from v_i to v_{i+1}, and $\mathrm{sign}(e_i) = -1$ otherwise.

Given a connected subgraph H of G with $E(H) \neq \emptyset$ and a vertex $v \in V_1(H)$, the *gain of* H *under* ψ *with base vertex* v (equivalently, the *gain of* $E(H)$ *under* ψ *with base vertex* v) is the group generated by

$$\{\psi(W) : W \text{ is a closed walk starting at } v \text{ and not containing fixed vertices}\}.$$

We denote this group by $\langle E(H)\rangle_{v,\psi}$ (or $\langle H\rangle_{v,\psi}$).

Notice that the definition of $\langle H\rangle_{v,\psi}$ does not take into account walks which contain fixed vertices of H. This is because, by applying a type II switching, a closed walk W in H which contains a fixed vertex can have any gain. For example, the Γ-symmetric graph in Figure 3.4(a) can have the two different type II equivalent Γ-gain graphs given in Figure 3.4(b,c). (Here, $\Gamma = \langle\gamma\rangle \simeq \mathbb{Z}_2, v_4 = \gamma v_1^\star, v_5 = \gamma v_2^\star$, and $v_6 = \gamma v_3^\star$.) In (b), the cycle $(v_0,v_1),(v_1,v_2),(v_0,v_2)$ has gain id, whereas in (c) it has gain γ.

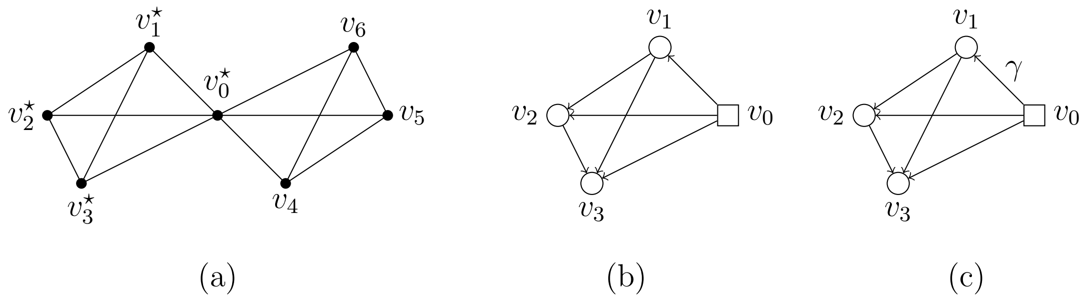


Figure 3.4: Two different Γ-gain graphs (b,c) of the same Γ-symmetric graph, where $\Gamma = \langle\gamma\rangle \simeq \mathbb{Z}_2$. Here, the unlabelled edges have identity gain.

The proofs given in [29] to prove Propositions 3.1.3 and 3.1.4 for the case where $V(G) = V_1(G)$ extends to our setting, and show the following two results.

Proposition 3.2.8. For a cyclic group Γ, let (G, ψ) be a Γ-gain graph. For any connected subgraph H of G such that $H - V_{|\Gamma|}(H)$ is also connected, and for all vertices $u, v \in V_1(H)$, we have $\langle H \rangle_{u,\psi} = \langle H \rangle_{v,\psi}$.

Proof. Since $H - V_{|\Gamma|}(H)$ is connected, there exists a $u - v$ path P in H which does not contain fixed vertices. Given a closed walk W in H starting at u and not containing fixed vertices, $P^{-1} \circ W \circ P$ is a closed walk in H starting at v and not containing fixed vertices. Hence, $\psi(P)\psi(W)\psi(P)^{-1} \in \langle H \rangle_{v,\psi}$. The result follows from the fact that Γ is abelian. $\square$

Proposition 3.2.9. Let Γ be a cyclic group and $(G, \psi), (G, \psi')$ be equivalent Γ-gain graphs. For all connected subgraphs H of G and all $v \in V_1(H)$, we have $\langle H \rangle_{v,\psi} = \langle H \rangle_{v,\psi'}$.

Proof. It suffices to show that the result holds if (G, ψ') is obtained from (G, ψ) by applying a switching operation at a vertex $v \in V(G)$ with some gain $\gamma \in \Gamma$, since type II switchings only effect edges incident to a fixed vertex. Hence, assume that this is the case. Notice that $\psi'(e)\psi'(f) = \psi(e)\psi(f)$ for all pairs of edges e, f, where e is directed to v and f is directed from v. Notice also that $\psi'(e) = \psi(e)$ for all edges e not incident to v. Then, for all closed walks W starting at v, we have $\psi'(W) = \gamma\psi(W)\gamma^{-1}$. Moreover, for all closed walks W not starting at v, we have $\psi'(W) = \psi(W)$. The result follows from the fact that Γ is abelian. $\square$

Proposition 3.2.8 need not hold if H is connected but $H - V_{|\Gamma|}(H)$ is disconnected: let $\Gamma = \langle \gamma \rangle$ be a (non-trivial) group and consider the Γ-gain graph (G, ψ) in Figure 3.5. Here $\langle G \rangle_{u,\psi} = \Gamma$, whereas $\langle G \rangle_{v,\psi} = \{\mathrm{id}\}$.

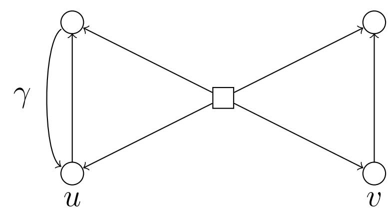


Figure 3.5: Connected Γ-gain graph (G, ψ) such that $G - V_{|\Gamma|}(G)$ is disconnected. Here, all unlabelled edges have identity gain.

However, provided that $H - V_{|\Gamma|}(H)$ is connected Propositions 3.2.8 and 3.2.9 allow us to drop v, ψ from the notation of $\langle H \rangle_{v,\psi}$ whenever the gain graph (G, ψ) is clear from the context. If $H - V_{|\Gamma|}(H)$ is not connected (but H is), and (G, ψ) is clear from the context, we let $\langle H \rangle$ (and $\langle E(H) \rangle$) denote the group generated by $\langle H_1 \rangle, \ldots, \langle H_t \rangle$, where $H_1, \ldots, H_t$ are the connected components of $H - V_{|\Gamma|}(H)$.

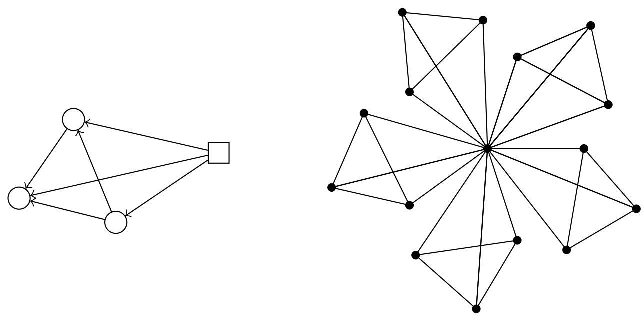
Definition 3.2.10. Let Γ be a group and (G, ψ) be a Γ-gain graph. We say a connected subgraph H of G (equivalently, $E(H), (H, \psi|_{E(H)})$) is *balanced (under ψ)* if all closed walks in H only containing free vertices have identity gain under ψ. Otherwise, we say H (equivalently, $E(H), (H, \psi|_{E(H)})$) is *unbalanced (under ψ)*. We say a disconnected graph H is *balanced* if all of its connected components are balanced.

Notice that H is balanced under ψ if and only if it is balanced under ψ' for all ψ' which are equivalent to ψ. Hence, we sometimes drop ψ and simply say that H (equivalently, $E(H), (H, \psi|_{E(H)})$) is balanced.

The usual notion of balancedness forces all closed walks in H to have identity gain, whereas here we allow closed walks which contain fixed vertices to have any gain. For instance, both Γ-gain graphs given in Figure 3.4 are balanced, even though the Γ-gain graph in Figure 3.4(c) contains the cycle $(v_0, v_1), (v_1, v_2), (v_0, v_2)$ of non-identity gain.

Given a group Γ, let (G, ψ) be a Γ-gain graph with Γ-lifting $\tilde{G}$. Let $\tau : \Gamma \to O(\mathbb{R}^2)$ be an injective homomorphism and $(\tilde{G}, \tilde{p})$ be a $\tau(\Gamma)$-generic realisation of $\tilde{G}$. Given a balanced subgraph H of G which is not $(2,3)$-sparse, $(\tilde{H}, \tilde{p}|_{V(\tilde{H})})$ has a non-zero equilibrium stress, regardless of the size of $V_{|\Gamma|}(G)$. (Here, $\tilde{H}$ denotes the Γ-lifting of H.) In fact, if $|V_{|\Gamma|}(G)| \leq 1$, then $(\tilde{G}, \tilde{p})$ has at least $|\Gamma|$ non-zero equilibrium stresses with mutually disjoint support. See e.g. Figure 3.6 (a) and (b).

Consider a connected Γ-gain graph (G, ψ) with $V(G) = V_1(G)$, and take a spanning tree T of G. By Proposition 3.2.6, there is a gain map ψ' equivalent to ψ such that $\psi'(e) = \mathrm{id}$ for all $e \in E(T)$. [29] provides a useful way to deduce the gain of a connected subgraph H of G (under ψ'), by considering only the edges of



(a) (b)

Figure 3.6: Balanced Γ-gain graph and its Γ-lifting, where $\Gamma = \langle\gamma\rangle \simeq \mathbb{Z}_5$. Here, all edges in the Γ-gain graphs are labelled id.

H which are not contained in T.

Lemma 3.2.11 ([29], Lemma 2.4)**.** Let Γ be a cyclic group and (G, ψ) be a Γ-gain graph with $V(G) = V_1(G)$. Let H be a connected subgraph of G with non-empty edge set, and let T be a spanning tree of H. Assume that $\psi(e) = \mathrm{id}$ for all $e \in E(T)$. Then, $\langle H\rangle = \langle \psi(e) : e \in E(H - T)\rangle$.

(a) (b)

Figure 3.7: Two connected Γ-gain graphs, each with a spanning tree given in red.

If we allow T to have a fixed vertex v, Lemma 3.2.11 is not generally true. For example, let Γ be an arbitrary group with some non-identity element $\gamma \in \Gamma$, and consider the Γ-gain graphs $(G_1, \psi_1), (G_2, \psi_2)$ in Figure 3.7(a) and (b), respectively. For $1 \leq i \leq 2$, let T_i be the spanning tree of G_i given by the edges in red. Both graphs are balanced, since they do not contain cycles with only free vertices. Hence, $\langle G_1\rangle = \langle G_2\rangle = \{\mathrm{id}\}$. However, for $1 \leq i \leq 2$, $G_i - T_i$ has an edge e_i with $\psi_i(e_i) \neq \mathrm{id}$.

(Note, e_1 is incident to a fixed vertex, and so we could directly apply a type II switching at e_1 induced by γ^{-1}.)

Let (G, ψ) be a Γ-gain graph for some group Γ. Let $H_1, \dots, H_t$ be the connected components of $G - V_{|\Gamma|}(G)$ and for each $1 \leq i \leq t$ let T_i be a spanning tree of H_i. Take a gain map ψ' which is equivalent to ψ and such that $\psi'(e) = \mathrm{id}$ for all $e \in E(T_i), 1 \leq i \leq t$. Take an edge $e \in E(G - (T_1 \cup \dots \cup T_t))$. If both end-vertices of e are free, then $e \in E(H_i - T)$ for some $1 \leq i \leq t$. Since $\langle H_i \rangle_{v,\psi'} \leq \langle G \rangle_{v,\psi'}$ for all $v \in V(H_i)$, it follows from Lemma 3.2.11 that $\psi'(e) \in \langle G \rangle_{v,\psi'}$. If one of the end-vertices of e is fixed, then we may apply a type II switching at e induced by $\psi'(e)^{-1}$, so that e has identity gain. In particular, if (G, ψ) is balanced, then there is always a gain map ψ'' equivalent to ψ such that $\psi''(e) = \mathrm{id}$ for all $e \in E(G)$. Therefore, by Lemma 3.2.11, the following result holds.

Lemma 3.2.12. Let Γ be a cyclic group and (G, ψ) be a Γ-gain graph with $V(G) = V_1(G)$. Let H be a connected subgraph of G with non-empty edge set. There is a gain map ψ' equivalent to ψ such that $\psi'(e) \in \langle H \rangle$ for all $e \in E(H)$.

Lemma 3.2.11 was used in [29] to show the following.

Lemma 3.2.13 ([29], Lemma 2.5)**.** For a cyclic group Γ, let (G, ψ) be a Γ-gain graph with $V(G) = V_1(G)$. Let H_1, H_2 be connected subgraphs of G such that $H_1 \cap H_2$ is connected. If H_1 is balanced, then $\langle H_1 \cup H_2 \rangle = \langle H_2 \rangle$. In particular, if H_2 is balanced, then so is $H_1 \cup H_2$.

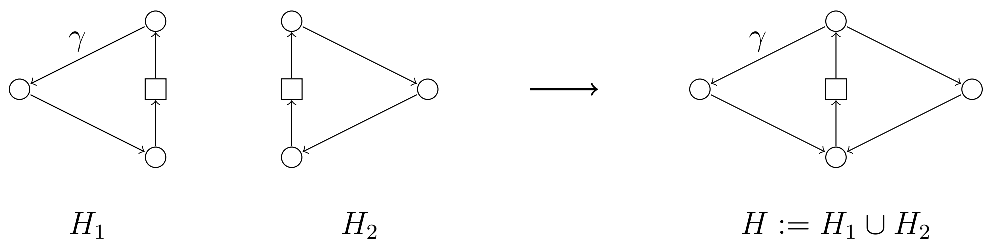


Figure 3.8: Unbalanced union of two connected balanced Γ-gain graphs, whose intersection is also connected. Here Γ is a cyclic group and $\gamma \in \Gamma$ is not the identity. All unlabelled edges have identity gain.

In general, this is not true if $V(G) \neq V_1(G)$, as shown in Figure 3.8. However, if we force $H_1 \cap H_2 - V_{|\Gamma|}(H_1 \cap H_2)$ to be connected, we may use Lemma 3.2.11 (in an analogous way as was done in [29]) to show the following.

Lemma 3.2.14. Let Γ be a cyclic group and (G, ψ) be a Γ-gain graph. Let H_1, H_2 be connected subgraphs of G. If the graph $H_1 \cap H_2 - V_{|\Gamma|}(H_1 \cap H_2)$ is connected and H_1 is balanced, then $\langle H_1 \cup H_2 \rangle = \langle H_2 \rangle$. In particular, if H_2 is balanced, then so is $H_1 \cup H_2$.

Proof. Let H' be the graph obtained from $H_1 \cup H_2$ by removing its fixed vertices. Since $H_1 \cap H_2 - V_{|\Gamma|}(H_1 \cap H_2)$ is connected, there is a connected component of H' which contains $H_1 \cap H_2$. Use H to denote such a connected component, and take a spanning tree T of H such that $T \cap H_1$ is a spanning tree for $H_1 \cap H$, $T \cap H_2$ is a spanning tree for $H_2 \cap H$ and $T \cap H_1 \cap H_2$ is a spanning tree for $H_1 \cap H_2$. By Proposition 3.2.6, there is a Γ-gain map ψ' type I equivalent to ψ such that $\psi'(e) = \text{id}$ for all $e \in E(T)$. By Lemma 3.2.11, for all edges $e \in E(H - T)$, $\psi'(e) = \text{id}$ if $e \in E(H_1)$ and $\psi'(e) \in \langle H_2 \rangle$ if $e \in E(H_2)$. Hence, $\langle H \rangle = \langle H_2 \rangle$ by Lemma 3.2.11. Note that all connected components of H' which are not H are either in H_1 or in H_2. Hence, the result follows. $\square$

We conclude the subsection with another useful result. Here, we adapt a proof which was given in [27] for the free action case.

Lemma 3.2.15. Let Γ be a cyclic group, (G, ψ) be a Γ-gain graph, and H_1, H_2 be connected subgraphs of G such that $H_1 \cap H_2$ is unbalanced and $H_1 \cap H_2 - V_{|\Gamma|}(H_1 \cap H_2)$ is connected. Suppose that $\langle H_2 \rangle \simeq \mathbb{Z}_p$ for some prime p. Then, $\langle H_2 \rangle = \langle H_1 \cap H_2 \rangle$ and $\langle H_1 \rangle = \langle H_1 \cup H_2 \rangle$.

Proof. Let H' be the graph obtained from $H_1 \cup H_2$ by removing its fixed vertices. Since $H_1 \cap H_2 - V_{|\Gamma|}(H_1 \cap H_2)$ is connected, there is a connected component of H' which contains $H_1 \cap H_2$. Use H to denote such a connected component, and take a spanning tree T of H such that $T \cap H_1$ is a spanning tree for $H_1 \cap H$, $T \cap H_2$

is a spanning tree for $H_2 \cap H$ and $T \cap H_1 \cap H_2$ is a spanning tree for $H_1 \cap H_2$. By Proposition 3.2.6, there is a Γ-gain map ψ' type I equivalent to ψ such that $\psi'(e) = \mathrm{id}$ for all $e \in E(T)$. By Proposition 3.2.11, there is some $\gamma \in \Gamma$ such that $\psi'(e) \in \{\mathrm{id}, \gamma, \dots, \gamma^{p-1}\}$ for all $e \in E(H_2)$. Since $H_1 \cap H_2$ is unbalanced, $\psi'(e) = \gamma^k$ for some $e \in E(H_1 \cap H_2) \setminus E(T)$ and some $1 \leq k \leq p-1$. It follows that $\langle H_2 \rangle = \langle H_1 \cap H_2 \rangle$. Therefore, $\langle H_2 \rangle \subseteq \langle H_1 \rangle$. It then follows that $\langle H_1 \rangle = \langle H_1 \cup H_2 \rangle$, as required. $\square$

3.2.3 Near-balanced subgraphs

In this subsection we present the notion of near-balancedness. The first example of how near-balancedness effects the infinitesimal rigidity of a symmetric framework can be found in [[56], Section 6.2.3]. The notion of near-balancedness was later developed by R. Ikeshita and S. Tanigawa in [27] and [28]. All results stated in this subsection were proved in R. Ikeshita's thesis (see [27]) for the case where the group acts freely on the vertex set. As we will see, we define near-balancedness only for graphs with no fixed vertices. This allows us to directly apply the existing proofs. Since R. Ikeshita's thesis is hard to access, we include all proofs in Appendix A using our terminology.

Definition 3.2.16. Let Γ be a cyclic group and (G, ψ) be a Γ-gain graph. Suppose H is a connected subgraph of G with $V(H) = V_1(H)$ and $E(H) \neq \emptyset$. We say H (equivalently, $E(H), (H, \psi|_{E(H)})$) is *near-balanced (under ψ)* if it is unbalanced, and there exists a vertex v of H, called the *base vertex of H*, and $\gamma \in \Gamma$ such that, for all closed walks W in H starting from v and not containing v as an internal vertex, $\psi(W) \in \{\mathrm{id}, \gamma, \gamma^{-1}\}$. We also say H (equivalently, $E(H), (H, \psi|_{E(H)})$) is *near-balanced (under ψ) with base vertex v and gain γ (and γ^{-1}).*

If $\langle H \rangle \simeq \mathbb{Z}_2$ or $\langle H \rangle \simeq \mathbb{Z}_3$, then it is easy to see that H is always near-balanced. Hence, we say H (equivalently, $E(H)$) is *proper near-balanced* if it is near-balanced and $\langle H \rangle \not\simeq \mathbb{Z}_2, \mathbb{Z}_3$.

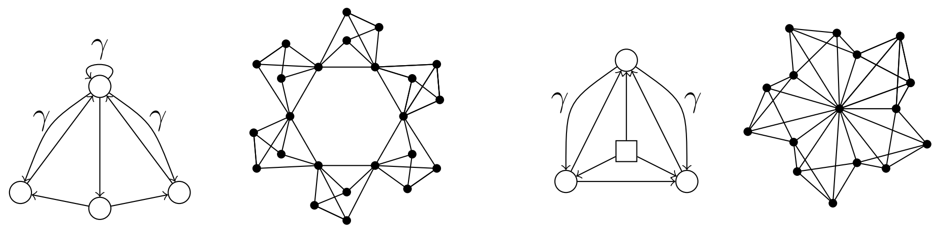


Figure 3.9: (a) is a proper near-balanced Γ-gain graph with Γ-lifting (b). (c) is a proper near-balanced Γ-gain graph with an additional fixed vertex, and (d) is its Γ-lifting. In (a,c), the unlabelled edges have gain id.

Given a cyclic group Γ, let (G, ψ) be a Γ-gain graph with Γ-lifting $\tilde{G}$. For some injective homomorphism $\tau : \Gamma \to O(\mathbb{R}^2)$, let $(\tilde{G}, \tilde{p})$ be a $\tau(\Gamma)$-generic realisation of $\tilde{G}$. Given a near-balanced subgraph H of G which is not $(2,1)$-sparse, $(\tilde{H}, \tilde{p}|_{V(H)})$ has a non-zero equilibrium stress (see Figure 3.9(a,b)). Similarly, if we add a fixed vertex to a near-balanced graph, resulting in a graph H with $|E(H)| \geq 2|V(H)|$, then its lifting has a non-zero equilibrium stress (see Figure 3.9(c,d)). The graph in Figure 3.9(c) is not $(2,1)$-sparse and contains an equilibrium stress. However, this equilibrium stress is already detected by the fact that it contains a balanced K_4. So, at least in this example, the near-balanced condition is not necessary. Though this is just an example, we will see in Subsection 3.2.5 that near-balancedness need not be defined on graphs with fixed vertices.

For some cyclic group Γ, let (G, ψ) be a Γ-gain graph, and H a proper near-balanced subgraph of G with base vertex v and gain $\gamma \in \Gamma$. It is easy to see that all unbalanced cycles must pass through v, and that γ is uniquely determined up to taking its multiplicative inverse. Moreover, the following result gives a useful way to see proper near-balanced gain graphs.

Lemma 3.2.17 ([27], Lemma 4.1)**.** Let Γ be a group, (G, ψ) be a connected proper near-balanced Γ-gain graph. Then, G is unbalanced and there is some $\gamma \in \Gamma$ and a gain map ψ' equivalent to ψ such that $\psi'(e) \in \{\text{id}, \gamma\}$ for all edges $e \in E(G)$ directed

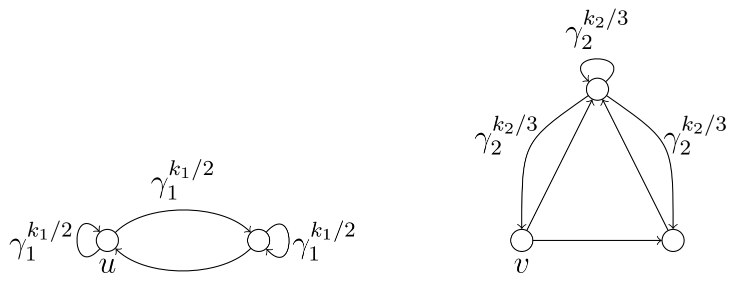


Figure 3.10: (a) is a near-balanced Γ_1-gain graph (G,ψ), where $\Gamma_1 = \langle\gamma_1\rangle \simeq \mathbb{Z}_{k_1}$ for some even integer $k_1 \geq 2$. (b) is a near-balanced Γ_2-gain graph (G,ψ), where $\Gamma_2 = \langle\gamma_2\rangle \simeq \mathbb{Z}_{k_2}$ for some integer $k_2 \geq 3$ divisible by 3.

to v, and $\psi'(e) = \mathrm{id}$ for all edges $e \in E(G)$ not directed to v.

Suppose that (G,ψ) is a Γ-gain graph for some group Γ, and that there are $v \in V(G), \gamma \in \Gamma$ such that $\psi(e) \in \{\mathrm{id}, \gamma\}$ for all edges $e \in E(G)$ directed to v, and $\psi(e) = \mathrm{id}$ for all edges $e \in E(G)$ not incident to v. Then, every closed walk W containing v as an initial vertex but not as an internal vertex clearly has gain id, γ or γ^{-1}. Hence, the converse to Lemma 3.2.17 is also true. Notice this is also true if $\langle\gamma\rangle \simeq \mathbb{Z}_2$ or $\langle\gamma\rangle \simeq \mathbb{Z}_3$. However, Lemma 3.2.17 need not hold if G is not proper near-balanced. In Figure 3.10, we show two examples of non-proper near-balanced graphs: (a) shows a Γ_1-gain graph (G_1,ψ_1) where $\langle G_1\rangle \simeq \mathbb{Z}_2$; (b) shows a Γ_2-gain graph (G_2,ψ_2) where $\langle G_2\rangle \simeq \mathbb{Z}_3$. We can take u and v to be the base vertices of near-balancedness of the graphs in (a) and (b), respectively. In both cases, there is a non-identity loop not incident to a base vertex.

It is important to note that, in both graphs given in Figure 3.10, every vertex may be considered as a base vertex of near-balancedness. In general, the base vertex of near-balancedness is not unique. However, the base vertex of a proper near-balanced graph is unique, provided the graph contains sufficiently many edges and all of its balanced subgraphs are $(2,3)$-sparse.

Lemma 3.2.18 ([27], Lemma 4.2)**.** Let Γ be a group and (G,ψ) be a proper near-balanced Γ-gain graph which satisfies $|E(G)| \geq 2|V(G)| - 1$. Suppose that all balanced subgraphs of (G, ψ) are $(2,3)$-sparse. Then, the base vertex of G is unique. Moreover, for all near-balanced subgraphs H of G with $|E(H)| = 2|V(H)| - 1$, the unique base vertex of H coincides with the base vertex of G.

Lemmas 4.4 to 4.10 in [27] study the union of a proper near-balanced graph H_1 together with some other graph H_2. (Except for Lemma 4.6, which shows that the union of two balanced graphs is near-balanced under certain conditions.) All such results are fundamental for the proofs in Chapter 7. Whereas [27] takes a matroidal approach, the arguments in this thesis will not. Hence, we slightly modify the statements given in [27] to fit our setting. Even though the statements given in [27] differ from ours, the same proofs apply.

Lemma 3.2.19 ([27], Lemma 4.4)**.** Let Γ be a group, (G, ψ) be a Γ-gain graph and H_1, H_2 be proper near-balanced subgraphs of G such that $H_1 \cap H_2$ is $(2,1)$-tight and proper near-balanced. Assume that for $1 \leq i \leq 2$ there is an edge $f_i \in E(H_i)$ such that $H_i - f_i$ is $(2,1)$-tight. Assume further that every balanced subgraph of $H_1 - f_1, H_2 - f_2$ is $(2,3)$-sparse. Then $H_1 \cup H_2$ is proper near-balanced.

Lemma 3.2.20 ([27], Lemma 4.5)**.** Let Γ be a group, (G, ψ) be a Γ-gain graph and H_1, H_2 be subgraphs of G such that $H_1 \cap H_2$ is connected, balanced and $(2,3)$-tight. Assume that there is an edge $f_1 \in E(H_1)$ such that $H_1 - f_1$ is $(2,1)$-tight and that H_1 is proper near-balanced. Assume further that H_2 is connected and balanced, and that $V_{|\Gamma|}(H_2) = \emptyset$. Then $H_1 \cup H_2$ is proper near-balanced.

Lemma 3.2.21 ([27], Lemma 4.6)**.** Let Γ be a group, (G, ψ) be a Γ-gain graph and H_1, H_2 be balanced subgraphs of G such that $H_1 \cap H_2$ consists of two connected components, one of which is an isolated vertex v. Suppose that there is an edge $f_1 \in E(H_1)$ such that $H_1 - f_1$ is $(2,3)$-tight, and that H_2 is connected. Suppose further that $V_{|\Gamma|}(H_1 \cup H_2) = \emptyset$. Then $H_1 \cup H_2$ is near-balanced with base vertex v.

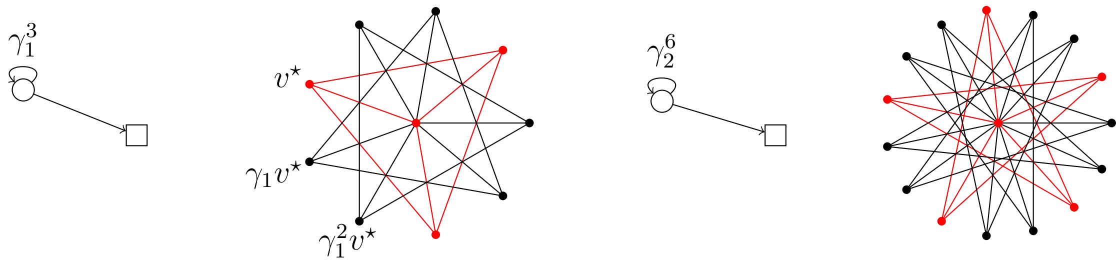
Lemma 3.2.22. Let Γ be a group, (G, ψ) be a Γ-gain graph and H_1, H_2 be connected subgraphs of G such that $H_1 \cap H_2$ is connected and unbalanced. Assume that H_1

is proper near-balanced and that $\langle H_2 \rangle \simeq \mathbb{Z}_p$ for some prime p. Then, we have $\langle H_1 \rangle = \langle H_2 \rangle = \langle H_1 \cap H_2 \rangle = \langle H_1 \cup H_2 \rangle$.

3.2.4 Subgroups of a cyclic group

Let $\tilde{G}$ be a Γ-symmetric graph for some cyclic group Γ of non-prime order $k \geq 4$. So, $\tilde{G}$ is $\mathbb{Z}_k$-symmetric. For all $n|k$, $\mathbb{Z}_n$ is a subgroup of $\mathbb{Z}_k$, and so $\tilde{G}$ is also $\mathbb{Z}_n$-symmetric. It follows that the $\mathbb{Z}_k$-gain graph (G, ψ) of $\tilde{G}$ can also be considered as a $\mathbb{Z}_n$-gain graph.

For example, let $\Gamma_1 = \langle \gamma_1 \rangle, \Gamma_2 = \langle \gamma_2 \rangle$ be cyclic groups of order 9 and 15, respectively. Consider the Γ_1-gain graph (G_1, ψ_1) given in Figure 3.11(a). Its Γ_1-lifting (b) is also a $\mathbb{Z}_3$-symmetric graph: the triangle in red is rotated by $2\pi/3$ and $4\pi/3$ anti-clockwise around the origin. In fact, replacing the gain γ_1^3 in Figure 3.11(a) with the generator of $\mathbb{Z}_3$, we obtain the $\mathbb{Z}_3$-gain graph of the $\mathbb{Z}_3$-symmetric graph in Figure 3.11(b). Similarly, the Γ_2-lifting of the Γ_2-gain graph (G_2, ψ_2) given in Figure 3.11(c) can be seen as a $\mathbb{Z}_3$-symmetric graph: the star in red is rotated by $2\pi/3$ and $4\pi/3$ anti-clockwise around the origin.



(a) (b) (c) (d)

Figure 3.11: (a) is a Γ_1-gain graph with Γ_1-lifting (b), where $\Gamma_1 = \langle \gamma_1 \rangle$ is a cyclic group of order 9. (c) is a Γ_2-gain graph with Γ_2-lifting (d), where $\Gamma_2 = \langle \gamma_2 \rangle$ is a cyclic group of order 15.

Similarly, any $\mathcal{C}_k$-symmetric framework may be seen as a $\mathcal{C}_n$-symmetric framework for all $n|k$. As we will see in Chapter 4, infinitesimal motions can be classified:

for all $0 \leq j \leq k-1$, we will define ρ_j-symmetric infinitesimal motions, i.e. motions which exhibit symmetry described by the irreducible representation ρ_j of $\mathcal{C}_k$. (In particular, if ρ_j is the trivial representation, then the ρ_j-symmetric infinitesimal motions are the fully-symmetric infinitesimal motions.) Each class can be studied separately. However, for some $0 \leq j \leq k-1, 0 \leq i \leq n-1$, a ρ_j-symmetric infinitesimal motion of a $\mathcal{C}_k$-symmetric framework $(\tilde{G}, \tilde{p})$ may also be a ρ_i-symmetric infinitesimal motion of $(\tilde{G}, \tilde{p})$ when this is seen as a $\mathcal{C}_n$-symmetric framework. When we combinatorially characterise $\mathcal{C}_k$-generic frameworks, we must take this fact into account. If $\langle G \rangle \not\simeq \mathbb{Z}_2$ or j is even, we do so through the notion of $S_i(k, j)$ gain graphs, which may also be found in [[28], Section 2]. (We will consider the case where $\langle G \rangle \simeq \mathbb{Z}_2$ and j is odd separately.) Unless $2 \leq j \leq k-2$ and $-1 \leq i \leq 1$, and unless $\langle G \rangle \simeq \mathbb{Z}_2$ and j is odd, the combinatorial counts corresponding to the ρ_j-symmetric infinitesimal motions of a $\mathcal{C}_k$-generic framework coincide with the combinatorial counts corresponding to the ρ_i-symmetric infinitesimal motions of a $\mathcal{C}_n$-generic framework. Hence, we need only define $S_i(k, j)$ for $2 \leq j \leq k-2$ and $-1 \leq i \leq 1$.

Definition 3.2.23. Let (G, ψ) be a Γ-gain graph for some cyclic group Γ of order $k \geq 4$. For $2 \leq j \leq k-2, -1 \leq i \leq 1$, we define the following sets:

$$S_i(k, j) = \begin{cases} \{n \in \mathbb{N} : 2 \leq n, n|k, j \equiv i (\bmod n)\} & \text{if } j \text{ is even} \\ \{n \in \mathbb{N} : 2 < n, n|k, j \equiv i (\bmod n)\} & \text{if } j \text{ is odd} \end{cases}$$

We say a connected subset F of $E(G)$ (equivalently, a connected subgraph H of G) is $S_0(k, j)$ if $\langle F \rangle \simeq \mathbb{Z}_n$ (equivalently, $\langle H \rangle \simeq \mathbb{Z}_n$) for some $n \in S_0(k, j)$. Similarly, we say F (equivalently, H) is $S_{\pm 1}(k, j)$ if $\langle F \rangle \simeq \mathbb{Z}_n$ (equivalently, $\langle H \rangle \simeq \mathbb{Z}_n$) for some $n \in S_{-1}(k, j) \cup S_1(k, j)$. We say F (equivalently, H) is $S(k, j)$ if it is either $S_0(k, j)$ or $S_{\pm 1}(k, j)$.

Example 3.2.24. We consider the case where $k = 6$, and we compute $S_i(6, j)$ for all $2 \leq j \leq 4$ and all $-1 \leq i \leq 1$. Since the only divisors of 6 are $1, 2, 3$ and 6, we know that $S_i(6, j) \subseteq \{2, 3\}$ for all $2 \leq j \leq 4$ and all $-1 \leq i \leq 1$. When $j = 2$, we have $j \equiv 0(\bmod 2)$ and $j \equiv -1(\bmod 3)$. So, $S_0(6,2) = \{2\}$, $S_1(6,2) = \emptyset$ and $S_{-1}(6,2) = \{3\}$. When $j = 3$, we have $j \equiv 0(\bmod 3)$ and $j \equiv 1(\bmod 2)$. However, since $j = 3$ is odd, we do not include 2 in $S_1(6,3)$. Hence, $S_0(6,3) = \{3\}$, and $S_1(6,3) = S_{-1}(6,3) = \emptyset$. When $j = 4$, we have $j \equiv 0(\bmod 2)$ and $j \equiv 1(\bmod 3)$. So, $S_0(6,2) = \{2\}$, $S_1(6,2) = \{3\}$ and $S_{-1}(6,2) = \emptyset$.

If k is even, then the Γ-lifting $\tilde{G}$ of (G,ψ) is $\mathbb{Z}_2$-symmetric. Moreover, if j is odd, then $j \equiv 1(\bmod 2)$. This implies that the ρ_j-symmetric infinitesimal motions of a $\mathcal{C}_k$-symmetric realisation $(\tilde{G},\tilde{p})$ of $\tilde{G}$ can be seen as ρ_1-symmetric infinitesimal motions of $(\tilde{G},\tilde{p})$, when seen as a $\mathcal{C}_2$-symmetric framework. However, $2 \notin S_{\pm 1}(k,j)$ for odd j. This is because the combinatorial count that a $\mathbb{Z}_2$-gain graph must satisfy in order for its lifting to have an infinitesimally rigid $\mathcal{C}_2$-generic realisation differs from the count that a $\mathbb{Z}_k$-gain graph must satisfy in order for its lifting to have an infinitesimally rigid $\mathcal{C}_k$-generic realisation for all $k \geq 3$. Hence, when defining a combinatorial count of a $\mathbb{Z}_k$-gain graph for $k \geq 4$, we let $S_{\pm 1}(k,j)$ gain graphs have different counts than the gain graphs (G,ψ) with $\langle G \rangle \simeq \mathbb{Z}_2$.

Lemma 3.2.25. Let $k \geq 4, 2 \leq j \leq k-2$. Let $n,m \in S_{-1}(k,j) \cup S_0(k,j) \cup S_1(k,j)$. If $\gcd(n,m) \neq 1$, then $n,m \in S_i(k,j)$ for some $-1 \leq i \leq 1$.

Proof. For all integers $n' \geq 1$, $\gcd(n',n'+1) = 1$, $\gcd(n',n'+2) = 1$ if n' is odd, and $\gcd(n',n'+2) = 2$ if n' is even. It follows that $S_{i_1}(k,j) \cap S_{i_2}(k,j) = \emptyset$ for all $-1 \leq i_1 \neq i_2 \leq 1$. Then, since $\gcd(n,m) \neq 1$, $n,m \in S_i(k,j)$ for some $-1 \leq j \leq 1$. □

Corollary 3.2.26. Let $k \geq 4, 2 \leq j \leq k-2$ and let (G,ψ) be a Γ-gain graph for some cyclic group Γ of order k. Let $H_1 \leq H_2 \leq G$. If H_1, H_2 are $S(k,j)$, then H_1, H_2 are either both $S_{\pm 1}(k,j)$ or they are both $S_0(k,j)$.

Proof. There are some $i_1, i_2 \in \{-1,0,1\}$, and some $n \in S_{i_1}(k,j), m \in S_{i_2}(k,j)$ such that $\langle H_1 \rangle \simeq \mathbb{Z}_n$ and $\langle H_2 \rangle \simeq \mathbb{Z}_m$. Since $H_1 \leq H_2$, $\mathbb{Z}_n \leq \mathbb{Z}_m$. By Lagrange's Theorem $n|m$, so $\gcd(n,m) = n \neq 1$. By Lemma 3.2.25, $i_1 = i_2$. This proves our result. □

Lemma 3.2.27. Let (G,ψ) be a Γ-gain graph for some cyclic group Γ of order $k \geq 4$. Let $H_1, H_2 \leq G$ be such that $H_1 \cap H_2 - V_{|\Gamma|}(H_1 \cap H_2)$ is connected with non-empty edge set. Then the following hold:

(i) if H_1, H_2 are $S_0(k,j)$ (respectively, $S_{\pm 1}(k,j)$), then so is $H_1 \cup H_2$; and

(ii) if H_1 is $S_0(k,j)$ (respectively, $S_{\pm 1}(k,j)$), H_2 is near-balanced and $H_1 \cap H_2$ is unbalanced, then $H_1 \cup H_2$ is $S_0(k,j)$ (respectively, $S_{\pm 1}(k,j)$).

Proof. Since $H_1 \cap H_2 - V_{|\Gamma|}(H_1 \cap H_2)$ is connected, each closed walk W in $H_1 \cup H_2$ only containing free vertices can be decomposed into $W = W_1 \circ \cdots \circ W_n$ such that each W_i is a closed walk in H_1 or H_2 only containing free vertices. Hence, $\langle H_1 \cup H_2 \rangle$ is the group generated by the elements of $\langle H_1 \rangle \cup \langle H_2 \rangle$, and (i) holds.

For (ii), suppose that $\langle H_1 \rangle \simeq \mathbb{Z}_n$ for some $n \in S_{-1}(k,j) \cup S_0(k,j) \cup S_1(k,j)$, that H_2 is near balanced with base vertex v and gain γ, and that $H_1 \cap H_2$ is unbalanced. Since $H_1 \cap H_2$ is unbalanced, $\gamma \in \langle H_1 \cap H_2 \rangle \leq \langle H_1 \rangle$. Hence, $\langle H_2 \rangle \leq \langle H_1 \rangle$ and so $\langle H_1 \cup H_2 \rangle \simeq \mathbb{Z}_n$, as required. $\square$

3.2.5 Gain sparsity

In this section we will introduce the criteria we will use to characterise infinitesimally rigid Γ-symmetric graphs. Depending on the symmetry group $\tau(\Gamma)$, the sparsity count required for a full characterisation of $\tau(\Gamma)$-generic infinitesimally rigid frameworks is more or less complex. If $\tau(\Gamma)$ is one of $\mathcal{C}_s, \mathcal{C}_2$ and $\mathcal{C}_3$, the following definition of sparsity suffices to describe $\tau(\Gamma)$-generic infinitesimally rigid frameworks.

Definition 3.2.28. Let Γ be a cyclic group and (G,ψ) be a Γ-gain graph. Let m, l be integers such that $0 \leq m \leq 2, 0 \leq l \leq 3, m \leq l$. (G,ψ) is called $(2,m,3,l)$-*gain sparse* if the following hold:

- Any balanced subgraph H of (G,ψ) with non-empty edge set is $(2,3)$-sparse.

- For any subgraph H (G,ψ) with $E(H) \neq \emptyset$, we have

$$|E(H)| \leq 2|V_1(H)| + m|V_{|\Gamma|}(H)| - l. \tag{3.1}$$

(G,ψ) is called $(2,m,3,l)$-*gain tight* if it is $(2,m,3,l)$-gain sparse and it satisfies $|E(G)| = 2|V_1(G)| + m|V_{|\Gamma|}(G)| - l$.

Let (G,ψ) be a Γ-gain graph for some cyclic group Γ. Suppose that (G,ψ) is $(2,m,3,l)$-gain sparse for some integers m,l such that $0 \leq m \leq 2, 0 \leq l \leq 3, m \leq l$, and let H be a balanced subgraph of G with non-empty edge set. Then, in addition to H being $(2,3)$-sparse, Equation (3.1) must also hold for H. If $V_{|\Gamma|}(H) = \emptyset$, then $(2,3)$-sparsity is always a stronger condition than that given in Equation 3.1. However, this need not be the case if $V_{|\Gamma|}(H) \neq \emptyset$.

Lemma 3.2.29. Let Γ be a cyclic group and (G,ψ) be a Γ-gain graph. Let m,l be integers such that $0 \leq m \leq 2, 0 \leq l \leq 3, m \leq l$, and let H be a subgraph of G. Then, $2|V_1(H)|+m|V_{|\Gamma|}(H)|-l \leq 2|V(H)|-3$ if and only if $3-l \leq (2-m)|V_{|\Gamma|}(H)|$.

Proof. Recall that $V(H) = V_1(H) \dot\cup V_{|\Gamma|}(H)$. Hence,

$$2|V_1(H)| + m|V_{|\Gamma|}(H)| - l \leq 2|V(H)| - 3 = 2|V_1(H)| + 2|V_{|\Gamma|}(H)| - 3$$

if and only if $m|V_{|\Gamma|}(H)| - l \leq 2|V_{|\Gamma|}(H)| - 3$. Rearranging, we obtain the result. □

An argument similar to the proof of [[27], Lemma 4.13] shows the following.

Lemma 3.2.30. For $0 \leq m \leq 2, 1 \leq l \leq 3$ such that $m \leq l$, any $(2,m,3,l)$-gain tight graph G with non-empty edge-set has exactly one connected component with non-empty edge set, and it has no isolated free vertex. Moreover, if $m \geq 1$, then G has no isolated fixed vertex.

Proof. Fix $0 \leq m \leq 2, 1 \leq l \leq 3$ such that $m \leq l$. Let $c_0 \geq 0, c \geq 1$ be integers such that $c - c_0 \geq 1$, and (G,ψ) be a $(2,m,3,l)$-gain tight graph with connected components $H_1,\ldots,H_c$, of which $H_1,\ldots,H_{c_0}$ are isolated vertices, and $H_{c_0+1},\ldots,H_c$

have non-empty edge sets. Notice that $c - c_0 = 1$ if and only if G has exactly one connected component with non-empty edge set. We have

$$\begin{aligned}|E(G)| &= \sum_{i=c_0+1}^{c} |E(H_i)| \\ &\leq 2 \sum_{i=c_0+1}^{c} |V_1(H_i)| + m \sum_{i=c_0+1}^{c} |V_{|\Gamma|}(H_i)| - (c - c_0)l \\ &= 2(|V_1(G)| - \sum_{i=1}^{c_0} |V_1(H_i)|) + m(|V_{|\Gamma|}(G)| - \sum_{i=1}^{c_0} |V_{|\Gamma|}(H_i)|) - (c - c_0)l.\end{aligned}$$

By $(2, m, 3, l)$-gain tightness, this is not strictly less than $2|V_1(G)| + m|V_{|\Gamma|}(G)| - l$. Since $l \geq 1$, it follows that $|V_1(H_i)| = 0$ for all $1 \leq i \leq c_0$ and that $c - c_0 = 1$. Moreover, if $m \geq 1$ then $V_{|\Gamma|}(H_i)$ must be empty for all $1 \leq i \leq c_0$. Hence, the result holds. □

Now, let $|\Gamma| \geq 4$, and $\tau : \Gamma \to O(\mathbb{R}^2)$ be an injective homomorphism. In this case, the sparsity count given in Definition 3.2.28 does not suffice to characterise the infinitesimal rigidity of $\tau(\Gamma)$-generic frameworks. Hence, we introduce the more refined notions of $(2, m, 3, l)'$-gain sparsity and $\mathbb{Z}_k^j$-gain sparsity. Both consider subgraphs which may be seen as $\mathbb{Z}_2$-gain graphs. In addition, $\mathbb{Z}_k^j$-gain sparsity considers near-balanced and $S(k, j)$ subgraphs. We start with the simpler notion of $(2, m, 3, l)'$-gain sparsity.

Definition 3.2.31. Let $k \geq 4$ be even and Γ be a cyclic group of order k. Let m, l be integers such that $0 \leq m \leq 2, 0 \leq l \leq 3, m \leq l$. We say a Γ-gain graph (G, ψ) is $(2, m, 3, l)'$-gain tight if it is $(2, m, 3, l)$-gain tight and for all connected subgraphs H of G with $E(H) \neq \emptyset$ and $\langle H \rangle \simeq \mathbb{Z}_2$, H is $(2, 2)$-sparse.

Let Γ be a cyclic group of order $k \geq 4$, let (G, ψ) be a Γ-gain graph with non-empty edge set, and recall that $|V_{|\Gamma|}(G)| \leq 1$. Let $2 \leq j \leq k - 2$. We now define the notion of $\mathbb{Z}_k^j$-gain sparsity. In order to do so, we first define the function f_k^j on $2^{E(G)}$ by

$$f_k^j(F) = \sum_{X \in C(F)} \left\{2|V(X)| - 3 + \alpha_k^j(X)\right\},$$

where F is a subset of $E(G)$, $C(F)$ denotes the set of connected components of the graph spanned by F, and

$$\alpha_k^j(X) = \begin{cases} 0 & \text{if } X \text{ is balanced,} \\ 1 & \text{if } j \text{ is odd and } \langle X \rangle \simeq \mathbb{Z}_2, \\ 2 - |V_{|\Gamma|}(X)| & \text{if } X \text{ is } S_{\pm 1}(k,j), \\ 2 - 2|V_{|\Gamma|}(X)| & \text{if } X \text{ is } S_0(k,j) \text{ or } X \text{ is proper near-balanced,} \\ 3 - 2|V_{|\Gamma|}(X)| & \text{otherwise.} \end{cases}$$

Since α_k^j depends on $|V_{|\Gamma|}(X)|$, it is not straightforward to see that α_k^j and f_k^j are well-defined functions. In the following result we show that both α_k^j and f_k^j are well-defined, and that f_k^j is monotone.

Lemma 3.2.32. Let $k \geq 4, 2 \leq j \leq k-2$, and Γ be a cyclic group of order k. For any Γ-gain graph (G, ψ), f_k^j is a monotone function.

Proof. We first show that f_k^j is a well-defined function. To do so, it suffices to show that α_k^j is well-defined. So, take an arbitrary connected non-empty subset X of $E(G)$. We show that $\alpha_k^j(X)$ can take exactly one value in the set $\{0, 1, 2 - |V_{|\Gamma|}(X)|, 2 - 2|V_{|\Gamma|}(X)|, 3 - 2|V_{|\Gamma|}(X)|\}$.

If X is balanced, then it is neither near-balanced nor $S(k,j)$. Moreover, if j is odd, then $2 \notin S(k,j)$ by definition, and if $\langle X \rangle \simeq \mathbb{Z}_2$, then X is near-balanced but not proper near-balanced. Hence, we may assume that $\alpha_k^j(X)$ lies in the set $\{2 - |V_{|\Gamma|}(X)|, 2 - 2|V_{|\Gamma|}(X)|, 3 - 2|V_{|\Gamma|}(X)|\}$.

Suppose that X is $S(k,j)$. If $V_{|\Gamma|}(X) = \emptyset$, then $2 - |V_{|\Gamma|}(X)| = 2 - 2|V_{|\Gamma|}(X)|$. Hence, we may assume that $V_{|\Gamma|}(X) \neq \emptyset$. In particular, this implies that X is not near-balanced. By Corollary 3.2.26, X can be either $S_0(k,j)$ or $S_{\pm 1}(k,j)$, but it cannot be both. Hence, α_k^j, f_k^j are well-defined.

Now, we show that f_k^j is monotone. It suffices to show that, for all connected non-empty subsets $X \subseteq Y$ of $E(G)$,

$$f_k^j(X) \leq f_k^j(Y). \tag{3.2}$$

Hence, take any two such subsets X, Y of $E(G)$. If $V_{|\Gamma|}(Y) = V_{|\Gamma|}(X) = \emptyset$, this is clearly true. So, assume that $|V_{|\Gamma|}(Y)| = 1$. In particular, this implies that Y is not proper near-balanced. If Y is balanced, then so is X and so Equation (3.2) holds since $|V(X)| \leq |V(Y)|$. Similarly, if j is odd and $\langle Y \rangle \simeq \mathbb{Z}_2$, then either X is balanced or $\langle X \rangle \simeq \mathbb{Z}_2$. In both cases $f_k^j(X) \leq 2|V(X)| - 2$ and so Equation (3.2) holds, since $|V(X)| \leq |V(Y)|$. Hence, we may assume that Y is not balanced and that $\langle Y \rangle \not\simeq \mathbb{Z}_2$ whenever j is odd. Since $|V_\Gamma(X)| \leq 1$, it is easy to see that $f_k^j(X) \leq 2|V_1(X)|$ whether X is balanced, near-balanced, $S(k, j)$, or $\langle X \rangle \simeq \mathbb{Z}_2$ and j is odd. Hence,

$$f_k^j(X) \leq 2|V_1(X)| \leq 2|V_1(Y)|.$$

In particular, if $f_k^j(Y) = 2|V_1(Y)|$, then Equation (3.2) holds. Therefore, we may assume that Y is $S(k, j)$.

So, let $\langle Y \rangle \simeq \mathbb{Z}_n$ for some $n \in S(k, j)$. If X is also $S(k, j)$, then Equation (3.2) holds by Corollary 3.2.26 and the fact that $V(X) \subseteq V(Y)$. If $n \in S_{-1}(k, j) \cup S_1(k, j)$, by the fact that $|V_{|\Gamma|}(Y)| = 1$, we have $f_k^j(Y) = 2|V_1(Y)| + |V_{|\Gamma|}(Y)| - 1 = 2|V_1(Y)|$. If $n \in S_0(k, j)$, then $f_k^j(Y) = 2|V_1(Y)| - 1$. If X is balanced, then

$$f_k^j(X) = 2|V(X)| - 3 \leq 2|V_1(X)| - 1 \leq 2|V_1(Y)| - 1 < 2|V_1(Y)|.$$

Hence, whether Y is $S_{\pm 1}(k, j)$ or $S_0(k, j)$, Equation (3.2) holds. Similarly, if X is proper near-balanced then $f_k^j(X) = 2|V_1(X)| - 1 \leq 2|V_1(Y)| - 1$ and Equation (3.2) holds. Hence, we may assume that j is odd and $\langle X \rangle \simeq \mathbb{Z}_2$.

In particular, $f_k^j(X) = 2|V(X)| - 2$. We show that $n \in S_{-1}(k, j) \cup S_1(k, j)$. To see this, assume for a contradiction that $n \in S_0(k, j)$. Since $X \subseteq Y$, we know that $\mathbb{Z}_2 \simeq \langle X \rangle \leq \langle Y \rangle \simeq \mathbb{Z}_n$. Hence, n is even. By the definition of $S_0(k, j)$, $n|j$, and so j is even, a contradiction. Hence, $n \in S_{-1}(k, j) \cup S_1(k, j)$. We have

$$f_k^j(X) = 2|V(X)| - 2 \leq 2|V_1(X)| \leq 2|V_1(Y)| = f_k^j(Y),$$

Therefore, Equation (3.2) holds, as required. □

Definition 3.2.33. Let Γ be a cyclic group of order $k \geq 4$ and let (G, ψ) be a Γ-gain graph. For $2 \leq j \leq k - 2$, we say (G, ψ) is $\mathbb{Z}_k^j$*-gain sparse* if $|E(H)| \leq f_k^j(E(H))$ for all subgraphs H of G with non-empty edge set. We say (G,ψ) is $\mathbb{Z}_k^j$*-gain tight* if it is $\mathbb{Z}_k^j$-gain sparse and $|E(G)| = f_k^j(E(G))$.

Recall that Lemma 3.2.32 says f_k^j is a monotone function. In particular, this implies that any connected subgraph H of G with non-empty edge set must satisfy $|E(H)| \leq 2|V_1(H)|$, whether it is balanced, near-balanced, $S(k,j)$ or $\langle H\rangle_\psi \simeq \mathbb{Z}_2$. Recall also that the concept of near-balancedness is only defined on graphs with no fixed vertices. This is because, if there is some connected H with a fixed vertex v such that $H - v$ is near-balanced, then $|E(H)| \leq 2|V_1(H)| = 2|V(H)| - 2 \leq 2|V(H)| - 1$ is always true. Hence, if X is near-balanced, we assume by default that it has no fixed vertices.

It will follow from one of the main results of this thesis (see Theorem 7.3.3) that f_k^j induces a matroid for all odd $4 \leq k < 1000$ or $k = 4, 6$. We expect that the same is true for all other symmetry groups in the plane. By Lemma 3.2.32 f_k^j is monotone. Hence, only the submodularity of f_k^j must be checked.

3.3 Gain graphs for all groups

In this section we further generalise the notion of gain graph. For the rest of the chapter we allow Γ to be any abstract group of finite order, unless stated otherwise. In Section 3.2 we defined a gain graph as a directed multigraph with labelled edges and whose vertex set is the disjoint union of two sets, one corresponding to the free vertices of a symmetric graph, and one corresponding to its fixed vertices. This definition fits the setting where the group Γ is cyclic because knowing if a joint is free/fixed under $\tau(\Gamma)$ also tells us exactly the stabiliser of the joint. In general, joints with different stabilisers have different degrees of freedom. It follows that knowing the stabiliser of a given vertex is crucial. In this section we introduce a new notion of gain graph, which also retains information on the stabilisers of the vertices of $\tilde{G}$. In this section, and for the rest of the thesis, we let $\mathrm{Sub}(\Gamma)$ denote the set of all subgroups of Γ.

Let $\tilde{G}$ be a Γ-symmetric graph, and $G = (V, E)$ be the quotient of $\tilde{G}$ with respect to Γ. We label the vertices in V, and we orient and label the edges in E. We do so in the following way.

For each vertex orbit $v := \Gamma v^\star$ we fix a vertex orbit representative $v^\star$, and we define the map $\varphi : V \to \mathrm{Sub}(\Gamma)$ by letting $\varphi(v) = \mathrm{Stab}_\Gamma(v^\star)$. We also fix an orientation on the edges of G. Pick an arbitrary directed edge $e = (u, v)$ in the directed quotient graph, and let $u^\star, v^\star$ be the vertex orbit representatives of $u = \Gamma u^\star$ and $v = \Gamma v^\star$, respectively. For all $\gamma \in \Gamma$, let $u^\star_\gamma := \gamma u^\star$ and $v^\star_\gamma := \gamma v^\star$.

There is some $\gamma \in \Gamma$ such that the edge $u^\star v^\star_\gamma$ lies in the edge orbit uv. Hence, we may write uv as $\Gamma u^\star v^\star_\gamma$. Then, for all $\alpha \in \varphi(u), \beta \in \varphi(v)$, $u^\star v^\star_{\alpha\gamma\beta} = u^\star_\alpha v^\star_{\alpha\gamma} \in uv$. Conversely, if $uv = \Gamma u^\star v^\star_\gamma = \Gamma u^\star v^\star_\delta$ for some $\delta \in \Gamma$, then there is some $g \in \Gamma$ such that $u^\star v^\star_\gamma = u^\star_g v^\star_{g\delta}$, i.e. $u^\star = g u^\star$ and $\gamma v^\star = g\delta v^\star$. So, $\alpha := g^{-1} \in \varphi(u)$ and $\beta := \gamma^{-1} g\delta \in \varphi(v)$. Hence, $\delta = g^{-1}\gamma(\gamma^{-1}g\delta) = \alpha\gamma\beta$ for some $\alpha \in \varphi(u), \beta \in \varphi(v)$. We define the *gain on* e to be any element in $\varphi(u)\gamma\varphi(v)$.

In the construction above, we could re-direct e from v to u and re-label it with the group inverse of the original label chosen. Up to this operation, and up to the choice of representatives, and of the gains for edges incident to a non-free vertex as described above, this process gives a unique directed multigraph with labels on its vertices and on its edges. The process gives rise to a new class of directed multigraphs with labelled vertices and edges, which are a generalisation of the Γ-gain graphs given both in Section 3.1 and in Section 3.2.

Definition 3.3.1. Let Γ be an abstract group. A Γ-*gain graph* is a triple (G, φ, ψ), where G is a directed multigraph, and $\varphi : V(G) \to \mathrm{Sub}(\Gamma), \psi : E(G) \to \Gamma$ are maps such that the following conditions are satisfied:

1. For all parallel edges $e = (u, v), f \in E(G)$ and elements $\gamma_u \in \varphi(u), \gamma_v \in \varphi(v)$, we have $\psi(e) \neq \gamma_u \psi(f) \gamma_v$ whenever $f = (u, v)$, and $\psi(e) \neq [\gamma_u \psi(f) \gamma_v]^{-1}$ whenever $f = (v, u)$.

2. If $e \in E(G)$ is a loop at a vertex $u \in V(G)$, then $\psi(e) \notin \varphi(u)$.

See Figure 3.12 for examples of parallel edges which are not allowed. We call φ and ψ the *vertex-gain map* and *edge-gain map* of (G, φ, ψ), respectively. For each $u \in V(G), e \in E(G)$, we call $\varphi(u), \psi(e)$ the *gains* or *labels* of u, e, respectively.

Since for all $u \in V(G)$, $\varphi(u)$ is a group, $\mathrm{id} \in \varphi(u)$, and so $|\varphi(u)| \geq 1$. For $1 \leq i \leq |\Gamma|$, we define $V_i(G)$ to be the subset $\{u \in V(G) : |\varphi(u)| = i\}$ of $V(G)$. As already defined in Section 3.2, we call the vertices in $V_1(G)$ and $V_{|\Gamma|}(G)$ the *free* and *fixed vertices* of G, respectively. For $2 \leq i \leq |\Gamma| - 1$, we call the vertices in $V_i(G)$ the *semi-free vertices* of G. When drawing a Γ-gain graph, we use squares to represent the fixed vertices, white circles to represent the free vertices, and black circles to represent the semi-free vertices.

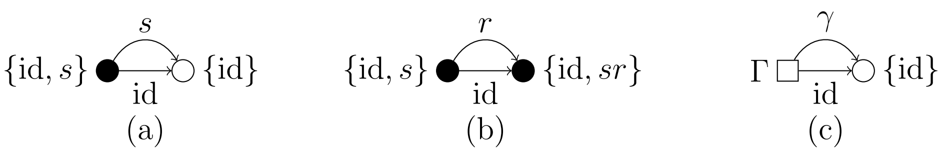


Figure 3.12: Three examples of labelling which are not allowed by the definition of Γ-gain graph. In (a,b), $\Gamma = \mathbb{D}_{2k}$ for some $k \geq 2$. In (c), Γ is any group and $\gamma \in \Gamma$ is arbitrary.

Given a group Γ and a Γ-gain graph (G, φ, ψ), Definitions 3.3.1 and 3.1.1 are equivalent whenever $V(G) = V_1(G)$, and Definitions 3.3.1 and 3.2.1 are equivalent whenever $V(G) = V_1(G) \,\dot{\cup}\, V_{|\Gamma|}(G)$. Therefore, given a group Γ, we will henceforth use the terminology Γ-gain graph to refer to the combinatorial object given in Definition 3.3.11. If Γ is cyclic, then we will abbreviate the notation (G, φ, ψ) to (G, ψ).

Conversely, given a Γ-gain graph (G, φ, ψ), we may construct a Γ-symmetric graph $\tilde{G}$. We do so in the following way. For each $v \in V(G)$, $V(\tilde{G})$ contains the vertices γv for all $\gamma \in \Gamma$. For all $\delta \in \varphi(v), \gamma \in \Gamma$, we let the vertices γv and $\gamma \delta v$ coincide. For each edge $e = (u, v) \in E(G)$ and for all $\gamma \in \Gamma$, $E(\tilde{G})$ contains the edge $u_\gamma v_{\gamma\psi(e)}$, where $u_\gamma := \gamma u$ and $v_{\gamma\psi(e)} := \gamma\psi(e)v$.

The graph obtained by applying this process is simple and it is unique up to isomorphism. Moreover, given any Γ-symmetric graph $\tilde{G}$, if we construct a Γ-gain graph (G, φ, ψ) from $\tilde{G}$ using the process described at the beginning of the section, and we then construct a Γ-symmetric graph $\tilde{H}$ from (G, φ, ψ) by applying the process just described, then $\tilde{G} \simeq \tilde{H}$. Hence, with a slight abuse of terminology, we call (G, φ, ψ) the *(quotient)* Γ-*gain graph* of $\tilde{G}$ and we call $\tilde{G} \simeq \tilde{H}$ the Γ-*covering graph* (or Γ-*lifting*) of (G, φ, ψ).

3.3.1 Switchings

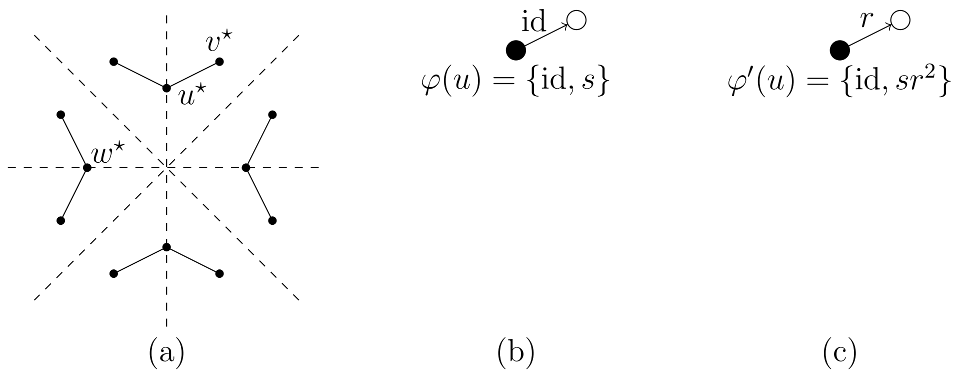


Figure 3.13: (a) is a $\mathbb{D}_8$-symmetric graph with two different $\mathbb{D}_8$-gain graphs (b,c). The representatives chosen for (b) are $u^\star$ and $v^\star$. The graph in (c) can be obtained from the graph in (b) by applying a switching at u induced by r.

We now generalise Definitions 3.2.2 and 3.2.3 of switchings. Similarly as described in Subsection 3.2.1, switchings of type I and of type II correspond to changing the choice of vertex orbit representatives and the choice of edge labels, respectively, during the construction of a Γ-gain graph.

Definition 3.3.2. For a group Γ, let (G, φ, ψ) be a Γ-gain graph. Let $v \in V(G)$. A *(type I) switching at v induced by the element* $\gamma \in \Gamma$ is an operation which generates new gain maps φ', ψ' by letting $\varphi'(v) = \gamma\varphi(v)\gamma^{-1}$ and

$$\psi'(e) = \begin{cases} \gamma\psi(e)\gamma^{-1} & \text{if } e \text{ is a loop incident to } v \\ \gamma\psi(e) & \text{if } e \text{ is a non-loop edge directed from } v \\ \psi(e)\gamma^{-1} & \text{if } e \text{ is a non-loop edge directed to } v \\ \psi(e) & \text{otherwise} \end{cases}$$

for all $e \in E(G)$.

Figure 3.13 shows an example of a type I switching applied to a vertex of a $\mathbb{D}_8$-gain graph. (Note, the conjugacy classes of s and sr are distinct in $\mathbb{D}_8$.) The $\mathbb{D}_8$-gain graph (G, φ, ψ) in (b) is obtained from the $\mathbb{D}_8$-symmetric graph in (a) by choosing the vertex orbit representatives $u^\star$ and $v^\star$. Applying a type I switching at u induced by r, we obtain the $\mathbb{D}_8$-gain graph (G, φ', ψ') in Figure 3.13(c), where $\varphi'(u) = r\{\text{id}, s\}r^{-1} = \{\text{id}, sr^2\}$. Note that (c) can also be obtained directly from (a) by choosing the vertex orbit representatives $w^\star$ (i.e. $ru^\star$) and $v^\star$.

Definition 3.3.3. For a group Γ, let (G, φ, ψ) be a Γ-gain graph. A *(type II) switching at an edge $e = (u, v) \in E(G)$ induced by the elements $\gamma_u \in \varphi(u)$ and* $\gamma_v \in \varphi(v)$ is an operation which generates a new gain map $\psi' : E(G) \to \Gamma$ defined by letting $\psi'(e) = \gamma_u\psi(e)\gamma_v$ and $\psi'(f) = \psi(f)$ for all other $f \in E(G)$.

Sometimes we apply type II switchings at edges of the form $e = (u, v)$ induced by elements $\gamma \in \varphi(u)$ and $\text{id} \in \varphi(v)$, or induced by $\text{id} \in \varphi(v)$ and $\gamma \in \varphi(v)$ (e.g., if one of u, v is free). In these cases, if clear from the context, we simply say that we apply a *type II switching at e induced by γ*.

Definition 3.3.4. Given a group Γ and a Γ-gain graph (G, φ, ψ), we define a *switching of* (G, φ, ψ) to be any type I switching at a vertex $v \in V(G)$ or type II switching at an edge $e \in E(G)$.

Similarly as in the case where Γ is a cyclic group, applying a series of switchings to a Γ-gain graph defines an equivalence relation. More precisely, we have the following.

Definition 3.3.5. For a group Γ, let $(G, \varphi_1, \psi_1), (G, \varphi_2, \psi_2)$ be two Γ-gain graphs. We say ψ_1, ψ_2 (or $(\varphi_1, \psi_1), (\varphi_2, \psi_2)$) are *equivalent/type I equivalent/type II equivalent* if one can be obtained from the other by applying a series of switchings/type I switchings/type II switchings. We say (G, φ_1, ψ_1) and (G, φ_2, ψ_2) are *equivalent/type I equivalent/type II equivalent* if $(\psi_1, \varphi_1)(\psi_2, \varphi_2)$ are equivalent/type I equivalent/type II equivalent.

Given a group Γ and a Γ-gain graph (G, φ, ψ) with $V(G) = V_1(G) \dot\cup V_{|\Gamma|}(G)$, Definitions 3.3.2 and 3.2.2 are equivalent, as are Definitions 3.3.3 and 3.2.3. If $V_{|\Gamma|}(G) = \emptyset$, Definition 3.3.2 is also equivalent to the definition of switching given in [29]. However, type II switchings are only defined in this thesis.

Let Γ be a group and (G, φ, ψ) be a Γ-gain graph. Suppose there are vertices $u, v \in V(G)$ with conjugate labels. It is easy to see that $\varphi'(u) = \varphi'(v)$ for some vertex-gain map φ' equivalent to φ: since $\varphi(u) = \delta^{-1}\varphi(v)\delta$ for some $\delta \in \Gamma$, applying a type I switching at u induced by δ generates the desired vertex-gain map.

In Subsection 3.2.1 we showed how to obtain an edge-gain map which assigns the identity gain to all edges of a forest (see Proposition 3.2.6, and the paragraph just before), provided Γ is cyclic. It is easy to see that the same process can be used also when Γ is not cyclic. Moreover, by applying the same process while avoiding switchings at semi-free vertices, we may obtain an edge-gain map which assigns the identity gain to all edges not directed to a semi-free vertex, without altering the vertex-gain map.

Proposition 3.3.6. Let Γ be a group and (G, φ, ψ) be a Γ-gain graph. For any forest F in G, there is a Γ-gain graph (G, φ', ψ') type I equivalent to (G, φ, ψ) such that $\psi'(e) = \mathrm{id}$ for all $e \in E(F)$.

Proposition 3.3.7. Let Γ be a group and (G, φ, ψ) be a Γ-gain graph. For any forest F in G, there is a Γ-gain graph (G, φ', ψ') type I equivalent to (G, φ, ψ) such that $\varphi'(v) = \varphi(v)$ for all $v \in V(G)$ and $\psi'(e) = \mathrm{id}$ for all $e \in E(F)$ not directed to a semi-free vertex.

3.3.2 Balanced subgraphs

In this section we further generalise the notion of balancedness given in Section 3.1 for gain graphs containing only free vertices and in Subsection 3.2.2 for Γ-gain graphs where Γ is cyclic.

Definition 3.3.8. Let Γ be a group and (G, φ, ψ) be a Γ-gain graph. Let W be a walk in G of the form $W = e_1, \dots, e_t$, where e_i has end-vertices $v_i, v_{i+1} \in V(G)$ for all $1 \leq i \leq t$. We say the *gain of* W *under* ψ is $\psi(W) = \prod_{i=1}^{t} \psi(e_i)^{\text{sign}(e_i)}$, where $\text{sign}(e_i) = 1$ if e_i is directed from v_i to v_{i+1}, and $\text{sign}(e_i) = -1$ otherwise.

Given a connected subgraph H of G with $E(H) \neq \emptyset$ and a non-fixed vertex $v \in V(H)$, the *gain of* H *under* ψ *with base vertex* v (equivalently, the *gain of* $E(H)$ *under* ψ *with base vertex* v) is the group generated by

$$\{\psi(W) : W \text{ is a closed walk starting at } v \text{ and not containing fixed vertices}\}.$$

We denote such a group by $\langle E(H)\rangle_{v,\psi}$ (or $\langle H\rangle_{v,\psi}$).

Notice that, unlike for the case where $V(G) = V_1(G)$ or Γ is cyclic, this definition strongly depends on the gain map ψ. In fact, given two equivalent Γ-gain graphs $(G, \varphi_1, \psi_1), (G, \varphi_2, \psi_2)$, a connected subgraph H of G, and some non-fixed vertex $v \in V(H) \setminus V_{|\Gamma|}(H)$, $\langle H\rangle_{v,\psi_1}$ and $\langle H\rangle_{v,\psi_2}$ need not coincide, or even be conjugate. For example, let Γ be any group, and let (G, φ_1, ψ_1) and (G, φ_2, ψ_2) be the Γ-gain graphs in Figure 3.4(a,b), respectively. Here we let Γ' be a non trivial subgroup of Γ which contains some non-trivial element $\gamma \in \Gamma$. Then (G, φ_1, ψ_1) and (G, φ_2, ψ_2) are equivalent, but $\langle G\rangle_{v,\psi_1} = \{\text{id}\} \neq \langle\gamma\rangle = \langle G\rangle_{v,\psi_2}$ for any $v \in V(G)$.

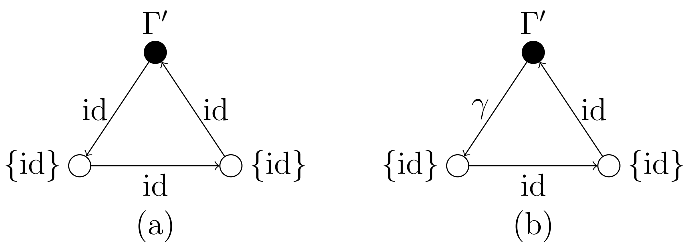


Figure 3.14: Two Γ-gain graphs which are type II equivalent.

However, some results which hold for the case where $V(G) = V_1(G)$ or Γ is cyclic extend to our new setting. For instance, the same proofs as Proposition 3.2.8 and 3.2.9 can be used to show that the following two results hold.

Proposition 3.3.9. For a group Γ, let (G, φ, ψ) be Γ-gain graph. For any connected subgraph H of G such that $H - V_{|\Gamma|}(H)$ is connected and for all non-fixed $u, v \in V(H)$, $\langle H \rangle_{u,\psi}$ and $\langle H \rangle_{v,\psi}$ are conjugate.

Proposition 3.3.10. Let Γ be a group and $(G, \varphi, \psi), (G, \varphi', \psi')$ be type I equivalent Γ-gain graphs. For any connected subgraph H of G and any non-fixed $v \in V(H)$, $\langle H \rangle_{v,\psi}$ and $\langle H \rangle_{v,\psi'}$ are conjugate.

Definition 3.3.11. Let Γ be a group and (G, φ, ψ) be a Γ-gain graph. We say a connected subgraph H of G (equivalently, $E(H), (H, \varphi|_{V(H)}, \psi|_{E(H)})$) is *balanced under* ψ if all closed walks in H only containing free and semi-free vertices have identity gain under ψ. We say $H, E(H)$ are *balanced* if there is some edge-gain map ψ' type II equivalent to ψ such that H is balanced under ψ'. Otherwise, we say $H, E(H)$ are *unbalanced*. We say a graph is *balanced* if all of its connected components are balanced.

By Proposition 3.3.10, given two type I equivalent Γ-gain graphs (G, φ_1, ψ_1) and (G, φ_2, ψ_2) and given a connected subgraph H of G, H is balanced under ψ_1 if and only if it is balanced under ψ_2. Hence, in Definition 3.3.11, we may also say that $H, E(H)$ are balanced if there is some edge-gain map ψ' equivalent to ψ such that H is balanced under ψ'.

Since $\langle H \rangle_{v,\psi}$ now changes when applying type II switchings, some of the results from Section 3.2 which hold for cyclic groups do not automatically transfer to our new setting. Notably, given a Γ-gain graph (G, φ, ψ), we cannot determine whether G is balanced just by looking at the spanning trees of the connected components of $G - V_{|\Gamma|}(G)$. For example, consider the Γ-gain graph (G, φ, ψ) given in Figure 3.15. The spanning tree T of G given by the edges in red is such that $\psi(e) = \mathrm{id}$ for all $e \in E(T)$, and the edge which does not lie in T has non-identity gain. However, we may apply a type II switching at one of the edges incident to the semi-free vertex induced by $\gamma \in \Gamma'$ so that the only closed cycle in (G, φ, ψ) has gain id.

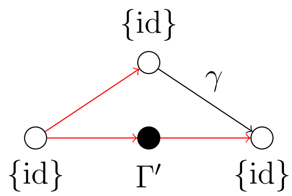


Figure 3.15: Connected Γ-gain graph (G, φ, ψ) and a spanning tree of G, given in red. Here $\Gamma' < \Gamma$ is non-trivial, $\gamma \in \Gamma'$ and all unlabelled edges have identity gain.

However, Lemma 3.2.11 still holds in our setting, as the same proof can be applied to show Lemma 3.3.12. In particular, this implies that for all balanced Γ-gain graphs (G, φ, ψ), there is an equivalent Γ-gain graph (G, φ', ψ') such that $\psi'(e) = \text{id}$ for all $e \in E(G)$.

Lemma 3.3.12. For a group Γ, let (G, φ, ψ) be a Γ-gain graph with $V_{|\Gamma|}(G) = \emptyset$. Let H be a connected subgraph of G with non-empty edge set, and let T be a spanning tree of H. Assume that $\psi(e) = \text{id}$ for all $e \in E(T)$. Then, for all vertices $v \in V(H)$, $\langle H \rangle_{v,\psi} = \langle \psi(e) : e \in E(H - T) \rangle$.

It is still not straightforward to determine whether a given Γ-gain graph is balanced or not, since there might be an edge $e \in E(H - T)$ with non-identity gain even if (G, φ, ψ) is balanced (see Figure 3.15).

3.3.3 Dihedral groups

Let $k \geq 2$ be an integer and $\tilde{G}$ be a $\mathbb{D}_{2k}$-symmetric graph. Recall that, given a $v^\star \in V(\tilde{G})$, we assume that $v^\star$ is either free under $\mathbb{D}_{2k}$, fixed by $\mathbb{D}_{2k}$ or fixed exactly by id and sr^j for some $0 \leq j \leq k-1$. Hence, the construction given at the beginning of this section gives a $\mathbb{D}_{2k}$-gain graph (G, φ, ψ) such that $V(G) = V_1(G) \,\dot\cup\, V_2(G) \,\dot\cup\, V_{2k}(G)$. (See Figure 3.16 for an example of the construction applied to a $\mathbb{D}_6$-symmetric graph.)

Whenever k is odd, for all $u^\star, v^\star \in V_2(\tilde{G})$, $\mathrm{Stab}_{\mathbb{D}_{2k}}(u^\star)$ and $\mathrm{Stab}_{\mathbb{D}_{2k}}(v^\star)$ are conjugate. In particular, when constructing (G, φ, ψ), we may always choose vertex orbit representatives in a way that gives $\varphi(u) = \{\mathrm{id}, s\}$ for all $u \in V_2(G)$. This will sometimes be useful when studying such gain graphs.

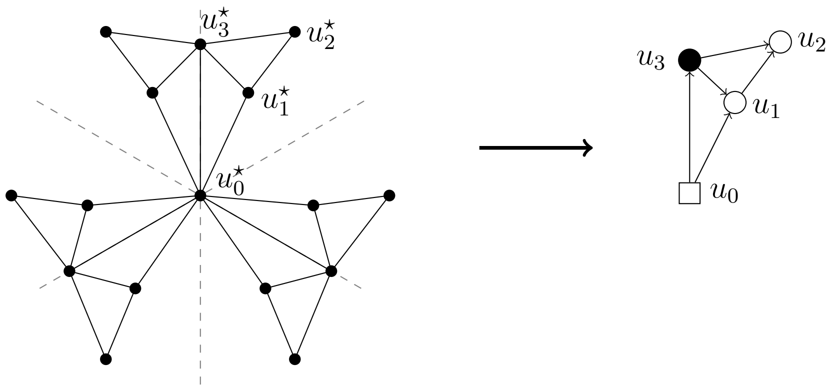


Figure 3.16: $\mathbb{D}_6$-symmetric graph and its corresponding $\mathbb{D}_6$-gain graph. Here, all edges have identity gain.

For the rest of the chapter we consider $\mathbb{D}_{2k}$-gain graphs for all integers $k \geq 2$. For $k \geq 2$, the group $\mathbb{D}_{2k}$ has at least $k+1$ subgroups: the subgroup $\langle r \rangle$; and for $0 \leq j \leq k-1$, the subgroups $\langle sr^j \rangle = \{\mathrm{id}, sr^j\}$. As we will see in Chapter 5, the infinitesimal rigidity of a $\mathbb{D}_{2k}$-symmetric framework $(\tilde{G}, \tilde{p})$ also depends on the 'sub-frameworks' of $(\tilde{G}, \tilde{p})$ which are symmetric with respect to $\mathcal{C}_s$ and $\mathcal{C}_k$. Hence, we define the rotational and reflectional subgraphs of a $\mathbb{D}_{2k}$-gain graph. We start by defining rotational gain graphs.

Definition 3.3.13. Let $k \geq 2$ be an integer, and (G, φ, ψ) be a connected $\mathbb{D}_{2k}$-gain graph. We say G is *rotational* if there is a type II equivalent $\mathbb{D}_{2k}$-gain graph (G, φ', ψ') of (G, φ, ψ) such that $\psi'(W) \in \langle r \rangle$ for all closed walks W in G not containing fixed vertices.

Definition 3.3.13 is independent of the fixed vertices. Moreover, Lemma 3.3.14 shows that Definition 3.3.13 is also independent of the semi-free vertices.

Lemma 3.3.14. For some $k \geq 2$, let (G, φ, ψ) be a connected $\mathbb{D}_{2k}$-gain graph. Suppose that there is some $v \in V(G)$ such that $\varphi(v) = \langle \mathrm{id}, sr^t \rangle$ for some $0 \leq t \leq k-1$. G is rotational if and only if each connected component of $G - v$ is rotational.

Proof. Clearly, each connected component of $G - v$ is rotational whenever G is rotational. So, we assume that each connected component of $G-v$ is rotational and show that G is also rotational. For some $m \geq 1$, let $G_1, \dots, G_m$ be the connected components of $G - v$. Since each neighbour of v lies in exactly one connected component of G, it is enough to show that $G_i +_G v$ is rotational for an arbitrary $1 \leq i \leq m$. Without loss of generality, we show that $G_1 +_G v$ is rotational.

Write the vertices incident to v in $G_1 +_G v$ as $v_1, \dots, v_n$ and, for $1 \leq i \leq n$, let $e_i := (v, v_i)$. Since G_1 is rotational, there is a Γ-gain graph (G, φ, ψ') type II equivalent to (G, φ, ψ) such that every closed walk in G_1 containing no fixed vertex has gain in $\langle r \rangle$ under ψ'. Fix some $1 \leq i \neq j \leq n$ and suppose there are two $v_i - v_j$ walks W_{ij}^1, W_{ij}^2 in G_1.

By assumption, there are some $0 \leq a, b \leq k-1$ such that either $\psi'(W_{ij}^1) = r^a$ and $\psi'(W_{ij}^2) = r^b$ or $\psi'(W_{ij}^1) = sr^a$ and $\psi'(W_{ij}) = sr^b$. Since G_1 is connected, we may define the map $f : \{2, \dots, n\} \to \{\mathrm{id}, s\}$ by letting

$$f(i) = \begin{cases} \mathrm{id} & \text{for some } 0 \leq m \leq k-1 \text{ and a } v_1 - v_j \text{ walk } W_{1,i},\ \psi'(W_{1,i}) = r^m \\ s & \text{for some } 0 \leq m \leq k-1 \text{ and a } v_1 - v_j \text{ walk } W_{1,i},\ \psi'(W_{1,i}) = sr^m. \end{cases}$$

Clearly, $f^2(i) = \mathrm{id}$ for all $2 \leq i \leq n$. If $\psi(e_1) = sr^a$ for some $0 \leq a \leq k-1$, then we may apply a type II switching at e_1 induced by sr^j and id such that e_1 has gain $(sr^j)(sr^a) = r^{a-j}$. Hence, we may always apply a (possibly trivial) type II switching at e_1 such that e_1 has gain r^{a_1} for some integer $0 \leq a_1 \leq k-1$. Similarly, for all $2 \leq i \leq n$, we may apply type II switchings at e_i such that e_i has gain $f(i)r^{a_i}$. Let ψ'' be the edge-gain map obtained from ψ' by applying such switchings.

Notice that for all $2 \leq i \neq j \leq n$ and all $v_i - v_j$ walks W_{ij}, $\psi''(W_{ij})$ has the form $f(i)f(j)r^{a_{ij}}$ for some $0 \leq a_{ij} \leq k-1$. Take a closed walk W starting at v. For some indices $\{i_1, \dots, i_t\} \subseteq \{1, \dots, n\}$, W has the form $(u, v_{i_1}) \circ W_{i_1 i_2} \circ \cdots \circ W_{i_{t-1} i_t} \circ (v_{i_t}, u)$, where W_{ij} denotes a $v_i - v_j$ walk in G_1. Hence, for some $a, b, a_1, \ldots, a_t$,

$$\begin{aligned}\psi''(W) &= \psi''(e_{i_1})\psi''(W_{i_1i_2})\psi''(W_{i_2i_3})\ldots\psi''(W_{i_{t-i}i_t})\psi''(e_{i_t})^{-1}\\ &= f(i_1)r^a(f(i_1)f(i_2)r^{a_1})\ldots(f(i_{t-1})f(i_t)r^{a_t})r^bf(i_t),\end{aligned}$$

which equals r^m for some $0 \le m \le k-1$. Since W was arbitrary, $G_1 +_G v$ is rotational, as required. $\square$

Lemma 3.3.14 allows us to redefine rotational gain graphs independently of the semi-free vertices. Therefore, we have the following equivalent definition of rotational.

Definition 3.3.15. Let $k \ge 2$ be an integer, and (G, φ, ψ) be a connected $\mathbb{D}_{2k}$-gain graph. We say G is *rotational* if every closed walk in G only containing free vertices has gain r^j for some $0 \le j \le k-1$.

We now define reflectional gain graphs.

Definition 3.3.16. Let $k \ge 2$ be an integer and (G, φ, ψ) be a connected $\mathbb{D}_{2k}$-gain graph. We say G is *reflectional (with reflection sr^j and base vertex $v \in V(G)$)* if v is not fixed and there is an integer $0 \le j \le k-1$ and a $\mathbb{D}_{2k}$-gain graph (G, φ, ψ') type II equivalent to (G, φ, ψ) such that $\langle G\rangle_{v,\psi'} \in \langle sr^j\rangle$.

Let G be reflectional with reflection sr^j and base vertex v. By Proposition 3.3.7 and Lemma 3.3.12, there is always a $\mathbb{D}_{2k}$-gain graph (G, φ', ψ') equivalent to (G, φ, ψ) such that a $\psi'(e) \in \{\mathrm{id}, s\}$ for all $e \in E(G)$ and $\varphi'(v) = \varphi(v)$ (the root of any spanning tree can be chosen to be v). We define $V_{sr^j}(G) = \{v \in V(G) : \varphi'(v) = \mathbb{D}_{2k}$ or $\varphi'(v) = \{\mathrm{id}, sr^j\}\}$.

When realising the $\mathbb{D}_{2k}$-lifting of (G, φ, ψ) as a $\mathcal{C}_{kv}$-symmetric framework $(\tilde{G}, \tilde{p})$, it can also be seen as a $\mathcal{C}_s$-symmetric framework, for which the reflection line l is the y-axis rotated anti-clockwise by the angle $2\pi j/k$. The joints fixed by the symmetry group $\mathcal{C}_s$ are exactly the joints which correspond to representatives of the vertices in $V_{sr^j}(G)$. All joints of $(\tilde{G}, \tilde{p})$ which correspond to the vertex representatives of vertices in $V_2(G - V_{sr^j}(G))$ are free joints of the $\mathcal{C}_s$-symmetric $(\tilde{G}, \tilde{p})$, since they lie on a reflection line but not the reflection line l. (We will see this more in detail in Section 5.2.)

Definition 3.3.17. Let $k \geq 2$ and let (G, φ, ψ) be a $\mathbb{D}_{2k}$-gain graph. We say (G, φ, ψ) is $\mathbb{D}_{2k}$-*gain sparse* if for all subgraphs H of G with non-empty edge set we have:

(i) $|E(H)| \leq 2|V_1(H)| + |V_2(H)|$.

(ii) $|E(H)| \leq 2|V(H)| - 3$ if H is balanced.

(iii) $|E(H)| \leq 2|V_1(H)| + 2|V_2(H)| - 1$ if H is rotational.

(iv) $|E(H)| \leq 2|V(H - V_{sr^j}(H))| + |V_{sr^j}(H)| - 1$, if H is reflectional with reflection sr^j for some $0 \leq j \leq k-1$.

(G, φ, ψ) is $\mathbb{D}_{2k}$-*gain tight* if it is $\mathbb{D}_{2k}$-gain sparse and $|E(G)| = 2|V_1(H)| + |V_2(H)|$.

Let (G, φ, ψ) be a $\mathbb{D}_{2k}$-gain tight graph for some $k \geq 2$. Then, a subgraph H of G must satisfy $|E(H)| \leq 2|V_1(H)| + |V_2(H)|$, regardless of whether it is balanced, rotational, reflectional, or none of these. Similarly, if H is balanced, then it is also rotational and reflectional. Therefore, it must satisfy all four counts given in Definition 3.3.17. Using the fact that $V(H)$ is the disjoint union $V_1(H) \dot{\cup} V_2(H) \dot{\cup} V_{2k}(H)$, it is easy to show the following lemma, and see how the strength of the counts depends on $V_2(H)$ and $V_{2k}(H)$.

Lemma 3.3.18. Let $k \geq 2$ be an integer and (G, ψ) be a $\mathbb{D}_{2k}$-gain tight graph, and let H be a subgraph of G. Then,

(i) $2|V(H)| - 3 \leq 2|V_1(H)| + |V_2(H)|$ if and only if $|V_2(H)| + 2|V_{2k}(H)| \leq 3$.

(ii) $2|V_1(H)| + 2|V_2(H)| - 1 \leq 2|V_1(H)| + |V_2(H)|$ if and only if $|V_2(H)| \leq 1$.

(iii) For all $0 \leq j \leq k-1$, $2|V(H - V_{sr^j}(H))| + |V_{sr^j}(H)| - 1 \leq 2|V_1(H)| + |V_2(H)|$ if and only if $(|V_2(H)| - |V_{sr^j}(H)|) + 2|V_{2k}(H)| \leq 1$.

Chapter 4

Orbit rigidity matrices

In this chapter we consider the rigidity matrix of a symmetric framework. Since symmetric frameworks may not be generic, introducing symmetry to a generic framework may drop the rank of its rigidity matrix.

For the forced symmetric case, B. Schulze and W.J. Whiteley introduced the orbit rigidity matrix, a matrix whose kernel is isomorphic to the space of fully-symmetric infinitesimal motions of the framework [61]. The orbit rigidity matrix is defined in terms of vertex and edge orbits, rather than vertices and edges. Hence, it loses all redundancies which are inherent of the symmetry. Moreover, it was defined in arbitrary dimension and for any symmetry group. Most interesting for this thesis, it does not assume that the group action is free on the vertex set of the underlying graph. Hence, it can be applied directly to our setting.

In 2000, engineers R.D. Kangwai and S. Guest observed that the rigidity matrix of a symmetric framework can be block-diagonalised in such a way that each block corresponds to an irreducible representation of the symmetry group which acts on the framework [19]. The result was later proved by B. Schulze [56]. The same result was proved independently by J.C. Owen and S. Power [45]. This was a breakthrough result, as it implied that the rigidity properties of a symmetric framework can be studied by considering each block separately, at least in theory.

In practice, the entries of each block of the rigidity matrix are not explicit.

However, the dimension, rank and kernel of each block can be deduced. Hence, we may define 'phase-symmetric' orbit rigidity matrices, one corresponding to each irreducible representation of the symmetry group. The orbit rigidity matrix corresponding to an irreducible representation ρ of the symmetry group has the same dimension, rank and nullity as the block in the (block-diagonalised) rigidity matrix corresponding to ρ. Moreover, as we will see, the orbit rigidity matrix corresponding to the trivial representation coincides with the orbit rigidity matrix defined in [61] for the study of forced symmetric rigidity. Phase-symmetric orbit rigidity matrices were defined in [56] for the case where the symmetry group is abelian and it acts freely on the joints of the framework. This chapter is aimed at extending the definition given in [56] to the case where the symmetry group need not act freely on the joints of the framework.

If the symmetry group is not abelian, e.g. if it is a dihedral group of order at least 6, then it has 2-dimensional irreducible representations. For such representations, it is still unclear how to define phase-symmetric orbit matrices in an explicit way which allows us to characterise the infinitesimal rigidity of symmetry-generic frameworks.

We structure the chapter as follows. In Section 4.1 we consider the forced symmetric case, and we present the orbit rigidity matrix which was introduced in [61], together with some key results. In Section 4.2 we give some background on representation theory, which is crucial to understand the block-diagonalisation of the rigidity matrix. In Section 4.3 we present the main result given in [19] and [49], which asserts that the rigidity matrix may be block-diagonalised. In Section 4.4, we study the dimension and kernel of each block in the rigidity matrix for the case where the symmetry group is cyclic, in order to construct phase-symmetric orbit rigidity matrices. In Section 4.5, we use the information gathered in Section 4.4 to introduce a generalisation of the phase-symmetric orbit matrices for cyclic groups given in [56], which does not assume that the symmetry group acts freely on the joints of the framework. We also prove some results which will be useful for the characterisation of infinitesimally rigid symmetric frameworks. In Section 4.6 we use the same ideas used in Sections 4.4 and 4.5 to define phase-symmetric orbit rigidity matrices for the dihedral group of order 4.

4.1 The orbit rigidity matrix

Definition 4.1.1. Let Γ be a group, $\tau : \Gamma \to O(\mathbb{R}^d)$ be an injective homomorphism and $(\tilde{G}, \tilde{p})$ be a $\tau(\Gamma)$-symmetric framework. Let (G, φ, ψ) be the Γ-gain graph of $\tilde{G}$, and define the map $p : V(G) \to \mathbb{R}^d$ by letting $p(v) = \tilde{p}(v^\star)$ for all $v \in V(G)$ with vertex orbit representative $v^\star \in V(\tilde{G})$. We say (G, φ, ψ, p) is the $\tau(\Gamma)$-*gain framework of* $(\tilde{G}, \tilde{p})$. If Γ is cyclic, we simply write (G, ψ, p).

Recall Definition 2.3.2 of $U(x)$ for $x \in \mathbb{R}^d$ and consider a $\tau(\Gamma)$-gain framework (G, φ, ψ, p). Notice that, for all $\gamma, \delta \in \Gamma$, we have

$$\begin{aligned}\tau(\gamma)F_{\tau(\delta)} &= \{x \in \mathbb{R}^d : x = \tau(\gamma)y, \tau(\delta)y = y\} = \{x \in \mathbb{R}^d : \tau(\delta)\tau(\gamma^{-1})x = \tau(\gamma^{-1})x\} \\ &= \{x \in \mathbb{R}^d : \tau(\gamma\delta\gamma^{-1})x = x\} = F_{\tau(\gamma\delta\gamma^{-1})},\end{aligned}$$

and so

$$\begin{aligned}U(\tau(\gamma)p(v)) &= \bigcap_{\substack{\tau(\delta)\in\tau(\Gamma) \\ \tau(\delta\gamma)p(v)=\tau(\gamma)p(v)}} F_{\tau(\delta)} = \bigcap_{\substack{\tau(\delta)\in\tau(\Gamma) \\ \tau(\gamma^{-1}\delta\gamma)p(v)=p(v)}} F_{\tau(\delta)} \\ &= \bigcap_{\substack{\tau(\delta)\in\tau(\Gamma) \\ \tau(\delta)p(v)=p(v)}} F_{\tau(\gamma\delta\gamma^{-1})} = \bigcap_{\substack{\tau(\delta)\in\tau(\Gamma) \\ \tau(\delta)p(v)=p(v)}} \tau(\gamma)F_{\tau(\delta)} = \tau(\gamma)U(p(v)). \qquad (4.1)\end{aligned}$$

We now present the orbit rigidity matrix which was introduced in [61]. Here, we use the terminology 'ρ_0-orbit rigidity matrix', in order to be consistent with a more general definition given in Section 4.5. We will later see that ρ_0 refers to the 'trivial representation' of Γ, i.e. the injective homomorphism which maps each element of Γ to the 1×1 identity matrix.

Definition 4.1.2. Let Γ be a group, $\tau : \Gamma \to O(\mathbb{R}^d)$ be an injective homomorphism, and $(\tilde{G}, \tilde{p})$ be a $\tau(\Gamma)$-symmetric framework with $\tau(\Gamma)$-gain framework (G, φ, ψ, p). For each $v \in V(G)$, choose a basis $\mathcal{B}_v^0$ of $U(p(v))$ and let M_v^0 be the matrix whose columns are the coordinate vectors of $\mathcal{B}_v^0$ relative to the canonical basis of $\mathbb{R}^d$. The *ρ_0-orbit rigidity matrix of* (G,φ,ψ,p) is a matrix denoted by $O_0(G,\varphi,\psi,p)$ which has exactly $|E(G)|$ rows and exactly $c_v^0 := \dim U(p(v))$ columns for each $v \in V(G)$. The row representing an edge $e=(u,v) \in E(G)$ in $O_0(G,\varphi,\psi,p)$ is

$$\begin{array}{ccccccc} & & u & & v & & \\ \Big(0 & \dots & [p(u)-\tau(\psi(e))p(v)]^T M_u^0 & \dots & [p(v)-\tau(\psi(e)^{-1})p(u)]^T M_v^0 & \dots & 0\Big) \end{array}$$

if $u \neq v$, and it is

$$\begin{array}{ccccc} & & u & & \\ \Big(0 & \dots & [2p(v)-\tau(\psi(e))p(v)-\tau(\psi(e)^{-1})p(v)]^T M_v^0 & \dots & 0\Big) \end{array}$$

otherwise. If Γ is cyclic, we abbreviate $O_0(G,\varphi,\psi,p)$ to $O_0(G,\psi,p)$.

The rank of the orbit rigidity matrix is independent of the choice of the bases $\mathcal{B}_v^0$ for $v \in V(G)$. Moreover, if $\tau(\Gamma)$ acts freely on the joints of $(\tilde{G},\tilde{p})$, then $O_0(G,\varphi,\psi,p)$ is an $|E(G)|\times d|V(G)|$ matrix, and each M_v^0 may be chosen to be the identity matrix.

Theorem 4.1.3 ([61], Theorem 6.1)**.** For a group Γ and an injective homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$, let $(\tilde{G},\tilde{p})$ be a $\tau(\Gamma)$-symmetric framework with $\tau(\Gamma)$-gain framework (G,φ,ψ,p). Then $\ker O_0(G,\varphi,\psi,p)$ is isomorphic to the space of fully-symmetric infinitesimal motions of $(\tilde{G},\tilde{p})$.

Since equivalent Γ-gain graphs have the same Γ-lifting, we expect that, for all equivalent Γ-gain graphs $(G,\varphi,\psi),(G',\varphi',\psi')$ and for some configurations p,p' of G in $\mathbb{R}^d$, $O_0(G,\varphi,\psi,p)$ and $O_0(G,\varphi',\psi',p')$ share the same rank. We show this in Propositions 4.1.4 and 4.1.5.

Proposition 4.1.4. Let (G,φ,ψ,p) be a $\tau(\Gamma)$-gain framework for some group Γ and some injective homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$. Let $v \in V(G), \gamma \in \Gamma$, and let (G,φ',ψ') be obtained from (G,φ,ψ) by applying a switching at v induced by γ. Then, there is a map $p' : V(G) \to \mathbb{R}^d$ such that $O_0(G,\varphi,\psi,p)$ and $O_0(G,\varphi',\psi',p')$ share the same rank.

Proof. Let $p' : V(G) \to \mathbb{R}^d$ be defined by letting $p'(v) = \tau(\gamma)p(v)$ and $p'(u) = p(u)$ for all other $u \in V(G)$. For each $u \in V(G)$, choose a basis $\mathcal{B}_u^0$ for $U(p(u))$ and let M_u^0 be the matrix whose columns are the coordinate vectors of $\mathcal{B}_u^0$ relative to the canonical basis of $\mathbb{R}^d$. Notice that for each $u \in V(G)$ with $u \neq v$, we may choose the same basis $\mathcal{B}_u^0$ for $U(p'(u))$. Moreover, by Equation (4.1), we may choose a basis $\mathcal{B}'_v$ for $U(p'(v))$ such that $\tau(\gamma)M_v^0$ is the matrix whose columns are the coordinate vectors of $\mathcal{B}'_v$ relative to the canonical basis of $\mathbb{R}^d$.

We use such bases to construct $O_0(G, \varphi, \psi, p)$ and $O_0(G, \varphi', \psi', p')$. Since all rows corresponding to edges not incident to v are the same in $O_0(G, \varphi, \psi, p)$ as they are in $O_0(G, \varphi', \psi', p')$, it suffices to consider the rows in $O_0(G, \varphi, \psi, p)$ and $O_0(G, \varphi', \psi', p')$ which represents edges incident to v. Moreover, we may assume that each edge incident to v is directed to v. So, consider an arbitrary edge $e = (u, v) \in E(G)$, and let r, r' be the rows representing e in $O_0(G, \varphi, \psi, p)$ and $O_0(G, \varphi', \psi', p')$, respectively. Let $\psi(e) = \delta$. If $u = v$, then $\psi'(e) = \gamma\delta\gamma^{-1}$ and, since $p'_v = \tau(\gamma)p_v$,

$$
\begin{aligned}
r' &= \begin{pmatrix} 0 & \dots & [2\tau(\gamma)p_v - \tau(\gamma\delta)p_v - \tau(\gamma\delta)p_v]^T\tau(\gamma)M_v^0 & \dots & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \dots & [2p_v - \tau(\delta)p_v - \tau(\delta)p_v]^T\tau(\gamma)^T\tau(\gamma)M_v^0 & \dots & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \dots & [2p_u - \tau(\delta)p_u - \tau(\delta)p_u]^T M_u^0 & \dots & 0 \end{pmatrix} = r,
\end{aligned}
$$

where the second to last equality follows from the fact that $\tau(\gamma)$ is orthogonal. If $u \neq v$, then $\psi'(e) = \delta\gamma^{-1}$ and, since $p'_v = \tau(\gamma)p_v$

$$
\begin{aligned}
r' &= \begin{pmatrix} 0 & \dots & [p_u - \tau(\delta)p_v]^T M_u^0 & \dots & [\tau(\gamma)p_v - \tau(\gamma\delta^{-1})p_u]^T\tau(\gamma)M_v^0 & \dots & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \dots & [p_u - \tau(\delta)p_v]^T M_u^0 & \dots & [p_v - \tau(\delta^{-1})p_u]^T\tau(\gamma)^T\tau(\gamma)M_v^0 & \dots & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \dots & [p_u - \tau(\delta)p_v]^T M_u^0 & \dots & [p_v - \tau(\delta^{-1})p_u]^T M_v^0 & \dots & 0 \end{pmatrix} = r.
\end{aligned}
$$

Since the rank of $O_0(G, \varphi, \psi, p)$ and $O_0(G, \varphi', \psi', p')$ is independent of the choice of bases for $U(p(u))$ and $U(p'(u))$ for all $u \in V(G)$, the result follows. □

Proposition 4.1.5. Let (G, φ, ψ, p) be a $\tau(\Gamma)$-gain framework for some group Γ and some injective homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$. Let $e = (u, v) \in E(G)$ for some $u, v \in V(G)$, and let $g \in \varphi(u), h \in \varphi(v)$. Further, let $\psi' : V(G) \to \Gamma$ be obtained by applying a switching at e induced by g and h. Then, $O_0(G, \varphi, \psi, p)$ and $O_0(G, \varphi, \psi', p)$ share the same rank.

Proof. For each $w \in V(G)$, choose a basis $\mathcal{B}_w^0$ for $U(p(w))$ and let M_w^0 be the matrix whose columns are the coordinate vectors of $\mathcal{B}_w^0$ relative to the canonical basis of $\mathbb{R}^d$. Use such bases to construct $O_0(G, \varphi, \psi, p)$ and $O_0(G, \varphi, \psi', p)$. Since $g \in \varphi(u)$, $\tau(g^{-1})p_u = p_u$ and so $\tau(g^{-1})q = q$ for all $q \in U(p(u))$. In particular, this implies that $\tau(g)^T M_u^0 = M_u^0$. Similarly, $\tau(h) M_v^0 = M_v^0$.

Let r and r' be the rows which represent e in $O_0(G, \varphi, \psi, p)$ and $O_0(G, \varphi, \psi', p)$, respectively. Since the rank of $O_0(G, \varphi, \psi, p)$ and $O_0(G, \varphi, \psi', p)$ is independent of the bases chosen for $U(p(w))$ for all $w \in V(G)$, it suffices to show that $r = r'$. If $u = v$, since $p_u = \tau(g)^{-1}p_u = \tau(h)p_u$ and $\tau(g), \tau(h)$ are orthogonal, we have

$$\begin{aligned} r' &= \begin{pmatrix} \dots & [2p_u - \tau(g\psi(e)h)p_u - \tau(g\psi(e)h)^{-1}p_u]^T M_u^0 & \dots \end{pmatrix} \\ &= \begin{pmatrix} \dots & [2p_u - \tau(\psi(e))p_u]^T \tau(g)^T M_u^0 - [\tau(\psi(e))^{-1}p_u]^T \tau(h) M_u^0 & \dots \end{pmatrix} \\ &= \begin{pmatrix} \dots & [2p_u - \tau(\psi(e))p_u - \tau(\psi(e))^{-1}p_u]^T M_u^0 & \dots \end{pmatrix} = r. \end{aligned}$$

Similarly, if $u \neq v$, then we have

$$\begin{aligned} r' &= \begin{pmatrix} \dots & [p_u - \tau(g\psi(e)h)p_v]^T M_u^0 & \dots & [p_v - \tau(g\psi(e)h)^{-1}p_u]^T M_v^0 & \dots \end{pmatrix} \\ &= \begin{pmatrix} \dots & p_u^T M_u^0 - [\tau(\psi(e))p_v]^T \tau(g)^T M_u^0 & \dots & [p_v - \tau(\psi(e))^{-1}p_u]^T \tau(h) M_v^0 & \dots \end{pmatrix} \\ &= \begin{pmatrix} \dots & p_u^T M_u^0 - [\tau(\psi(e))p_v]^T M_u^0 & \dots & [p_v - \tau(\psi(e))^{-1}p_u]^T M_v^0 & \dots \end{pmatrix} \\ &= \begin{pmatrix} \dots & [p_u - \tau(\psi(e))p_v]^T M_u^0 & \dots & [p_v - \tau(\psi(e))^{-1}p_u]^T M_v^0 & \dots \end{pmatrix} = r. \end{aligned}$$

This concludes the proof. □

Proposition 4.1.4 and 4.1.5 imply that, given a $\tau(\Gamma)$-symmetric framework $(\tilde{G}, \tilde{p})$ with two (equivalent) $\tau(\Gamma)$-gain frameworks $(G_1, \varphi_1, \psi_1, p_1)$ and $(G_2, \varphi_2, \psi_2, p_2)$, the orbit matrices $O_0(G_1, \varphi_1, \psi_1, p_1)$ and $O_0(G_1, \varphi_1, \psi_1, p_1)$ share the same dimension, rank and nullity. Hence, we may define the following.

Definition 4.1.6. Let $(\tilde{G}, \tilde{p})$ be a $\tau(\Gamma)$-symmetric framework for some group Γ and some injective homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$. Let (G, φ, ψ, p) be a $\tau(\Gamma)$-gain framework of $(\tilde{G}, \tilde{p})$. We say that (G, φ, ψ, p) is *fully-symmetrically isostatic* if $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic.

4.2 Group representation theory

In this section we give a review of basic representation theory. Unless stated otherwise, all notions in this section can be found in any representation theory book (see e.g. [21]). Throughout the section we use $Gl(\mathbb{C}^d)$ and $Gl(d, \mathbb{C})$ to denote the (isomorphic) general linear group of $\mathbb{C}^d$ and general linear group of degree d over $\mathbb{C}$, respectively.

Definition 4.2.1. Let $d \geq 1$ and Γ be a finite group.

(i) A *linear representation* of Γ on $\mathbb{C}^d$ is a homomorphism $\rho : \Gamma \to Gl(\mathbb{C}^d)$.

(ii) A *matrix representation* of Γ in $\mathbb{C}$ is a homomorphism $\rho : \Gamma \to Gl(d, \mathbb{C})$.

The *degree (or dimension)*, denoted by $\dim \rho$, of ρ is d.

Given a linear representation ρ of a group Γ on $\mathbb{C}^d$ and an ordered basis $\mathcal{B}$ for $\mathbb{C}^d$, the map $\rho_{\mathcal{B}} : \Gamma \to Gl(d, \mathbb{C})$ defined by letting $\rho_{\mathcal{B}}(\gamma) = [\rho(\gamma)]_{\mathcal{B}}$ for all $\gamma \in \Gamma$ is a d-dimensional matrix representation of Γ. We call $\rho_{\mathcal{B}}$ the *matrix representation corresponding to ρ with respect to $\mathcal{B}$*. Conversely, if ρ_M is a d-dimensional matrix representation of Γ, the map $\rho : \Gamma \to Gl(\mathbb{C}^d)$ defined by letting $\rho(\gamma)v = \rho_M(\gamma)v$ for all $\gamma \in \Gamma, v \in \mathbb{C}^d$, is a linear representation of Γ on $\mathbb{C}^d$. We call ρ the *linear representation corresponding to ρ_M*. Due to this bijective correspondence, all notions defined in terms of linear representations may be defined in terms of matrix representations, and vice versa (see, e.g., Definition 4.2.2). We therefore sometimes refer to linear representations and/or matrix representations simply as *representations*.

Definition 4.2.2. Given an integer $d \geq 1$ and a group Γ, let $\rho_1 : \Gamma \to Gl(d, \mathbb{C})$ and $\rho_2 : \Gamma \to Gl(\mathbb{C}^d)$ be a matrix and a linear representation of Γ, respectively.

(i) The *character of* ρ_1 is the map $\chi_1 : \Gamma \to \mathbb{C}$ defined by letting $\chi_1(\gamma) = \mathrm{Tr}(\rho_1(\gamma))$ for all $\gamma \in \Gamma$.

(ii) Given two bases $\mathcal{B}_1, \mathcal{B}_2$ for $\mathbb{C}^d$, the characters of the matrix representations corresponding to ρ_2 with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$ coincide. The *character of* ρ_2 is the map $\chi_2 : \Gamma \to \mathbb{F}$ defined by letting $\chi_2(\gamma) = \mathrm{Tr}(\rho_{\mathcal{B}}(\gamma))$ for all $\gamma \in \Gamma$, where $\rho_{\mathcal{B}}$ is the matrix representation corresponding to ρ_2 with respect to an arbitrary basis $\mathcal{B}$ for V.

We will sometimes define certain notions only in terms of linear (or matrix) representations. This idea may be used to extend all definitions to consider both linear and matrix representations.

If A is an $n \times m$ matrix and B is an $p \times q$ matrix, we use $A \otimes B$ to denote *Kronecker product* of A and B, i.e. the $np \times mq$ matrix obtained from B by replacing its $(i, j)^{th}$ entry a_{ij} with $a_{ij}A$. Given integers $n, m \geq 1$ and a group Γ with two matrix representations $\rho_1 : \Gamma \to Gl(n, \mathbb{C})$ and $\rho_2 : \Gamma \to Gl(m, \mathbb{C})$, the homomorphism $\rho_1 \otimes \rho_2 : \Gamma \to Gl(nm, \mathbb{C})$ defined by letting $(\rho_1 \otimes \rho_2)(\gamma) = \rho_1(\gamma) \otimes \rho_2(\gamma)$ for all $\gamma \in \Gamma$ is a matrix representation of Γ. In the following definitions, we let δ_{ij} be the Kronecker delta symbol.

Definition 4.2.3 ([49], Definition 3.1)**.** Let $d \geq 1$ be an integer, Γ be a group and $G = (V, E)$ be a Γ-symmetric graph with respect to an action θ.

Let $P_V : \Gamma \to Gl(|V|, \mathbb{C})$ be the permutation representation induced by the action of Γ on V, i.e. the representation which maps an element $\gamma \in \Gamma$ to the matrix $\left[\delta_{u,\theta(\gamma)(v)}\right]_{u,v}$, where u, v are elements of V. Given a homomorphism $\tau : \Gamma \to O(\mathbb{R}^d) \subseteq Gl(d, \mathbb{C})$, the *external representation of* Γ *(with respect to* G*,* θ *and* τ*)* is the homomorphism $\tau \otimes P_V : \Gamma \to Gl(d|V|, \mathbb{C})$.

The *internal representation of* Γ *(with respect to* G *and* θ*)* is the permutation representation $P_E : \Gamma \to Gl(|E|, \mathbb{C})$ induced by the action of Γ on E, i.e. the representation which maps an element $\gamma \in \Gamma$ to the matrix $\left[\delta_{e,\theta(\gamma)(f)}\right]_{e,f}$, where e, f are elements of E.

We will often drop the action θ when referring to internal and external representations of a group Γ.

Example 4.2.4. Let $\Gamma = \{\mathrm{id}, \gamma\}$, and $\tau : \Gamma \to$ be the homomorphism which maps γ to $\sigma = \mathrm{diag}(-1, 1)$, so that $\tau(\Gamma) = \mathcal{C}_s$ is the group which describes the reflection with reflection line $x = 0$. We let $(\tilde{G}, \tilde{p})$ be the $\mathcal{C}_s$-symmetric framework given in Figure 4.1, and we describe the internal representation of Γ with respect to $\tilde{G}$ and τ. So, define an 8×8 matrix A to be

$$A := \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $P_{V(\tilde{G})} : \Gamma \to Gl(8, \mathbb{C})$ is the representation which maps id to I_8 and γ to A. (Here, the j^{th} row/column of A corresponds to the vertex j.) Therefore, the external representation of Γ with respect to $\tilde{G}$ and τ is defined by letting $\tau \otimes P_{V(\tilde{G})}(\mathrm{id}) = I_{16}$, and by letting $\tau \otimes P_{V(\tilde{G})}(\gamma)$ be the matrix obtained from A by replacing each 0 with a 4×4 zero matrix and each 1 with $\mathrm{diag}(-1, 1)$.

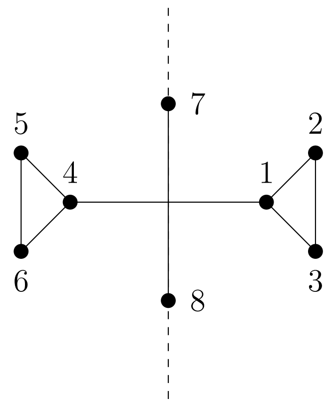


Figure 4.1: $\mathcal{C}_s$-symmetric framework.

Definition 4.2.5. Let Γ be a group of order k. The *regular representation of* Γ is the map $\rho_{\text{reg}} : \Gamma \to GL(k, \mathbb{C})$ which maps $\gamma \in \Gamma$ to the matrix $[\delta_{g,\gamma h}]_{g,h}$, where $g, h \in \Gamma$.

Most of the information about a group lies in its *irreducible representations.* Loosely speaking, a group representation is irreducible if it cannot be decomposed into representations of smaller dimension (in any non-trivial way). To formalise this concept, the notions of *equivalent representations* and *invariant subspaces* are required.

Definition 4.2.6. Let ρ_1, ρ_2 be linear representations of Γ on $\mathbb{C}^n, \mathbb{C}^m$, respectively. We say ρ_1 and ρ_2 are *equivalent* if there is an isomorphism $f : \mathbb{C}^n \to \mathbb{C}^m$ such that, for all $\gamma \in \Gamma$, $f \circ \rho_1(\gamma) \circ f^{-1} = \rho_2(\gamma)$.

Definition 4.2.7. For a group Γ, let ρ be a linear representation of Γ on $\mathbb{C}^d$. We say a subspace U of $\mathbb{C}^d$ is ρ*-invariant* if, for all $\gamma \in \Gamma$, $\rho(\gamma)(U) \subseteq U$. The restriction of ρ to U is a linear representation, and is called a *subrepresentation* of ρ. If the only ρ-invariant subspaces of $\mathbb{C}^d$ are the trivial space $\{0\}$ and $\mathbb{C}^d$ itself, then ρ is said to be *irreducible.*

A matrix representation ρ of a group Γ is irreducible if the only matrix representations ρ_1, ρ_2 of Γ in $\mathbb{C}$ such that $\rho = \rho_1 \oplus \rho_2$ are the *trivial representation* $\rho_0 : \Gamma \to Gl(1, \mathbb{R})$ defined by letting $\rho_0(\gamma) = (1)$ for all $\gamma \in \Gamma$, and ρ itself.

Example 4.2.8. A cyclic group $\Gamma = \langle \gamma \rangle$ of finite order $k \geq 1$ has exactly k pairwise non-equivalent irreducible representations $\rho_0, \ldots, \rho_{k-1}$ where, for $0 \leq j \leq k-1$, $\rho_j : \Gamma \to Gl(1, \mathbb{C})$ is defined by letting $\rho_j(\gamma) = \big(\exp(2\pi\sqrt{-1}j/k)\big)$ (see also Appendix B).

Example 4.2.9. Let $k \geq 2$ be an integer, and consider the group $\mathbb{D}_{2k}$. For all $1 \leq j \leq \lfloor (k-1)/2 \rfloor$, define a representation $\mu_j : \mathbb{D}_{2k} \to Gl(2, \mathbb{C})$ by letting

$$\mu_j(s) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \mu_j(r) = \begin{pmatrix} \exp(\frac{2\pi\sqrt{-1}j}{k}) & 0 \\ 0 & \exp(\frac{-2\pi\sqrt{-1}j}{k}) \end{pmatrix}.$$

Each μ_j is an irreducible representation of $\mathbb{D}_{2k}$. Also, the following hold:

(i) If k is odd, $\mathbb{D}_{2k}$ has 2 irreducible representations of order 1, the trivial representation and ρ_1 which maps s to (-1) and r to (1).

(ii) If k is even, $\mathbb{D}_{2k}$ has 4 irreducible representations of order 1: the trivial representation; the representation ρ_1 which maps s to (1) and r to (-1); the representation ρ_2 which maps both s and r to (-1); and the representation ρ_3 which maps s to (-1) and r to (1).

The group $\mathbb{D}_{2k}$ has no other irreducible representation (see also Appendix B).

Theorem 4.2.10. (Maschke's Theorem) Let Γ be a group and ρ be a linear representation of Γ on $\mathbb{C}^d$. Given a subrepresentation ρ_1 of ρ, there is another subrepresentation ρ_2 of ρ such that $\rho \simeq \rho_1 \oplus \rho_2$.

A direct consequence of Maschke's Theorem is that, given a group Γ with irreducible representations $\rho_1, \ldots, \rho_r$, any representation ρ is isomorphic to $\bigoplus_{j=1}^{r} a_j\rho_j$ for some $a_1, \ldots, a_r \in \mathbb{Z}^{\geq 0}$. For $0 \leq j \leq k$, a_j is known as the *multiplicity of* ρ_j *in* ρ.

Theorem 4.2.11. Let Γ be a group and let ρ_1 be a representation of Γ with character χ_1. Let ρ_2 be an irreducible representation of Γ with character χ_2. The multiplicity of ρ_2 in ρ_1 is

$$\langle \chi_2, \chi_1 \rangle = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \overline{\chi_2(\gamma)} \chi_1(\gamma).$$

Corollary 4.2.12. Let Γ be a group with irreducible representations $\rho_1, \ldots, \rho_r$. The regular representation of Γ can be written as $\rho_{\text{reg}} = \bigoplus_{j=1}^r (\dim \rho_j)\rho_j$.

Definition 4.2.13. Let Γ be a group, and ρ_1, ρ_2 be two matrix representations of Γ of order n, m, respectively. A matrix $M : \mathbb{R}^n \to \mathbb{R}^m$ is called a Γ-*linear map of* ρ_1 *and* ρ_2 if $M\rho_1(\gamma) = \rho_2(\gamma)M$ for all $\gamma \in \Gamma$. The set of all Γ-linear maps of ρ_1 and ρ_2 forms a linear space, which we denote $\mathrm{Hom}_\Gamma(\rho_1, \rho_2)$.

Lemma 4.2.14 (Schur's Lemma)**.** Let Γ be a group and ρ_1, ρ_2 be irreducible representations of Γ. If there is a Γ-linear map M of ρ_1 and ρ_2, then either $M = 0$ or M is an invertible square matrix. If $\rho_1 = \rho_2$, any Γ-linear map of ρ_1 and ρ_2 is a scalar multiple of the identity.

Often times, we will work with one-dimensional irreducible representations. Since such representations are simply 1×1 matrices, we sometimes treat them as scalars.

Remark 4.2.15. The theory of group representations is not restricted to considering complex numbers. All notions that we defined in this section, can defined analogously by substituting $\mathbb{C}$ with $\mathbb{R}$ (or any other field $\mathbb{F}$). Therefore, research in representation theory often distinguishes between *representations over* $\mathbb{R}$, and *representations over* $\mathbb{C}$. Note, for instance, that internal and external representations in Definition 4.2.3 can also be seen as representations over $\mathbb{R}$, as can the trivial representation of any group.

Clearly, each representation over $\mathbb{R}$ is also a representation over $\mathbb{C}$, since $\mathbb{R}$ is a subspace of $\mathbb{C}$. Given a group Γ, the irreducible representations of Γ over $\mathbb{R}$ may further decompose as representations over $\mathbb{C}$. For instance, $\mathbb{Z}_3$ has two irreducible representations over $\mathbb{R}$: the trivial representation, and $\tau : \mathbb{Z}_3 \to Gl(d, \mathbb{R})$ which maps the non-identity element of $\mathbb{Z}_3$ to $\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$. Through a complexification of the Euclidean plane, one can see that $\tau = \rho_1 \oplus \rho_2$ (see, e.g., Section 4.4). Therefore, working with representations over $\mathbb{C}$ gives us a further decomposition of certain representations over $\mathbb{R}$, and therefore allows us to study subspaces of $\mathbb{R}^d$ (for arbitrary $d \geq 1$) more in detail.

In previous literature, rigidity theorists have used irreducible representations over $\mathbb{C}$ to study symmetric frameworks in *Euclidean space*. Though this choice may seem counter-intuitive at first sight, it is in fact advantageous: for all abelian groups, all irreducible representations over $\mathbb{C}$ are one-dimensional. Since 1×1 matrices can be treated as scalars (through a slight abuse of notation), the theory simplifies significantly when considering irreducible representations over $\mathbb{C}$.

Therefore, in this Thesis we follow the past literature, and we assume that each representation is a representation over $\mathbb{C}$.

4.3 Block-diagonalisation of the rigidity matrix

Let $(\tilde{G}, \tilde{p})$ be a $\tau(\Gamma)$-generic framework for some finite group Γ and some injective homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$. A result in [49, 51] allows us to add structure to the rigidity matrix. We present it here as Theorem 4.3.1.

Theorem 4.3.1 ([49], Theorem 3.2)**.** For a group Γ and an injective homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$, let $(\tilde{G}, \tilde{p})$ be a $\tau(\Gamma)$-symmetric framework. The rigidity matrix $R(\tilde{G}, \tilde{p})$ is a Γ-linear map of $\tau \otimes P_{V(\tilde{G})}$ and $P_{E(\tilde{G})}$.

Let $\rho_1, \dots, \rho_r$ be the irreducible representations of Γ over $\mathbb{C}$. By Maschke's Theorem and Schur's Lemma, this implies that the rigidity matrix of $(\tilde{G}, \tilde{p})$ block-decomposes with respect to suitable symmetry-adapted bases, which subdivide the column space into the direct sum of the spaces $V^1, \dots, V^r$, where each V^j is the $(\tau \otimes P_{E(\tilde{V})})$-invariant subspace corresponding to an irreducible representation ρ_j of Γ. Similarly, the row space can be written as the direct sum of the spaces $W^1, \dots, W^r$, each W^j being the $P_{E(\tilde{G})}$-invariant subspace corresponding to ρ_j (for details, see [[49], Section 3.2] and [[51], Section 4.1.3]). Hence, we can write the rigidity matrix in the form

$$\tilde{R}(\tilde{G}, \tilde{p}) = \begin{pmatrix} \tilde{R}_1(\tilde{G}, \tilde{p}) & & 0 \\ & \ddots & \\ 0 & & \tilde{R}_r(\tilde{G}, \tilde{p}) \end{pmatrix},$$

where each $\tilde{R}_j(\tilde{G}, \tilde{p})$ is determined by the irreducible representation ρ_j. Since the multiplicity of each irreducible representation in the internal and external representation can be 0, we allow some blocks to have dimension 0. This decomposition into subspaces also allows us to define the following.

Definition 4.3.2. With the notation above, fix some $1 \leq j \leq r$. We say an infinitesimal motion $\tilde{m}$ of $(\tilde{G}, \tilde{p})$ is *symmetric with respect to* ρ_j (or simply ρ_j-*symmetric*) if it lies in V^j. We say $(\tilde{G}, \tilde{p})$ is ρ_j-*symmetrically infinitesimally rigid* (or simply ρ_j-*rigid*) if all of its ρ_j-symmetric infinitesimal motions are trivial. Otherwise, we say $(\tilde{G}, \tilde{p})$ is ρ_j-*symmetrically infinitesimally flexible* (or simply ρ_j-*flexible*). It is ρ_j-*symmetrically isostatic* (or simply ρ_j-*isostatic*) if it is ρ_j-rigid and removing any bar would make it ρ_j-flexible.

Due to the block-diagonalisation of the rigidity matrix, the infinitesimal motions of the framework $(\tilde{G}, \tilde{p})$ can be decomposed as a direct sum of its ρ_j-symmetric infinitesimal motions. Hence, $(\tilde{G}, \tilde{p})$ is infinitesimally rigid if and only if it is ρ_j-rigid for all $1 \leq j \leq r$. Though the entries of each block $\tilde{R}_j(\tilde{G}, \tilde{p})$ are tedious to compute, its dimension and kernel (and therefore its rank) are understood. For the case where $\tau(\Gamma)$ is abelian and it acts freely on the joints of $(\tilde{G}, \tilde{p})$, [56] constructed *phase-symmetric orbit matrices*, generalisations of the orbit rigidity matrix: for all $1 \leq j \leq r$, the ρ_j-orbit matrix is a matrix dependent on the $\tau(\Gamma)$-gain framework (G, φ, ψ, p) of $(\tilde{G}, \tilde{p})$ which shares the same dimension, rank and nullity as $\tilde{R}_j(\tilde{G}, \tilde{p})$. For cyclic groups and the dihedral group of order 4, we will generalise the notion of phase-symmetric orbit matrices defined in [56] to the case where the symmetry group need not act freely on the joints of the framework.

4.4 Dimensions of each block for cyclic groups

In order to construct phase-symmetric orbit rigidity matrices, it is crucial to understand the dimensions and kernels of each block in the rigidity matrix. We do so by considering the decomposition of the internal and external representations into irreducible representations. In this section we only consider finite cyclic groups. Throughout the section, let $k \geq 2$ be an integer and $\Gamma = \langle \gamma \rangle$ be a cyclic group of order k and let $\omega = \exp(2\pi\sqrt{-1}/k)$. We let $\tau : \Gamma \to O(\mathbb{R}^d)$ be an injective homomorphism and $(\tilde{G}, \tilde{p})$ be a $\tau(\Gamma)$-symmetric framework with $\tau(\Gamma)$-gain framework (G, ψ, p).

Recall that the irreducible representations of Γ over $\mathbb{C}$ are $\rho_0, \rho_1, \ldots, \rho_{k-1}$, where each ρ_j maps γ to the 1×1 matrix (ω^j). By Maschke's Theorem, Theorem 4.2.11 and Corollary 4.2.12, $\tau \otimes \rho_{\mathrm{reg}} \simeq \bigoplus_{j=0}^{k-1} d\rho_j$. By definition, $P_{V(\tilde{G})}$ is the direct sum of $|V_1(G)|$ copies of ρ_{reg} and $|V_k(G)|$ copies of the trivial representation and so

$$\tau \otimes P_{V(\tilde{G})} \simeq |V_1(G)|[\tau \otimes \rho_{\mathrm{reg}}] \oplus |V_k(G)|\tau \simeq \bigoplus_{j=0}^{k-1} d|V_1(G)|\rho_j \oplus |V_k(G)|\tau. \tag{4.2}$$

So, for each free vertex $v \in V_1(G)$, every block $\tilde{R}_j(\tilde{G}, \tilde{p})$ contains d columns. Where the columns corresponding to the fixed vertices lie depends on the homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$. For $d = 2$, we have the following result.

Proposition 4.4.1. Let $\Gamma = \langle \gamma \rangle$ be a cyclic group of finite order k and let $(\tilde{G}, \tilde{p})$ be a $\tau(\Gamma)$-symmetric framework for some injective homomorphism $\tau : \Gamma \to O(\mathbb{R}^2)$. Let (G, ψ) be the Γ-gain graph of $\tilde{G}$. The following statements hold:

(i) If $\tau(\Gamma) = \mathcal{C}_s$, $\tilde{R}_0(\tilde{G}, \tilde{p})$ and $\tilde{R}_1(\tilde{G}, \tilde{p})$ both have $2|V_1(G)| + |V_2(G)|$ columns.

(ii) If $\tau(\Gamma) = \mathcal{C}_2$, $\tilde{R}_0(\tilde{G}, \tilde{p})$ has $2|V_1(G)|$ columns and $\tilde{R}_1(\tilde{G}, \tilde{p})$ has $2|V(G)|$ columns.

(iii) If $\tau(\Gamma) = \mathcal{C}_k$ for some $k \geq 3$, $\tilde{R}_1(\tilde{G}, \tilde{p}), \tilde{R}_{k-1}(\tilde{G}, \tilde{p})$ have $2|V_1(G)| + |V_k(G)|$ columns, and all the other blocks have $2|V(G)|$ columns.

Proof. First, let $|\Gamma| = 2$ (so, $\tau(\Gamma)$ is either $\mathcal{C}_s$ or $\mathcal{C}_2$), and recall that Γ has irreducible representations ρ_0, ρ_1, where ρ_0 is the identity representation, and ρ_1 maps the non-identity element γ of Γ to (-1). Let $\tau_{\mathrm{ref}} : \Gamma \to O(\mathbb{R}^2)$ be the reflection homomorphism that maps γ to $\mathrm{diag}(-1, 1)$ and let $\tau_{\mathrm{rot}} : \Gamma \to O(\mathbb{R}^2)$ be the two-fold rotation homomorphism that maps γ to $\mathrm{diag}(-1, -1)$. It is easy to see that $\tau_{\mathrm{ref}} = \rho_0 \oplus \rho_1$ and so

$$\tau_{\mathrm{ref}} \otimes P_{V(\tilde{G})} \simeq \bigoplus_{j=0,1} 2|V_1(G)|\rho_j \oplus |V_2(G)|\tau_{\mathrm{ref}} \simeq \bigoplus_{j=0,1} (2|V_1(G)| + |V_2(G)|)\rho_j.$$

Similarly, we have $\tau_{\mathrm{rot}} = \rho_1 \oplus \rho_1$. So

$$\tau_{\mathrm{rot}} \otimes P_{V(\tilde{G})} \simeq \bigoplus_{j=0,1} 2|V_1(G)|\rho_j \oplus |V_2(G)|\tau_{\mathrm{rot}} \simeq (2|V_1(G)|)\rho_0 \oplus (2|V(G)|)\rho_1.$$

(i) and (ii) follow.

Now, let $|\Gamma| = k \geq 3$, so that $\tau(\Gamma) = \mathcal{C}_k$. Let $\alpha = 2\pi/k$. The standard k-fold rotation homomorphism $\tau : \Gamma \to O(\mathbb{R}^2)$ is given by

$$\tau(\gamma) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

We apply a complexification of the Euclidean plane through a change of basis from the canonical Euclidean basis $\mathcal{B}_1 = \left\{(1 \ \ 0)^T, (0 \ \ 1)^T\right\}$ to the basis $\mathcal{B}_2 = \left\{1/2\left(-1-\sqrt{-1} \ \ 1-\sqrt{-1}\right)^T, 1/2\left(-1+\sqrt{-1} \ \ 1+\sqrt{-1}\right)^T\right\}$, which has change of basis matrix

$$M_{1\to 2} = \frac{1}{2}\begin{pmatrix} -1-\sqrt{-1} & -1+\sqrt{-1} \\ 1-\sqrt{-1} & 1+\sqrt{-1} \end{pmatrix}.$$

Then, $\tau(\gamma)_{\mathcal{B}_2}$ is

$$M_{1\to 2}\tau(\gamma)_{\mathcal{B}_1}M_{1\to 2}^{-1} = \begin{pmatrix} \cos(\alpha) - \sqrt{-1}\sin(\alpha) & 0 \\ 0 & \cos(\alpha) + \sqrt{-1}\sin(\alpha) \end{pmatrix} = \begin{pmatrix} \overline{\omega} & 0 \\ 0 & \omega \end{pmatrix}.$$

It follows that $\tau = \rho_1 \oplus \rho_{k-1}$. Hence,

$$\tau \otimes P_{V(\tilde{G})} \simeq \textstyle\bigoplus_{j=0}^{k-1} 2|V_1(G)|\rho_j \oplus |V_k(G)|(\rho_1 \oplus \rho_{k-1}).$$

and (iii) holds. □

Equation (4.2) also gives us information on the ρ_j-symmetric infinitesimal motions of $(\tilde{G}, \tilde{p})$ for all $0 \leq j \leq k-1$. For all such j, the d-dimensional space

$$J_j^{(1)} = \left\{ \begin{pmatrix} \tau(\mathrm{id}) \\ \overline{\rho_j(\gamma)}\tau(\gamma^j) \\ \vdots \\ \overline{\rho_j(\gamma)}^{k-1}\tau(\gamma^{j(k-1)}) \end{pmatrix} x : x \in \mathbb{C}^d \right\}$$

is the $\tau\otimes\rho_{\text{reg}}$-invariant subspace of $\mathbb{C}^{dk}$ corresponding to ρ_j [56]. For all $0 \leq j \leq k-1$, define the subspace $J_j^{(k)}$ of $\mathbb{C}^d$ by

$$J_j^{(k)} = \left\{x \in \mathbb{C}^d : x = \overline{\rho_j(\delta)}\tau(\delta)x \text{ for all } \delta \in \Gamma\right\}.$$

Since Γ is abelian, for all $g, h \in \Gamma, 0 \leq j \leq k-1$ and all $x \in J_j^{(k)}$ we have

$$\overline{\rho_j(g)}\tau(g)(\tau(h)x) = \tau(h)(\overline{\rho_j(g)}\tau(g)x) = \tau(h)x,$$

and so $\tau(h)x \in J_j^{(k)}$. Therefore, for all $0 \leq j \leq k-1$, $J_j^{(k)}$ is the τ-invariant subspace of $\mathbb{C}^d$ corresponding to ρ_j. It follows that $J_j^{\text{mo}} := [\bigoplus_{v\in V_1(G)} J_j^{(1)}] \oplus [\bigoplus_{v\in V_k(G)} J_j^{(k)}]$ is the $\tau \otimes P_{V(\tilde{G})}$-invariant subspace of $\mathbb{C}^{dk|V_1(G)|+d|V_k(G)|} = \mathbb{C}^{d|V(\tilde{G})|}$ corresponding to ρ_j. By Definition 4.3.2, for all $0 \leq j \leq k-1$, an infinitesimal motion $\tilde{m}$ of $(\tilde{G}, \tilde{p})$ is ρ_j-symmetric if and only if it lies in J_j^{mo}.

By the definition of J_j^{mo}, an infinitesimal motion $\tilde{m}$ of $(\tilde{G}, \tilde{p})$ is ρ_j-symmetric if and only if satisfies the equation

$$\tilde{m}(\delta v) = \overline{\rho_j(\gamma)}\tau(\delta)\tilde{m}(v) \tag{4.3}$$

for each $0 \leq j \leq k-1, v \in V(\tilde{G})$ and $\delta \in \Gamma$. (See also [56] for the free action case.) For each $0 \leq j \leq k-1$, we will use the notation $\mathcal{M}_j(\tilde{G}, \tilde{p})$ to denote the space of ρ_j-symmetric infinitesimal motions of $(\tilde{G}, \tilde{p})$. Notice that $\mathcal{M}_0(\tilde{G}, \tilde{p})$ coincides with the space of fully-symmetric infinitesimal motions of $(\tilde{G}, \tilde{p})$.

Suppose $k = 2$, so that the irreducible representations of Γ are ρ_0 and ρ_1. Then, the ρ_1-symmetric infinitesimal motion vectors of $(\tilde{G}, \tilde{p})$ are reversed by the non-trivial element of the group. Hence, we call the ρ_1-symmetric infinitesimal motions of $(\tilde{G}, \tilde{p})$ *anti-symmetric*. We say $(\tilde{G}, \tilde{p})$ is *anti-symmetrically infinitesimally rigid/infinitesimally flexible/isostatic* if it is ρ_1-symmetrically infinitesimally rigid/infinitesimally flexible/isostatic.

Recall from Section 2.4 that $E(\tilde{G}) = E_1(\tilde{G}) \dot\cup E_2(\tilde{G})$, and that the only elements of Γ which can fix an edge are id and $\delta := \gamma^{k/2}$. (Note, if k is odd then Γ acts freely on $E(\tilde{G})$.) It was shown in [[56], Section 4.3] that for each edge orbit of $\tilde{G}$ of size $k/2$, all the blocks $\tilde{R}_j(\tilde{G}, \tilde{p})$ such that $\rho_j(\delta) = 1$ have one row, and all the other blocks have no rows. It was also shown that each block $\tilde{R}_j(\tilde{G}, \tilde{p})$ has a row for each edge orbit of $\tilde{G}$ of size k. This argument does not use the fact that the action is free on the vertex set. Since $\rho_j(\delta) = \rho_j(\gamma^{k/2}) = \exp(2\pi\sqrt{-1}kj/(2k)) = \exp(\pi\sqrt{-1}j)$, it follows that $\rho_j(\delta)$ is (1) if and only if j is even. Hence, for all even j, $\tilde{R}_j(\tilde{G}, \tilde{p})$ has $|E(G)|$ rows, and for all odd j, $\tilde{R}_j(\tilde{G}, \tilde{p})$ has 1 row for each edge orbit of $\tilde{G}$ of size k. Explicitly, we have

$$P_{E(\tilde{G})} = \left[\bigoplus_{j=0}^{k-1} |E(G)|\rho_j\right] \oplus \left[\bigoplus_{j=0}^{\lceil \frac{k-1}{2} \rceil} |E_1(G)|\rho_{2j}\right].$$

In Figure 4.2 we show realisations of non-free edges in $\mathcal{C}_s$-symmetric (a,b,c) and $\mathcal{C}_2$-symmetric frameworks (d). Figures (a,b,d) show anti-symmetric infinitesimal motions of such bars, whereas (c) shows a fully-symmetric infinitesimal motion. For any anti-symmetric velocity assignment $\tilde{m}$ to the vertices, the equation

$$(\tilde{m}(v) - \tilde{m}(u)) \cdot (\tilde{p}(v) - \tilde{p}(u)) = 0$$

always holds. Hence, the edge e constitutes no constraint for anti-symmetric infinitesimal rigidity. This is not the case for a fully-symmetric velocity assignment. (Note that in (c) the equation only holds because the velocities are parallel to the mirror line.) The edge in (d) can also be seen as a subgraph of a $\mathcal{C}_k$-symmetric framework $(\tilde{G}, \tilde{p})$, where $k \geq 4$ is even. Given an odd j with $2 \leq j \leq k-2$, (d) shows a ρ_j-symmetric infinitesimal motion of $(\tilde{G}, \tilde{p})$, restricted to that edge.

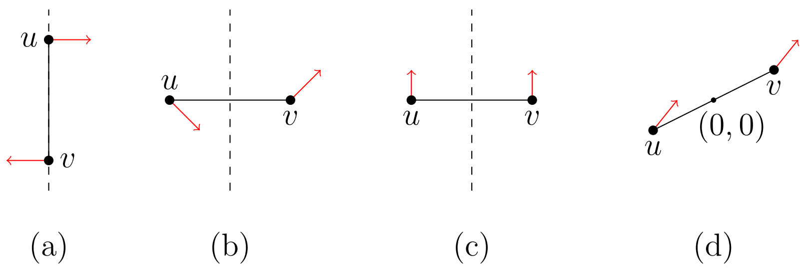


Figure 4.2: (a,b) Fixed bars of $\mathcal{C}_s$-symmetric frameworks with anti-symmetric infinitesimal motions. (c) Fixed bar of a $\mathcal{C}_s$-symmetric framework with a fully-symmetric infinitesimal motion. (d) Fixed bar of a $\mathcal{C}_2$-symmetric framework and an anti-symmetric infinitesimal motion of $(\tilde{G}, \tilde{p})$ applied to the bar.

4.5 Phase-symmetric orbit rigidity matrices for cyclic groups

Let $\Gamma = \langle\gamma\rangle$ be a cyclic group of finite order k and, for all $0 \leq j \leq k-1$, let ρ_j be the irreducible representation of Γ which maps γ to the 1×1 matrix (ω^j). Let $\tau : \Gamma \to O(\mathbb{R}^d)$ be an injective homomorphism and $(\tilde{G}, \tilde{p})$ be a $\tau(\Gamma)$-symmetric framework with $\tau(\Gamma)$-gain framework (G, ψ, p). Write $R(\tilde{G}, \tilde{p})$ as

$$R(\tilde{G}, \tilde{p}) = \begin{pmatrix} \tilde{R}_0(\tilde{G}, \tilde{p}) & & \\ & \ddots & \\ & & \tilde{R}_{k-1}(\tilde{G}, \tilde{p}) \end{pmatrix}$$

For each $0 \leq j \leq k-1$ we construct a matrix whose dimension coincides with the dimension of $R_j(\tilde{G}, \tilde{p})$, and whose kernel is isomorphic to the space of ρ_j-symmetric infinitesimal motions of $(\tilde{G}, \tilde{p})$.

For a cyclic group Γ of finite order and an injective homomorphism $\tau : \Gamma \to \mathbb{R}^d$, let $(\tilde{G}, \tilde{p})$ be a $\tau(\Gamma)$-symmetric framework with $\tau(\Gamma)$-gain framework (G, ψ, p). In [[56], Section 4.1.2], phase-symmetric orbit matrices were defined for the special case where $V(G) = V_1(G)$. We model our definition of ρ_j-orbit matrices based on this definition. For $v \in V(G)$ with representative $v^\star \in V(\tilde{G})$, we let $\mathcal{M}_j(p(v))$ denote the space $\{\tilde{m}(v^\star) : \tilde{m} \in \mathcal{M}_j(\tilde{G}, \tilde{p})\}$ and we let c_v^j denote the dimension of $\mathcal{M}_j(p(v))$.

Definition 4.5.1. With the same notation as above, fix some $0 \leq j \leq k-1$. For all $v \in V(G)$, choose a basis $\mathcal{B}_v^j$ for $\mathcal{M}_j(p(v))$ and let M_v^j be the matrix whose columns are the coordinate vectors of $\mathcal{B}_v^j$ relative to the canonical basis of $\mathbb{R}^d$. The *ρ_j-orbit rigidity matrix $O_j(G, \psi, p)$ of (G, ψ, p)* is a matrix with c_v^j columns for each $v \in V(G)$. If j is even $O_j(G, \psi, p)$ has $|E(G)|$ rows. Otherwise, it has $|E_1(G)|$ rows. Given an edge $e = (u, v) \in E(G)$, the row representing e in $O_j(G, \psi, p)$ is

$$\begin{array}{ccccc} & u & & v & \\ \Big(\dots & (p_u - \tau(\psi(e))p_v)^T M_u^j & \dots & \rho_j(\psi(e))(p_v - \tau(\psi^{-1}(e))p_u)^T M_v^j & \dots\Big) \end{array}$$

if $u \neq v$, and it is

$$\begin{array}{c} u \\ \begin{pmatrix} \dots & (p_u + \rho_j(\psi(e))p_u - \tau(\psi(e))p_u - \rho_j(\psi(e))\tau(\psi^{-1}(e))p_u)^T & \dots \end{pmatrix}. \end{array}$$

otherwise. If $c_u^j = 0$ (respectively, $c_v^j = 0$), then the columns corresponding to u (respectively, v) vanish.

Example 4.5.2. Let (G, ψ, p) be a $\mathcal{C}_s$-gain framework. Suppose that (G, ψ) is a single loop e at a vertex v. By the definition of gain graph, $\psi(e)$ is not the identity. Therefore, $\tau(\psi(e)) = \operatorname{diag}(-1, 1)$. It follows that $O_0(G, \psi, p) = \begin{pmatrix} 4x & 0 \end{pmatrix}$ and $O_1(G, \psi, p)$ is the empty matrix, where x denote the x-coordinate of $p(v)$.

For all $0 \leq j \leq k-1$, $\dim O_j(G, \psi, p) = \dim \tilde{R}_j(\tilde{G}, \tilde{p})$, as $O_j(G, \psi, p)$ contains exactly c_v^j columns for each vertex $v \in V(G)$, exactly one column for each $e \in E(G)$ if j is even and exactly one column for each $e \in E_1(G)$ if j is odd.

For all $0 \leq j \leq k-1$ the rank of $O_j(G, \psi, p)$ is independent of the choice of the bases $\mathcal{B}_v^j$ for $v \in V(G)$. If $v \in V_1(G)$, $\mathcal{B}_v^j$ can be chosen to be the identity matrix. Hence, if $V(G) = V_1(G)$, then Definition 4.5.1 coincides with the definition of the phase-symmetric orbit matrix $O_j(G, \psi, p)$ given in [56], for all $0 \leq j \leq k-1$. In the same paper, it was shown that $\ker O_j(G, \psi, p) \simeq \mathcal{M}_j(\tilde{G}, \tilde{p})$ for all such j (with $V(G) = V_1(G)$).

Note that for all $v \in V(G)$, an element $x \in \mathbb{R}^d$ lies in $U(p(v))$ if and only if $\tau(\gamma)x = x$ for all $\tau(\gamma) \in \tau(\Gamma)$ such that $\tau(\gamma)p(v) = p(v)$. By injectivity of p, this is equivalent to saying that $x \in U(p(v))$ if and only if $\tau(\gamma)x = x$ for all $\gamma \in \Gamma$ such that $\gamma \in \varphi(v)$. Therefore, by Equation (4.3), the spaces $U(p(v))$ and $\mathcal{M}_0(p(v))$ coincide. Hence, when $j = 0$, Definition 4.5.1 coincides with the definition of the orbit matrix given in [61] (see Definition 4.1.2). Since $\mathcal{M}_0(\tilde{G}, \tilde{p})$ coincides with the space of fully-symmetric infinitesimal motions of $(\tilde{G}, \tilde{p})$, Theorem 4.1.3 implies that $\ker O_0(G, \psi, p)$ and $\mathcal{M}_0(\tilde{G}, \tilde{p})$ are isomorphic. We now extend the results in [56] and [61] to the case where $V(G)$ need not be $V_1(G)$ and j need not be 0.

For the rest of the section, we make the following assumptions: we let Γ denote a cyclic group of finite order $k \geq 2$ with irreducible representations $\rho_0, \rho_1, \dots, \rho_{k-1}$; we let $\tau : \Gamma \to O(\mathbb{R}^2)$ be an injective homomorphism, and $(\tilde{G}, \tilde{p})$ be a $\tau(\Gamma)$-symmetric framework with $\tau(\Gamma)$-gain framework (G, ψ, p).

Let $\mathcal{O}$ denote the set of vertex orbit representatives of $\tilde{G}$. Define the subset $\mathcal{O}' \subseteq V_k(\tilde{G})$ to be $V_k(\tilde{G})$ if $\tau(\Gamma) = \mathcal{C}_k$ and $0 \leq j \leq k-2, j \neq 1$, and $\emptyset$ otherwise. For some fixed $u^\star \in \mathcal{O}$ and some free $v^\star \in \mathcal{O}$, let $u^\star v^\star \in E(\tilde{G})$. For each $\gamma \in \Gamma$, let (G, ψ_γ, p) be a Γ-gain framework of $(\tilde{G}, \tilde{p})$ such that the edge $e = (u, v)$ has gain γ under ψ_γ.

Lemma 4.5.3. Let $\gamma \in \Gamma$. A vector m lies in $\ker O_j(G, \psi_\gamma, p)$ if and only if $\tilde{m}' : \mathcal{O} \setminus \mathcal{O}' \to \mathbb{R}^2$ defined by $\tilde{m}'(w^\star) = M_w^j m(w)$ is the restriction of a ρ_j-symmetric infinitesimal motion $\tilde{m}$ of $(\tilde{G}, \tilde{p})$ to $\mathcal{O} \setminus \mathcal{O}'$.

Proof. Let $\tilde{m} : V(\tilde{G}) \to \mathbb{C}^2$ be defined by

$$\tilde{m}(\delta w^\star) = \begin{cases} \overline{\rho_j(\delta)}\tau(\delta)\tilde{m}'(w^\star) & \text{for all } w^\star \in \mathcal{O} \setminus \mathcal{O}', \delta \in \Gamma \\ (0 \quad 0)^T & \text{for all } w^\star \in \mathcal{O}', \delta \in \Gamma. \end{cases}$$

Clearly, $\tilde{m}'$ is a restriction of $\tilde{m}$ to $\mathcal{O} \setminus \mathcal{O}'$. Moreover, it is easy to see that $\tilde{m}$ is a ρ_j-symmetric infinitesimal motion of $(\tilde{G}, \tilde{p})$ if and only if it is an infinitesimal infinitesimal motion of $(\tilde{G}, \tilde{p})$, by Equation (4.3).

View m as a column vector. For each row r in $O_j(G, \psi_\gamma, p)$ that represents an edge $e = (u_1, u_2) \in E(G)$, we check that rm is zero if and only if $\tilde{m}$ satisfies the conditions of being an infinitesimal motion of the framework on the subgraph induced by the elements of the orbit e. Let $u_1^\star, u_2^\star$ be the vertex orbit representatives of u_1, u_2, respectively.

If $u_1, u_2 \in V_1(G)$, this has been shown in [[56], Section 4.1.2]. If $u_1, u_2 \in V_k(G)$, then $\tau(\Gamma) = \mathcal{C}_s$, since $\tau(\Gamma) = \mathcal{C}_k$ implies $|V_k(G)| \leq 1$ by definition of a framework. Since $\tilde{R}_1(\tilde{G}, \tilde{p})$ has no row corresponding to $u_1^\star u_2^\star$, we need only consider the case where $j = 0$. However, since $O_0(G, \psi, p)$ is the orbit rigidity matrix, this case was already proven in [61]. Hence, we may assume that $u_1 \in V_k(G), u_2 \in V_1(G)$. Without loss of generality, we consider the edge $e = (u, v)$, where $u^\star, v^\star$ are as defined in the statement. Note that the orbit of e is $\{u^\star v^\star_\delta : \delta \in \Gamma, v^\star_\delta = \delta v^\star\}$. Let r be the row of e in $O_j(G, \psi_\gamma, p)$.

The map $\tilde{m}$ satisfies the conditions of being an infinitesimal motion of the framework on the subgraph induced by the elements of the orbit e if and only if, for all $\delta \in \Gamma$

$$\langle \tilde{p}(u^\star) - \tilde{p}(\delta v^\star), \tilde{m}(u^\star) - \tilde{m}(\delta v^\star)\rangle = 0.$$

Since δ runs through all the elements of Γ, so does $\delta\gamma$. Hence, this is equivalent to saying that, for all $\delta \in \Gamma$,

$$\langle \tilde{p}(u^\star) - \tilde{p}(\delta\gamma v^\star), \tilde{m}(u^\star) - \tilde{m}(\delta\gamma v^\star)\rangle = 0.$$

Since u is fixed, $\tilde{m}(u^\star) = \tilde{m}(\delta u^\star)$ and so, by the definitions of $\tilde{m}$ and a $\tau(\Gamma)$-symmetric framework, this is equivalent to saying that for all $\delta \in \Gamma$

$$\left\langle p_u - \tau(\delta\gamma)p_v, \overline{\rho_j(\delta)}\tau(\delta)M_u^j m(u) - \overline{\rho_j(\delta\gamma)}\tau(\delta\gamma)m(v)\right\rangle = 0.$$

(If M_u^j has dimension 0, we ignore terms involving M_u^j.) This is equivalent to saying that for all $\delta \in \Gamma$

$$\overline{\rho_j(\delta)}\left(\left\langle p_u - \tau(\delta\gamma)p_v, \tau(\delta)M_u^j m(u)\right\rangle + \left\langle \tau(\delta\gamma)p_v - p_u, \overline{\rho_j(\gamma)}\tau(\delta\gamma)m(v)\right\rangle\right) = 0.$$

Notice that, since u is fixed, $p_u = \tau(\delta)p_u$ for all $\delta \in \Gamma$. Hence, since each $\tau(\delta)$ is an orthogonal matrix, we may remove the $\tau(\delta)$'s from the inner products, and multiply each equation by $\rho_j(\delta)$, to see that this set of equations holds if and only if

$$\left\langle p_u - \tau(\gamma)p_v, M_u^j m(u)\right\rangle + \left\langle \tau(\gamma)p_v - p_u, \overline{\rho_j(\gamma)}\tau(\gamma)m(v)\right\rangle = 0.$$

Similarly, since u is fixed, $p_u = \tau(\gamma)p_u$, and so we may remove $\tau(\gamma)$ from the second inner product, and move the factor of $\overline{\rho_j(\gamma)}$ in the second inner product to the left, to obtain the equivalent equation

$$\left\langle p_u - \tau(\gamma)p_v, M_u^j m(u)\right\rangle + \langle \rho_j(\gamma), m(v)\rangle = 0.$$

Again, since u is fixed, $p_u = \tau(\gamma^{-1})p_u$, and hence the above equation is equivalent to

$$[p_u - \tau(\gamma)p_v]^T M_u^j m(u) + \rho_j(\gamma)[p_v - \tau(\gamma^{-1})p_u]^T m(v) = 0,$$

i.e. $rm = 0$, as required. $\square$

Lemma 4.5.3 also shows that the nullity of $O_j(G, \psi, p)$ is independent of type II switchings. By the Rank-nullity Theorem, so is the rank of $O_j(G, \psi, p)$ (see Corollary 4.5.4).

Corollary 4.5.4. For some $u \in V_k(G)$ and some $v \in V_1(G)$, let $e = (u, v) \in E(G)$. For all $\gamma \in \Gamma$, let (G, ψ_γ, p) be a Γ-gain framework of $(\tilde{G}, \tilde{p})$ with $\psi_\gamma(e) = \gamma$. For all $0 \leq j \leq k-1, \gamma \in \Gamma$, $\operatorname{rank} O_j(G, \psi_\gamma, p) = \operatorname{rank} \tilde{R}_j(\tilde{G}, \tilde{p})$.

The rank (and hence nullity) of $O_j(G, \psi, p)$ is also independent of type I switchings for all $0 \leq j \leq k-1$ (see Proposition 4.5.5).

Proposition 4.5.5. Take an element $\gamma \in \Gamma$ and let ψ' be obtained from ψ by applying a switching at a vertex v with γ. Let $p' : V(G) \to \mathbb{R}^2$ be defined by $p'_v = \tau(\gamma)p_v$ and $p'_u = p_u$ for all $u \neq v$ in $V(G)$. Then for all $0 \leq j \leq k-1$,

$$\operatorname{rank}\ O_j(G, \psi, p) = \operatorname{rank}\ O_j(G, \psi', p').$$

Proof. For each $u \in V(G)$, choose a basis $\mathcal{B}_u^j$ for $\mathcal{M}_j(p(u))$ and let M_u^j be the matrix whose columns are the coordinate vectors of $\mathcal{B}_u^j$ with respect to the canonical basis of $\mathbb{R}^2$. Notice that for each $u \neq v$, we may choose the same basis $\mathcal{B}_u^j$ for $M_j(p'(u))$, and hence the same M_u^j. Moreover, we may choose a basis $\mathcal{B}'_v$ for $\mathcal{M}_j(p'(v))$ such that $\tau(\gamma)M_v^j$ is the matrix whose columns are the coordinate vectors for $\mathcal{B}'_v$ relative to the canonical basis of $\mathbb{R}^2$. We choose such bases to construct $O_j(G, \psi, p)$ and $O_j(G, \psi', p')$.

Clearly, any edge non-incident with v has the same row in $O_j(G, \psi, p)$ as in $O_j(G, \psi', p')$. Let $e := (u, v) \in E(G)$ with $\psi(e) = \delta$ for some $\delta \in \Gamma$ and, for $0 \leq j \leq k-1$, let r_j be the row representing e in $O_j(G, \psi', p')$. Notice that $\psi'(e) = \psi(e)\gamma^{-1} = \delta\gamma^{-1}$. Therefore,

$$\tau(\psi'(e))p'_v = \tau(\delta\gamma^{-1})\tau(\gamma)p_v = \tau(\delta)p_v,$$

and so, if $u \neq v$,

$$\begin{aligned} r_j &= \Big(\ldots \quad (p_u - \tau(\delta)p_v)^T M_u^j \quad \ldots \quad \rho_j(\delta\gamma^{-1})[\tau(\gamma)p_v - \tau((\delta\gamma^{-1})^{-1})p_u)]^T \quad \ldots\Big) \\ &= \Big(\ldots \quad (p_u - \tau(\delta)p_v)^T M_u^j \quad \ldots \quad \rho_j(\gamma^{-1})\rho_j(\delta)[\tau(\gamma)(p_v - \tau(\delta^{-1})p_u)]^T \quad \ldots\Big). \end{aligned}$$

(If $c_u^j = 0$, then there are no columns representing u.) If $u = v$, then $\psi'(\delta) = \psi(\delta)$, and so

$$\begin{aligned} r_j &= \Big(\ldots \quad [\tau(\gamma)p_v - \tau(\delta\gamma)p_v + \rho_j(\delta)\tau(\gamma)p_v - \rho_j(\delta)\tau(\delta^{-1}\gamma)p_v]^T \quad \ldots\Big) \\ &= \Big(\ldots \quad [p_v - \tau(\delta)p_v + \rho_j(\delta)p_v - \rho_j(\delta)\tau(\delta^{-1})p_v]^T\tau(\gamma)^T \quad \ldots\Big) \end{aligned}$$

Multiply each row representing a loop at v by the scalar $\rho_j(\gamma^{-1})$. Let s be the number of columns of $O_j(G,\psi,p)$, and $t, t+1$ be the columns representing v in $O_j(G,\psi,p)$. Define A to be the square matrix of dimension s such that the 2×2 submatrix with entries $A_{t,t}, A_{t,t+1}, A_{t+1,t}, A_{t+1,t+1}$ is $\rho_j(\gamma)\tau(\gamma)$, all other diagonal entries of A are 1, and all other entries 0. Then, $O_j(G,\psi',p')A = O_j(G,\psi,p)$. Since A is an orthogonal matrix, this implies that rank $O_j(G,\psi,p)$ = rank $O_j(G,\psi',p')$, as required. □

Hence, given a $\tau(\Gamma)$-symmetric framework $(\tilde{G},\tilde{p})$ with two (equivalent) $\tau(\Gamma)$-gain frameworks (G_1,ψ_1,p_1) and (G_2,ψ_2,p_2), $O_j(G_1,\psi_1,p_1)$ and $O_j(G_1,\psi_1,p_1)$ share the same rank, dimension and nullity. Therefore, we may define the following.

Definition 4.5.6. We say that (G,ψ,p) is *ρ_j-symmetrically isostatic* (or simply *ρ_j-isostatic*) if $(\tilde{G},\tilde{p})$ is ρ_j-isostatic.

Though the rank, nullity and dimension of $O_j(G,\psi,p)$ are independent of the choice of bases $\mathcal{B}_v^j$ for a vertex $v \in V(G)$, it is useful to set the matrices M_v^j which will be used throughout the thesis. For any free vertex $v \in V_1(G)$, we may always choose M_v^j to be the identity matrix. We conclude the section by studying the infinitesimal motions and phase-symmetric rigidity matrices for the specific cases where the symmetry groups are $\mathcal{C}_s, \mathcal{C}_2$ and $\mathcal{C}_k$ for $k \geq 3$, and hence setting the matrices M_v^j which will be used throughout the rest of the thesis. For all such groups, we also investigate the trivial infinitesimal motions of frameworks which manifest the symmetry, in order to understand which blocks of the rigidity matrix they correspond to.

4.5.1 Reflection group

Let $\Gamma = \{\mathrm{id}, \gamma\}$ be a cyclic group of order 2. Let $\tau : \Gamma \to O(\mathbb{R}^2)$ be the injective homomorphism which maps γ to $\mathrm{diag}(-1,1)$, and consider a $\tau(\Gamma)$-symmetric framework $(\tilde{G}, \tilde{p})$ with $\tau(\Gamma)$-gain framework (G, ψ, p). Recall that Γ has two irreducible representations: the trivial representation ρ_0 and the representation ρ_1 which maps γ to (-1). Fix some $j = 0, 1$ and let $\tilde{m}$ be a ρ_j-symmetric infinitesimal motion of $(\tilde{G}, \tilde{p})$. For all $v \in V(\tilde{G})$, we have

$$\tilde{m}(\gamma v) = (-1)^j \mathrm{diag}(-1,1)\tilde{m}(v).$$

In particular, if $v \in V_2(\tilde{G})$, then $\gamma v = v$, and so $\tilde{m}(v) = (-1)^j \mathrm{diag}(-1,1)\tilde{m}(v)$. If $j = 0$, $\tilde{m}(v) = \mathrm{diag}(-1,1)\tilde{m}(v)$ and so $\tilde{m}(v)$ has the form $(0 \;\; x)^T$ for some $x \in \mathbb{R}$. (Recall also Subsection 2.4.1.) If $j = 1$, then $\tilde{m}$ has the form $(x \;\; 0)^T$ for some $x \in \mathbb{R}$. As we would expect, the fully-symmetric infinitesimal motions of $(\tilde{G}, \tilde{p})$ assign velocity vectors on the symmetry line to all joints on the symmetry line (see Figure 4.3(a)), whereas the anti-symmetric infinitesimal motions of $(\tilde{G}, \tilde{p})$ assign vectors normal to the symmetry line to all joints on the symmetry line (see Figure 4.3(b)).

(a) (b)

Figure 4.3: $\mathcal{C}_s$-symmetric framework consisting of a single fixed joint. (a) shows a fully-symmetric infinitesimal motion of the framework and (b) shows an anti-symmetric infinitesimal motion of the framework.

Hence, given a fixed vertex $v \in V_2(G)$, M_v^0 can be chosen to be $(0 \;\; 1)^T$ and M_v^1 can be chosen to be $(1 \;\; 0)^T$. Throughout the rest of the thesis, we will assume that $M_v^0 = (0 \;\; 1)^T$ and $M_v^1 = (1 \;\; 0)^T$ for all $v \in V_2(G)$. Take an edge $e = (u, v) \in E(G)$

such that $u \in V_2(G)$. Let $p(u) = (0 \quad y_u)^T$ and $p(v) = (x_v \quad y_v)^T$. Since u is fixed, we may choose the label of e to be id (whether v is fixed or free). Then, the ρ_0-orbit rigidity matrix of $(\tilde{G}, \tilde{p})$ has row

$$\begin{array}{ccccc} & u & & v & \\ \big(\dots & y_u - y_v & \dots & x_v & y_v - y_u\big), \end{array}$$

where the column corresponding to v with entry x_v disappears if $v \in V_2(G)$. If e also has a column in $O_1(G, \psi, p)$, then $v \in V_1(G)$ and the row representing e in $O_1(G, \psi, p)$ is

$$\begin{array}{ccccc} & u & & v & \\ \big(\dots & -x_v & \dots & x_v & y_v - y_u\big). \end{array}$$

Using Equation (4.3), it is easy to see that the infinitesimal translation $m_{t_1} : V(\tilde{G}) \to \mathbb{R}^2$ given by $m_{t_1}(v) = (1 \quad 0)^T$ for all $v \in V(\tilde{G})$ is anti-symmetric, whereas the infinitesimal translation $m_{t_2} : V(\tilde{G}) \to \mathbb{R}^2$ given by $m_{t_2}(v) = (0 \quad 1)^T$ for all $v \in V(\tilde{G})$ is fully-symmetric. The infinitesimal rotation $m_r : V(\tilde{G}) \to \mathbb{R}^2$ given by $\tilde{m}(v) = (-y \quad x)^T$ for all $v \in V(\tilde{G})$ with $\tilde{p}(v) = (x \quad y)^T$ is anti-symmetric. Hence, we have the following (see also Figure 4.4).

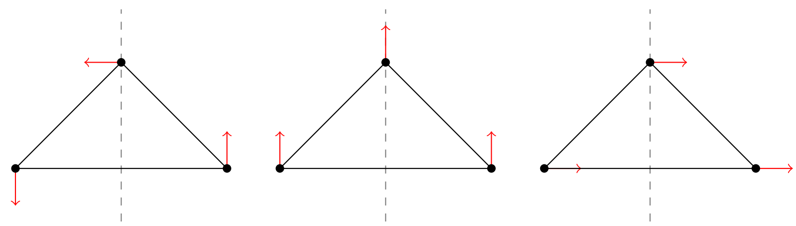


Figure 4.4: Trivial infinitesimal motions of a $\mathcal{C}_s$-symmetric framework $(\tilde{G}, \tilde{p})$. From left to right $(\tilde{G}, \tilde{p})$ is rotated, translated in the direction of the symmetry line and translated in the direction perpendicular to the symmetry line. The second infinitesimal motion maintains symmetry, the other two break the symmetry.

Lemma 4.5.7. For any $\mathcal{C}_s$-symmetric framework $(\tilde{G}, \tilde{p})$ with $\mathcal{C}_s$-gain framework (G, ψ, p), if $\tilde{p}(V(\tilde{G}))$ spans $\mathbb{R}^2$, then the following hold:

(i) $\operatorname{null} O_0(G, \psi, p) \geq 1$ with equality if and only if $(\tilde{G}, \tilde{p})$ is fully-symmetrically infinitesimally rigid.

(ii) $\operatorname{null} O_1(G,\psi,p) \geq 2$ with equality if and only if $(\tilde{G},\tilde{p})$ is anti-symmetrically infinitesimally rigid.

4.5.2 2-fold rotation group

Again, let $\Gamma = \{\mathrm{id},\gamma\}$ be a cyclic group of order 2. Let $\tau : \Gamma \to O(\mathbb{R}^2)$ be the injective homomorphism which maps γ to $-I_2$, and consider a $\tau(\Gamma)$-symmetric framework $(\tilde{G},\tilde{p})$ with $\tau(\Gamma)$-gain framework (G,ψ,p). Fix some $j = 0,1$ and let $\tilde{m}$ be a ρ_j-symmetric infinitesimal motion of $(\tilde{G},\tilde{p})$. For all $v \in V(G)$, we have

$$\tilde{m}(\gamma v) = (-1)^j(-I_2)\tilde{m}(v) = (-1)^{j+1}\tilde{m}(v).$$

In particular, if $v \in V_2(G)$, then $\gamma v = v$, and so $\tilde{m}(v) = (-1)^{j+1}\tilde{m}(v)$. This is always true if $j = 1$, whereas, when $j = 0$, it is only true if $\tilde{m}$ is the zero vector. (Recall also Subsection 2.4.2) Hence, any non-zero infinitesimal motion of a joint at the origin is anti-symmetric. Therefore, given a fixed vertex $v \in V_2(G)$, M_v^0 is the empty matrix and M_v^1 can be chosen to be I_2. Throughout the rest of the thesis, we let $M_v^1 = I_2$ for all $v \in V(G)$. Take an edge $e = (u,v) \in E(G)$ such that $u \in V_2(G)$, and notice that $v \in V_1(G)$, since $|V_2(G)| \leq 1$. Let $p(v) = (x \;\; y)^T$. Since u is fixed, we may choose the label of e to be id. Then, the row corresponding to e in the ρ_0-orbit rigidity matrix of $(\tilde{G},\tilde{p})$ is

$$\begin{array}{c} \qquad\quad v \\ \begin{pmatrix} \dots & x & y & \dots \end{pmatrix} \end{array},$$

whereas the row corresponding to e in the ρ_1-orbit rigidity matrix of $(\tilde{G},\tilde{p})$ is

$$\begin{array}{ccccccc} & \multicolumn{2}{c}{u} & & \multicolumn{2}{c}{v} & \\ \Big(\dots & -x & -y & \dots & x & y & \dots \Big) \end{array}.$$

The infinitesimal translations $m_{t_1} : V(G) \to \mathbb{R}^2$ and $m_{t_2} : V(G) \to \mathbb{R}^2$ given by $m_{t_1}(v) = (1 \;\; 0)^T$ and $m_{t_2}(v) = (0 \;\; 1)^T$ for all $v \in V(G)$, are both anti-symmetric. The infinitesimal rotation $m_r : V(G) \to \mathbb{R}^2$ given by $m_r(v) = (-y \;\; x)^T$ for all $v \in V(\tilde{G})$ with $\tilde{p}(v) = (x \;\; y)^T$ is fully-symmetric. Therefore, we have the following (see also Figure 4.5).

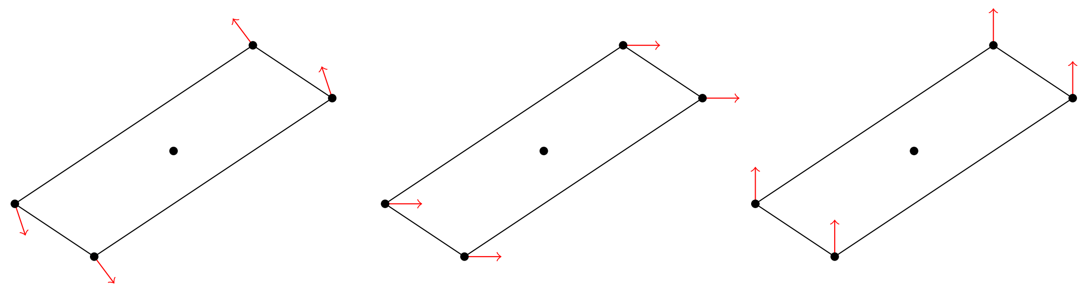


Figure 4.5: Trivial infinitesimal motions of a $\mathcal{C}_2$-symmetric framework $(\tilde{G}, \tilde{p})$. From left to right $(\tilde{G}, \tilde{p})$ is rotated around the origin, translated in the direction of the x-axis and translated in the direction of the y-axis. The first infinitesimal motion maintains symmetry, the other two break the symmetry.

Lemma 4.5.8. For any $\mathcal{C}_2$-symmetric framework $(\tilde{G}, \tilde{p})$ with $\mathcal{C}_2$-gain framework (G, ψ, p), if $\tilde{p}(V(\tilde{G}))$ spans $\mathbb{R}^2$, then the following hold:

(i) $\operatorname{null} O_0(G, \psi, p) \geq 1$ with equality if and only if $(\tilde{G}, \tilde{p})$ is fully-symmetrically infinitesimally rigid.

(ii) $\operatorname{null} O_1(G, \psi, p) \geq 2$ with equality if and only if $(\tilde{G}, \tilde{p})$ is anti-symmetrically infinitesimally rigid.

4.5.3 Rotation group of order 3 or higher

Now, let $\Gamma = \langle \gamma \rangle$ be a cyclic group of finite order $k \geq 3$. Let $\alpha = 2\pi/k, \omega = 2\pi\sqrt{-1}/k$ and let $\tau : \Gamma \to O(\mathbb{R}^2)$ be the injective homomorphism which maps γ to the matrix

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Consider a $\tau(\Gamma)$-symmetric framework $(\tilde{G}, \tilde{p})$ with $\tau(\Gamma)$-gain framework (G, ψ, p). Let $\rho_0, \rho_1, \ldots, \rho_{k-1}$ denote the irreducible representations of Γ as defined in Example 4.2.8. Fix $j = 1, k-1$ and let $\tilde{m}$ be a ρ_j-symmetric infinitesimal motion of $(\tilde{G}, \tilde{p})$. If there is a fixed vertex $v \in V_k(G)$, then for all $0 \leq t \leq k-1$ we have

$$\tilde{m}(v) = \overline{\omega}^{jt} R_\alpha^t \tilde{m}(v) \tag{4.4}$$

Let $\tilde{m}(v) = (m_1 \quad m_2)^T$. Equation (4.4) gives the system of two equations

$$\begin{cases} m_1 = \overline{\omega}^{jt}\left[\cos(\alpha t)m_1 - \sin(\alpha t)m_2\right] \\ m_2 = \overline{\omega}^{jt}\left[\sin(\alpha t)m_1 + \cos(\alpha t)m_2\right] \end{cases}$$

Noticing that $\overline{\omega}^t = \cos(\alpha t) - \sqrt{-1}\sin(\alpha t)$ and $\overline{\omega}^{(k-1)t} = \omega^t = \cos(\alpha t) + \sqrt{-1}\sin(\alpha t)$, and using the identity $\sin^2(\alpha) + \cos^2(\alpha) = 1$, these equations can be rearranged in the following way:

$$\begin{cases} m_1\left[-\sin^2(\alpha) \mp \sqrt{-1}\sin(\alpha t)\cos(\alpha t)\right] = m_2\left[\sin(\alpha t)\cos(\alpha t) \mp \sqrt{-1}\sin^2(\alpha t)\right] \\ m_2\left[\sin^2(\alpha) \pm \sqrt{-1}\sin(\alpha)\cos(\alpha)\right] = m_1\left[\sin(\alpha)\cos(\alpha) \mp \sqrt{-1}\sin^2(\alpha)\right] \end{cases}$$

(Here, $\pm, \mp$ depend on j: if $j = 1$, we take $+$ from $\pm$ and $-$ from $\mp$; if $j = k-1$, we take $-$ from $\pm$ and $+$ from $\mp$.) Taking out a factor of $-\sqrt{-1}$ on the left side of both equations, and dividing both sides of both equations by $\sin(\alpha)[\cos(\alpha) \mp \sqrt{-1}\sin(\alpha)]$ we obtain $m_2 = \mp\sqrt{-1}m_1$. Hence, M_v^1 can be chosen to be the matrix $(1 \quad -\sqrt{-1})^T$ and M_v^{k-1} can be chosen to be the matrix $(1 \quad \sqrt{-1})^T$. Throughout the rest of the thesis we let $M_v^1 = (1 \quad -\sqrt{-1})^T$ and $M_v^{k-1} = (1 \quad \sqrt{-1})^T$ if there is a vertex $v \in V_k(G)$. Since $\left\{(1 \quad -\sqrt{-1})^T, (1 \quad \sqrt{-1})^T\right\}$ forms a basis for $\mathbb{R}^2$ (under a complexification of the Euclidean space), it follows that M_v^j is the empty matrix for all $j \in \{0, 2, 3, \ldots, k-2\}$ and $v \in V_k(G)$. Let $e = (u, v) \in E(G)$ for some $u \in V_k(G), v \in V_1(G)$, and let $p(v) = (x \quad y)^T$. Since u is fixed, we may assume that $\psi(e) = \text{id}$. The row representing e in $O_1(G, \psi, p)$ is

$$\begin{array}{c} \begin{matrix} \quad & u & \quad & v & \quad & \quad \end{matrix} \\ \begin{pmatrix} \ldots & -x + \sqrt{-1}y & \ldots & x & y & \ldots \end{pmatrix}, \end{array}$$

whereas the row representing e in $O_{k-1}(G, \psi, p)$ is

$$\begin{array}{c} \begin{matrix} \quad & u & \quad & v & \quad & \quad \end{matrix} \\ \begin{pmatrix} \ldots & -x - \sqrt{-1}y & \ldots & x & y & \ldots \end{pmatrix}. \end{array}$$

For $j \in \{0, 2, \ldots, k-2\}$, the row representing e in $O_j(G, \psi, p)$ is obtained from the row representing e in $O_1(G, \psi, p)$ (equivalently, in $O_{k-1}(G, \psi, p)$) by removing the column corresponding to u.

Using Equation (4.3), it is easy to see that the infinitesimal translations $m_{t_1}: V(G) \to \mathbb{R}^2$ given by $m_{t_1}(v) = (1 \quad -\sqrt{-1})^T$ for all $v \in V(\tilde{G})$ is ρ_1-symmetric, whereas the infinitesimal translation $m_{t_2}: V(G) \to \mathbb{R}^2$ given by $m_{t_2}(v) = (1 \quad \sqrt{-1})^T$ for all $v \in V(\tilde{G})$ is ρ_{k-1}-symmetric. The infinitesimal rotation $m_r : V(G) \to \mathbb{R}^2$ given by $m_r(v) = (-y \quad x)^T$ for all $v \in V(\tilde{G})$ with $\tilde{p}(v) = (x \quad y)^T$ is fully-symmetric. Therefore, we have the following (see also Figure 4.6).

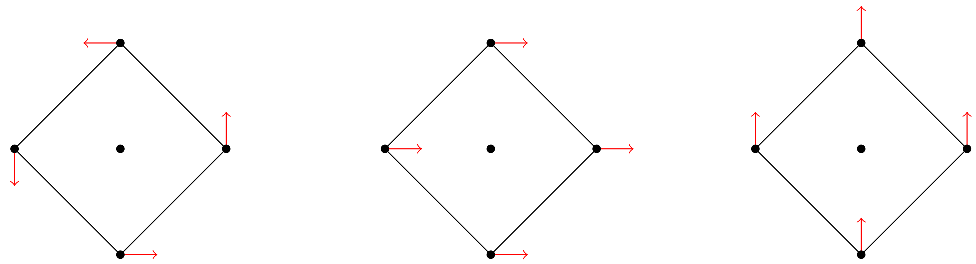


Figure 4.6: Trivial infinitesimal motions of a $\mathcal{C}_4$-symmetric framework $(\tilde{G}, \tilde{p})$. From left to right $(\tilde{G}, \tilde{p})$ is rotated around the origin, translated in the direction of the x-axis and translated in the direction of the y-axis. The first infinitesimal motion maintains symmetry, the other two break the symmetry.

Lemma 4.5.9. Let $k \geq 3$ be an integer. For any $\mathcal{C}_k$-symmetric framework $(\tilde{G}, \tilde{p})$ with $\mathcal{C}_k$-gain framework (G, ψ, p), if $\tilde{p}(V(\tilde{G}))$ spans $\mathbb{R}^2$, then the following hold:

(i) For $j = 0, 1, k-1$, $\operatorname{null} O_j(G, \psi, p) \geq 1$ with equality if and only if $(\tilde{G}, \tilde{p})$ is ρ_j-rigid.

(ii) If $k \geq 4, 2 \leq j \leq k-2$, $(\tilde{G}, \tilde{p})$ is ρ_j-rigid if and only if $\operatorname{null} O_j(G, \psi, p) = 0$.

4.6 The dihedral group of order 4

The blocks of the rigidity matrices are easier to understand for cyclic groups simply because all irreducible representations of cyclic groups over the complex numbers are 1-dimensional. In fact, the arguments in Sections 4.4 and 4.5 can be generalised to all abelian groups, as the irreducible representations of abelian groups are all

1-dimensional. This was done in [56] for the case where the symmetry group acts freely on the joints of the framework. Since the only non-cyclic abelian point group in $\mathbb{R}^2$ is $\mathcal{C}_{2v}$, we only consider $\mathcal{C}_{2v}$-symmetric frameworks in this section. For such frameworks, we extend the arguments in Sections 4.4 and 4.5, and define phase-symmetric orbit rigidity matrices. So, let $\mathbb{D}_4 = \{\mathrm{id}, s, r, sr\}$. $\mathbb{D}_4$ has four pair-wise non equivalent irreducible representations $\rho_0, \rho_1, \rho_2, \rho_3$ over $\mathbb{C}$, defined as follows: ρ_0 is the trivial representation; ρ_1 maps id, s to (1) and r, sr to (-1); ρ_2 maps id, sr to (1) and s, r to (-1); ρ_3 maps id, r to (1) and s, sr to (-1).

Let $\tau : \mathbb{D}_4 \to O(\mathbb{R}^2)$ be the injective homomorphism which maps s to $\mathrm{diag}(-1,1)$ and r to $-I_2$, so that $\tau(\mathbb{D}_4) = \mathcal{C}_{2v}$. Use τ to define a $\mathcal{C}_{2v}$-symmetric framework $(\tilde{G}, \tilde{p})$ with $\mathcal{C}_{2v}$-gain framework (G, φ, ψ, p). By Theorem 4.3.1, $R(\tilde{G}, \tilde{p})$ may be written as

$$\begin{pmatrix} \tilde{R}_0(\tilde{G},\tilde{p}) & 0 & 0 & 0 \\ 0 & \tilde{R}_1(\tilde{G},\tilde{p}) & 0 & 0 \\ 0 & 0 & \tilde{R}_2(\tilde{G},\tilde{p}) & 0 \\ 0 & 0 & 0 & \tilde{R}_3(\tilde{G},\tilde{p}) \end{pmatrix}$$

where each matrix $\tilde{R}_j(\tilde{G}, \tilde{p})$ corresponds to an irreducible representation ρ_j of $\mathbb{D}_4$.

Definition 4.6.1. Define $\tau_s : \mathbb{D}_4 \to O(\mathbb{R}^2)$ to be the homomorphism which maps s to I_2 and r to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Define $\tau_{sr} : \mathbb{D}_4 \to O(\mathbb{R}^2)$ to be the homomorphism which maps s and r to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Let $V_s := \{v \in V_2(G) : \varphi(v) = \{\mathrm{id}, s\}\}$ and $V_{sr} := \{v \in V_2(G) : \varphi(v) = \{\mathrm{id}, sr\}\}$. (So $V_s = V_s(G - V_4(G))$ and $V_{sr} = V_{sr}(G - V_4(G))$, where $V_s(G)$ and $V_{sr}(G)$ are as given in Subsection 3.3.3.) Then $P_{V(\tilde{G})}$ is the direct sum of $|V_1(G)|$ copies of ρ_{reg}, of $|V_s|$ copies of τ_s, of $|V_{sr}|$ copies of τ_{sr}, and of $|V_4(G)|$ copies of the trivial representation. Hence,

$$\tau \otimes P_{V(\tilde{G})} \simeq |V_1(G)|\,[\tau \otimes \rho_{\mathrm{reg}}] \oplus |V_s|\,[\tau \otimes \tau_s] \oplus |V_{sr}|\,[\tau \otimes \tau_{sr}] \oplus |V_4(G)|\tau.$$

Recall that $\tau \otimes \rho_{\text{reg}} \simeq \bigoplus_{j=0}^{3} 2\rho_j$ and notice that $\tau \simeq \rho_2 \oplus \rho_1$. We apply a change of basis of $\mathbb{R}^2$ from $\mathcal{B}_1 = \left\{(1\ \ 0)^T, (0\ \ 1)^T\right\}$ to $\mathcal{B}_2 = \left\{1/\sqrt{2}(-1\ \ 1)^T, 1/\sqrt{2}(1\ \ 1)^T\right\}$ with the change of basis matrix

$$M_{1\to 2} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},$$

so that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\mathcal{B}_2} = M_{1\to 2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M_{1\to 2}^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, $\tau_s \simeq \rho_1 \oplus \rho_0$ and $\tau_{sr} \simeq \rho_2 \oplus \rho_0$. Since $\rho_0, \rho_1, \rho_2, \rho_3$ are 1-dimensional, they can be treated as scalars and so

$$\tau \otimes \tau_s \simeq (\rho_2 \oplus \rho_1) \otimes (\rho_1 \oplus \rho_0) = \rho_2\rho_1 \oplus \rho_1\rho_1 \oplus \rho_2\rho_0 \oplus \rho_1\rho_0 = \rho_3 \oplus \rho_0 \oplus \rho_2 \oplus \rho_1$$

and

$$\tau \otimes \tau_{sr} \simeq (\rho_2 \oplus \rho_1) \otimes (\rho_2 \oplus \rho_0) = \rho_2\rho_2 \oplus \rho_1\rho_2 \oplus \rho_2\rho_0 \oplus \rho_1\rho_0 = \rho_0 \oplus \rho_3 \oplus \rho_2 \oplus \rho_1.$$

Therefore

$$\begin{aligned} \tau \otimes P_{V(\tilde{G})} &\simeq \left[\bigoplus_{j=0}^{3} 2|V_1(G)|\rho_j\right] \oplus \left[\bigoplus_{j=0}^{3} |V_s|\rho_j\right] \oplus \left[\bigoplus_{j=0}^{3} |V_{sr}|\rho_j\right] \oplus |V_4(G)|[\rho_2 \oplus \rho_1] \\ &\simeq \left[\bigoplus_{j=0}^{3} 2|V_1(G)| + |V_2(G)|\right] \oplus |V_4(G)|[\rho_1 \oplus \rho_2] \end{aligned}$$

It follows that each block of the rigidity matrix has two columns corresponding to each free vertex of G and 1 column for each semi-free vertex of G and that, if G has a fixed vertex v, then the columns corresponding to v split evenly between the block $\tilde{R}_1(\tilde{G}, \tilde{p})$ and $\tilde{R}_2(\tilde{G}, \tilde{p})$. For all $0 \le j \le 3$, the subspace

$$J_j^{(1)} = \left\{ \begin{pmatrix} \tau(\text{id}) \\ \rho_j(s)\tau(s) \\ \rho_j(r)\tau(r) \\ \rho_j(sr)\tau(sr) \end{pmatrix} x : x \in \mathbb{C}^2 \right\}$$

is the $\tau \otimes \rho_{\mathrm{reg}}$-invariant subspace of $\mathbb{C}^8$ corresponding to ρ_j [56]. Moreover, in the same way as we did in Section 4.4, we can see that for all $0 \leq j \leq 3$ the space $J_j^{(4)}$ defined by $\{x : \mathbb{C}^2 : x = \rho_j(s)\tau(s)x = \rho_j(r)\tau(r)x = \rho_j(sr)\tau(sr)x\}$ is the τ-invariant subspace of $\mathbb{C}^2$ corresponding to ρ_j, since $\mathbb{D}_4$ is abelian. Define the subspace $J_j^{(s)}$ of $\mathbb{C}^4$ to be

$$J_j^{(s)} = \left\{ \begin{pmatrix} \tau(\mathrm{id}) \\ \rho_j(r)\tau(r) \end{pmatrix} x : x \in \mathbb{C}^2, x = \rho_j(s)\tau(s)x \right\}.$$

For $0 \leq j \leq 3$, let $x \in J_j^{(s)}$. Some simple calculations show the following:

(i) If $j = 0$, then $x = (0 \quad a \quad 0 \quad -a)^T$ for some $a \in \mathbb{C}$, and so $(\tau \otimes \tau_s)(\gamma)x = x \in J_0^{(s)}$ for all $\gamma \in \mathbb{D}_4$.

(ii) If $j = 1$, then $x = (0 \quad a \quad 0 \quad a)^T$ for some $a \in \mathbb{C}$, and so $(\tau \otimes \tau_s)(\mathrm{id})x = (\tau \otimes \tau_s)(s)x = x \in J_1^{(s)}$ and $(\tau \otimes \tau_s)(sr)x = (\tau \otimes \tau_s)(r)x = -x \in J_1^{(s)}$.

(iii) If $j = 2$, then $x = (a \quad 0 \quad a \quad 0)^T$ for some $a \in \mathbb{C}$, and so $(\tau \otimes \tau_s)(\mathrm{id})x = (\tau \otimes \tau_s)(sr)x = x \in J_2^{(s)}$ and $(\tau \otimes \tau_s)(s)x = (\tau \otimes \tau_s)(r)x = -x \in J_2^{(s)}$.

(iv) If $j = 3$, then $x = (a \quad 0 \quad -a \quad 0)^T$ for some $a \in \mathbb{C}$, and so $(\tau \otimes \tau_s)(\mathrm{id})x = (\tau \otimes \tau_s)(r)x = x \in J_3^{(s)}$ and $(\tau \otimes \tau_s)(s)x = (\tau \otimes \tau_s)(sr)x = -x \in J_3^{(s)}$.

In all such cases, $J_j^{(s)}$ is the $\tau \otimes \tau_s$-invariant subspace of $\mathbb{C}^4$ corresponding to ρ_j. Similarly, it is easy to see that

$$J_j^{(sr)} = \left\{ \begin{pmatrix} \tau(\mathrm{id}) \\ \rho_j(r)\tau(r) \end{pmatrix} x : x \in \mathbb{C}^2, x = \rho_j(sr)\tau(sr)x \right\}$$

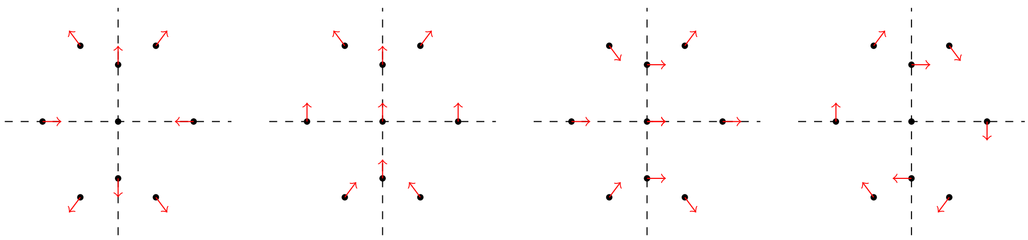
is the $\tau \otimes \tau_{sr}$-invariant subspace of $\mathbb{C}^4$ corresponding to ρ_j. Hence, the space

$$J_j^{\mathrm{mo}} := [\bigoplus_{v \in V_1(G)} J_j^{(1)}] \oplus [\bigoplus_{v \in V_s} J_j^{(s)}] \oplus [\bigoplus_{v \in V_{sr}} J_j^{(sr)}] \oplus [\bigoplus_{v \in V_4(G)} J_j^{(4)}]$$

is a $\tau \otimes P_{V(\tilde{G})}$-invariant subspace of $\mathbb{C}^{V(\tilde{G})}$. Therefore, in a similar way as for the cyclic group case, we have that an infinitesimal motion $\tilde{m}$ of $(\tilde{G}, \tilde{p})$ is ρ_j-symmetric if and only if for all $v \in V(G), \gamma \in \Gamma$, it satisfies

$$\tilde{m}(\gamma v) = \rho_j(\gamma)\tau(\gamma)\tilde{m}(v). \tag{4.5}$$

For $0 \leq j \leq 3$, we use $\mathcal{M}_j(\tilde{G},\tilde{p})$ to denote the space of ρ_j-symmetric infinitesimal motions of $(\tilde{G},\tilde{p})$. Similarly as with the case of cyclic groups, $\mathcal{M}_0(\tilde{G},\tilde{p})$ coincides with the space of fully-symmetric infinitesimal motions of $(\tilde{G},\tilde{p})$. (For a visual representation of ρ_j-symmetric infinitesimal motions on free, semi-free and fixed joints, see Figure 4.7.)



(a) (b) (c) (d)

Figure 4.7: Different infinitesimal motions applied to the same $\mathcal{C}_{2v}$-symmetric framework. The infinitesimal motions in (a,b,c,d) are, respectively, ρ_0-symmetric, ρ_1-symmetric, ρ_2-symmetric and ρ_3-symmetric.

Using Equation (4.5), it is easy to see that the infinitesimal translation $m_{t_1} : V(\tilde{G}) \to \mathbb{R}^2$ given by $m_{t_1}(v) = (1 \;\; 0)^T$ for all $v \in V(\tilde{G})$ is ρ_2-symmetric, whereas the infinitesimal translation $m_{t_2} : V(\tilde{G}) \to \mathbb{R}^2$ given by $m_{t_2}(v) = (0 \;\; 1)^T$ for all $v \in V(\tilde{G})$ is ρ_1-symmetric. The infinitesimal rotation $m_r : V(\tilde{G}) \to \mathbb{R}^2$ given by $\tilde{m}(v) = (-y \;\; x)^T$ for all $v \in V(\tilde{G})$ with $\tilde{p}(v) = (x \;\; y)^T$ is ρ_3-symmetric. Hence, we have the following.

Lemma 4.6.2. For any $\mathcal{C}_{2v}$-symmetric framework $(\tilde{G},\tilde{p})$ with $\mathcal{C}_{2v}$-gain framework (G,φ,ψ,p), the following hold:

(i) For $0 < j \leq 3$, $\operatorname{null} O_j(G,\varphi,\psi,p) \geq 1$ with equality if and only if $(\tilde{G},\tilde{p})$ is ρ_j-rigid.

(ii) The framework $(\tilde{G},\tilde{p})$ is fully-symmetrically infinitesimally rigid if and only $\operatorname{null} O_0(G,\varphi,\psi,p) = 0$.

Let $(\tilde{G},\tilde{p})$ be a $\mathcal{C}_{2v}$-symmetric framework with $\mathcal{C}_{2v}$-gain framework (G,φ,ψ,p). We now consider the edge sets of $\tilde{G}$ and of G.

Recall from Subsection 2.4.3 that $E(\tilde{G}) = E_1(\tilde{G}) \dot{\cup} E_2(\tilde{G}) \dot{\cup} E_4(\tilde{G})$. Define the subsets $E_s(\tilde{G}) = \{e \in E_2(\tilde{G}) : se = e\}$, $E_{sr}(\tilde{G}) = \{e \in E(\tilde{G}) : (sr)e = e\}$ and $E_r(\tilde{G}) = \{e \in E_2(\tilde{G}) : re = e\}$ of $V_2(\tilde{G})$. For $i = 1, 2, 4$, let $E_i(G)$ denote $\{e = (u, v) \in E(G) : u^\star v^\star_{\psi(e)} \in E_i(\tilde{G})\}$, where $u^\star, v^\star$ denote the vertex orbit representatives of u, v, respectively, and $v^\star_{\psi(e)}$ denotes $\psi(e)v^\star$. Further, for $\gamma \in \{s, sr, r\}$, define the set $E_\gamma := \{e = (u, v) \in E_2(G) : u^\star v^\star_{\psi(e)} \in E_\gamma(\tilde{G})\}$. It was shown in [[56], Section 4.3] that for each $e \in E_\gamma, \gamma \in \{s, sr, r\}$, all the blocks $\tilde{R}_j(\tilde{G}, \tilde{p})$ such that $\rho_j(\gamma) = (1)$ have one row, and all the other blocks have no rows. Hence, for all $e \in E_\gamma$, $\tilde{R}_0(\tilde{G}, \tilde{p})$ and $\tilde{R}_j(\tilde{G}, \tilde{p})$ have one row, where $j = 1$ if $\gamma = s$, $j = 2$ if $\gamma = sr$, and $j = 3$ if $\gamma = r$. Further, it follows from Definition 4.1.2 that $\tilde{R}_0(\tilde{G}, \tilde{p})$ has one row for each $e \in E_4(G)$. For all $e \in E_1(G)$ and all $0 \leq j \leq 3$, $\tilde{R}_j(\tilde{G}, \tilde{p})$ has one row. For $v \in V(G)$ with representative $v^\star \in V(\tilde{G})$, we let $\mathcal{M}_j(p(v))$ denote the space $\{\tilde{m}(v^\star) : \tilde{m} \in \mathcal{M}_j(\tilde{G}, \tilde{p})\}$ and we let c^j_v denote the dimension of $\mathcal{M}_j(p(v))$.

Definition 4.6.3. With the same notation as above, fix some $0 \leq j \leq 3$. For all $v \in V(G)$, choose a basis $\mathcal{B}^j_v$ for $\mathcal{M}_j(p(v))$ and let M^j_v be the matrix whose columns are the coordinate vectors of $\mathcal{B}^j_v$ relative to the canonical basis of $\mathbb{R}^2$. The *ρ_j-orbit rigidity matrix $O_j(G, \varphi, \psi, p)$ of (G, φ, ψ, p)* is a matrix with c^j_v columns for each $v \in V(G)$. For each edge $e \in E_1(G)$, $O_j(G, \varphi, \psi, p)$ has one row; for each $\gamma \in \{s, sr, r\}$ and each $e \in E_\gamma$, $O_0(G, \varphi, \psi, p)$ and $O_j(G, \varphi, \psi, p)$ have one row, where $j = 1$ if $\gamma = s$, $j = 2$ if $\gamma = sr$, and $j = 3$ if $\gamma = r$; in all other cases, $O_j(G, \varphi, \psi, p)$ has no rows corresponding to an edge $e \in E(G)$. Given an edge $e = (u, v) \in E(G)$, the row representing e in $O_j(G, \varphi, \psi, p)$ (provided it exists) is

$$\begin{array}{ccccc} & u & & v & \\ \Big(\dots & (p_u - \tau(\psi(e))p_v)^T M^j_u & \dots & \rho_j(\psi(e))(p_v - \tau(\psi^{-1}(e))p_u)^T M^j_v & \dots\Big) \end{array}$$

if $u \neq v$, and it is

$$\begin{array}{ccc} & u & \\ \Big(\dots & (p_u + \rho_j(\psi(e))p_u - \tau(\psi(e))p_u - \rho_j(\psi(e))\tau(\psi^{-1}(e))p_u)^T & \dots\Big) \end{array}$$

otherwise. If $c^j_u = 0$ (respectively, $c^j_v = 0$), then the columns corresponding to u (respectively, v) vanish.

For $0 \leq j \leq 3$, $\dim O_j(G, \varphi, \psi, p) = \dim \tilde{R}_j(\tilde{G}, \tilde{p})$ and the rank of $O_j(G, \varphi, \psi, p)$ is independent of the choice of the bases $\mathcal{B}_v^j$ for $v \in V(G)$. Moreover, when $j = 0$, Definitions 4.1.2 and 4.6.3 coincide. Therefore, Theorem 4.1.3 implies that $\ker O_0(G, \psi, p)$ and $\ker \mathcal{M}_0(\tilde{G}, \tilde{p})$ are isomorphic. Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_{2v}$-symmetric framework with $\mathcal{C}_{2v}$-gain framework (G, φ, ψ, p). Choose some $v \in V_2(G)$. By Equation (4.5), we may always choose bases $\mathcal{B}_v^j$ for $\mathcal{M}_j(p(v))$ such that

$$M_v^j = \begin{cases} (0 \quad 1)^T & \text{if } j = 0, 1 \text{ and } \varphi(v) = \{\text{id}, s\}, \text{ or if } j = 1, 3 \text{ and } \varphi(v) = \{\text{id}, sr\} \\ (1 \quad 0)^T & \text{if } j = 2, 3 \text{ and } \varphi(v) = \{\text{id}, s\}, \text{ or if } j = 0, 2 \text{ and } \varphi(v) = \{\text{id}, sr\} \end{cases}$$

Moreover, if $V_4(G) = \{u\}$, then we may always choose a basis $\mathcal{B}_u^j$ for $\mathcal{M}_j(p(v))$ such that

$$M_u^j = \begin{cases} (0 \quad 1)^T & \text{if } j = 1 \\ (1 \quad 0)^T & \text{if } j = 2 \\ 0 & \text{else} \end{cases}$$

For simplicity, we will always choose these M_v^j, M_u^j.

Lemma 4.6.4. Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_{2v}$-symmetric framework, and let $\mathcal{O}$ denote the set of vertex orbit representatives of $\tilde{G}$. Fix some $0 \leq j \leq 3$. Define the subset $\mathcal{O}' \subseteq V_4(\tilde{G})$ to be $V_4(\tilde{G})$ if $j = 0, 3$, and $\emptyset$ otherwise. For some $u^\star, v^\star \in \mathcal{O}$, let $u^\star v^\star \in E(\tilde{G})$. Let (G, φ, ψ, p) be the $\mathcal{C}_{2v}$-gain framework of $(\tilde{G}, \tilde{p})$. For each $g \in \varphi(u), h \in \varphi(v)$, let $(G, \varphi, \psi_{g,h}, p)$ be obtained from (G, φ, ψ, p) by applying a type II switching at e induced by g and h. Fix $g \in \varphi(u)$ and $h \in \varphi(v)$. Then a vector m lies in $\ker O_j(G, \varphi, \psi_{g,h}, p)$ if and only if $\tilde{m}' : \mathcal{O} \setminus \mathcal{O}' \to \mathbb{R}^2$ defined by $\tilde{m}'(w^\star) = M_w^j m(w)$ is the restriction of a ρ_j-symmetric infinitesimal motion $\tilde{m}$ of $(\tilde{G}, \tilde{p})$ to $\mathcal{O} \setminus \mathcal{O}'$.

Proof. Let $\tilde{m} : V(\tilde{G}) \to \mathbb{C}^2$ be defined by

$$\tilde{m}(\delta w^\star) = \begin{cases} \rho_j(\gamma)\tau(\gamma)\tilde{m}'(w^\star) & \text{for all } w^\star \in \mathcal{O} \setminus \mathcal{O}', \gamma \in \mathbb{D}_4 \\ (0 \quad 0)^T & \text{for all } w^\star \in \mathcal{O}', \gamma \in \mathbb{D}_4. \end{cases}$$

Then, $\tilde{m}'$ is a restriction of $\tilde{m}$ to $\mathcal{O} \setminus \mathcal{O}'$ and $\tilde{m}$ is a ρ_j-symmetric infinitesimal motion of $(\tilde{G}, \tilde{p})$ if and only if it is an infinitesimal infinitesimal motion of $(\tilde{G}, \tilde{p})$.

View m as a column vector. For each row r in $O_j(G, \varphi, \psi_{g,h}, p)$ that represents an edge $e = (u_1, u_2) \in E(G)$, we check that rm is zero if and only if $\tilde{m}$ satisfies the conditions of being an infinitesimal motion of the framework on the subgraph induced by the elements of the orbit e. Without loss of generality, we consider the edge $e = (u, v)$, where $u^\star, v^\star$ are as defined in the statement. Let $\psi(e) = \gamma$. The orbit of e is $\mathbb{D}_4\{u^\star v^\star_\gamma : v^\star_\gamma = \gamma v^\star\}$. Let r be the row of e in $O_j(G, \psi_{g,h}, p)$.

The map $\tilde{m}$ satisfies the conditions of being an infinitesimal motion of the framework on the subgraph induced by the elements of the orbit e if and only if, for all $\delta \in \mathbb{D}_4$

$$\langle \tilde{p}(\delta u^\star) - \tilde{p}(\delta\gamma v^\star), \tilde{m}(\delta u^\star) - \tilde{m}(\delta\gamma v^\star)\rangle = 0.$$

Since δ runs through all the elements of $\mathbb{D}_4$, so does $g\delta h$. Hence, this is equivalent to saying that for all $\delta \in \mathbb{D}_4$

$$\langle \tilde{p}(g\delta h u^\star) - \tilde{p}(g\delta h\gamma v^\star), \tilde{m}(g\delta h u^\star) - \tilde{m}(g\delta h\gamma v^\star)\rangle = 0.$$

Since $g \in \varphi(u), h \in \varphi(v)$ and $\mathbb{D}_4$ is abelian, we know that $\tilde{m}(g\delta h u^\star) = \tilde{m}(\delta h u^\star)$ and $\tilde{m}(g\delta h\gamma v^\star) = \tilde{m}(g\delta\gamma v^\star)$. Hence, by the definitions of $\tilde{m}$ and $\mathcal{C}_{2v}$-symmetric framework, this is equivalent to saying that for all $\delta \in \mathbb{D}_4$

$$\left\langle \tau(g\delta h)p_u - \tau(g\delta h\gamma)p_v, \rho_j(\delta h)\tau(\delta h)M_u^j m(u) - \rho_j(g\delta\gamma)\tau(g\delta\gamma)M_v^j m(v)\right\rangle = 0.$$

Since $\tau(\delta)$ is orthogonal and $\mathcal{C}_{2v}$ is abelian, we may remove $\tau(\delta)$ from the inner product. Moreover, we may take a factor of $\rho_j(h\delta)$ outside of the inner product, and divide both sides of the equation by $\rho_j(h\delta)$ to see that this is equivalent to

$$\left\langle \tau(gh)p_u - \tau(gh\gamma)p_v, \tau(h)M_u^j m(u) - \rho_j(g\gamma h)\tau(g\gamma)M_v^j m(v)\right\rangle = 0.$$

(Note that $\rho_j(h) = \rho_j(h)^{-1}$, since every element of $\mathbb{D}_4$ is its own inverse.) Since $\tau(gh)p_u = \tau(h)p_u$ and $\tau(gh\gamma)p_v = \tau(g\gamma)p_v$, multiplying both terms of the inner product by the orthogonal matrix $\tau(h)$, we obtain

$$\left\langle p_u - \tau(g\gamma h)p_v, M_u^j m(u) - \rho_j(g\gamma h)\tau(g\gamma h)M_v^j m(v)\right\rangle = 0,$$

which is equivalent to $rm = 0$. This completes the proof. $\square$

A direct consequence of Lemma 4.6.4 is that the nullity (and hence the rank) of $O_j(G, \psi, p)$ is independent of type II switchings. The same proof as that used for Proposition 4.1.4 shows that $O_j(G, \psi, p)$ is also independent of type I switchings, provided the configuration p of G is chosen adequately. Hence, we have the following.

Proposition 4.6.5. Let (G, φ, ψ, p) be a $\mathcal{C}_{2v}$-gain framework, and let (G, φ', ψ') be a $\mathbb{D}_4$-gain graph equivalent to (G, φ, ψ). There is a map $p' : V(G) \to \mathbb{R}^2$ such that rank $O_j(G, \varphi, \psi, p) =$ rank $O_j(G, \varphi', \psi', p')$.

Hence, given a $\mathcal{C}_{2v}$-symmetric framework $(\tilde{G}, \tilde{p})$ with two (equivalent) $\mathcal{C}_{2v}$-gain frameworks $(G_1, \varphi_1, \psi_1, p_1)$ and $(G_2, \varphi_2, \psi_2, p_2)$, the matrices $O_j(G_1, \varphi_1, \psi_1, p_1)$ and $O_j(G_2, \varphi_2, \psi_2, p_2)$ share the same rank, dimension and nullity. Hence, we may define the following.

Definition 4.6.6. Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_{2v}$-symmetric framework. Let (G, φ, ψ, p) be a $\mathcal{C}_{2v}$-gain framework of $(\tilde{G}, \tilde{p})$. We say that (G, φ, ψ, p) is *ρ_j-symmetrically isostatic* (or simply *ρ_j-isostatic*) if $(\tilde{G}, \tilde{p})$ is ρ_j-isostatic.

Chapter 5

Necessary conditions and graph extensions

Having defined phase-symmetric orbit rigidity matrices, we can establish necessary conditions for the infinitesimal rigidity of a $\tau(\Gamma)$-symmetric bar-joint framework $(\tilde{G}, \tilde{p})$, where Γ is a finite group and $\tau : \Gamma \to O(\mathbb{R}^2)$ is an injective homomorphism. In Section 5.1, we do so for the case where Γ is a cyclic group. In Section 5.2, we consider the case where $\Gamma = \mathbb{D}_{2k}$ for some $k \geq 2$. When $k = 2$, we provide necessary sparsity conditions for the infinitesimal rigidity of $(\tilde{G}, \tilde{p})$. For $k \geq 3$, we provide the necessary sparsity conditions for the forced symmetric rigidity of $(\tilde{G}, \tilde{p})$.

In Chapters 6 and 7 we will use combinatorial arguments to show that the conditions given in Section 5.1 are also sufficient, under genericity conditions and under some restrictions on the order of the group Γ. When proving the sufficiency of the sparsity conditions, we use an inductive argument on the order of the gain graph. To do so, we introduce certain operations on gain graphs, called extensions. For the inductive arguments to hold, extensions must maintain the symmetry-generic isostatic properties of a gain graph. In order to prove that this is the case, we adopt algebraic arguments, rather than combinatorial ones. We therefore present extensions in this chapter (see Section 5.3). In this chapter, and for the rest of the thesis, we work in $\mathbb{R}^2$.

5.1 Necessary conditions for cyclic symmetry

First, consider the case where Γ is a non-trivial cyclic group. Since we work in $\mathbb{R}^2$, the symmetry group $\tau(\Gamma)$ is either a reflection group $\mathcal{C}_s$ or a rotation group $\mathcal{C}_k$ for some $k \geq 2$. We consider the two cases separately, starting with the case where $\tau(\Gamma) = \mathcal{C}_s$ (see Subsection 5.1.1). If $\tau(\Gamma)$ is a rotation group, we consider the case where $k = 2$ and the case where $k \geq 3$ separately (see Subsections 5.1.2 and 5.1.3).

5.1.1 Reflection

Let (G, ψ, p) be a $\mathcal{C}_s$-gain framework and recall, from Subsection 4.5.1, that $O_0(G, \psi, p)$ and $O_1(G, \psi, p)$ both have exactly two columns for each $v \in V_1(G)$ and exactly one column for each $v \in V_2(G)$. Recall also that $O_0(G, \psi, p)$ has exactly one row for each $e \in E(G)$, and that $O_1(G, \psi, p)$ has exactly one row for each $e \in E_1(G)$.

Proposition 5.1.1. Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_s$-symmetric framework with $\mathcal{C}_s$-gain framework (G, ψ, p). The following hold:

(1) If $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic, then (G, ψ) is $(2, 1, 3, 1)$-gain tight.

(2) If $(\tilde{G}, \tilde{p})$ is anti-symmetrically isostatic, then (G, ψ) is $(2, 1, 3, 2)$-gain tight.

Proof. If $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic, then $\operatorname{null} O_0(G, \psi, p) = 1$ by Lemma 4.5.7. Since $O_0(G, \psi, p)$ is an $|E(G)| \times (2|V_1(G)| + |V_2(G)|)$ matrix, by the Rank-Nullity Theorem, we deduce that $|E(G)| = 2|V_1(G)| + |V_2(G)| - 1$. Moreover, there is no subgraph $\left(H, \psi|_{E(H)}\right)$ of (G, ψ) with non-empty edge set such that $|E(H)| > 2|V_1(H)| + |V_2(H)| - 1$ as this would imply a row dependency in the ρ_0-orbit rigidity matrix of (G, ψ, p). Similarly, if $(\tilde{G}, \tilde{p})$ is anti-symmetrically isostatic, then $|E_1(G)| = 2|V_1(G)| + |V_2(G)| - 2$ and $|E_1(H)| \leq 2|V_1(H)| + |V_2(H)| - 2$, for all subgraphs $\left(H, \psi|_{E(H)}\right)$ of (G, ψ) with $E(H) \neq \emptyset$. Moreover, if $(\tilde{G}, \tilde{p})$ is anti-symmetrically isostatic, then all of its bars are free, since removing any fixed bar from $(\tilde{G}, \tilde{p})$ does not change the rank of $O_1(G, \psi, p)$ (recall that $O_1(G, \psi, p)$ contains no rows corresponding to fixed bars). Therefore, if $(\tilde{G}, \tilde{p})$ is anti-symmetrically isostatic, then $|E(G)| = 2|V_1(G)| + |V_2(G)| - 2$ and, for all subgraphs $\big(H, \psi|_{E(H)}\big)$ of (G, ψ) with non-empty edge set, $|E(H)| \leq 2|V_1(H)| + |V_2(H)| - 2$.

Now, let $j = 0, 1$ and suppose for a contradiction that $(\tilde{G}, \tilde{p})$ is ρ_j-symmetrically isostatic and there is a balanced subgraph $\big(H, \psi|_{E(H)}\big)$ of (G, ψ) with non-empty edge set which satisfies $|E(H)| > 2|V(H)| - 3$. Let M be the submatrix of $O_j\,(G, \psi, p)$ obtained by removing all columns corresponding to the elements of $V(G) \setminus V(H)$, together with the rows corresponding to their incident edges. By Proposition 3.2.6, Lemma 3.2.11, Corollary 4.5.4 and Proposition 4.5.5, we can assume that $\psi(e) = \mathrm{id}$ for all $e \in E(H)$. M is a submatrix of a standard rigidity matrix for a graph F with $|E(F)| > 2|V(F)| - 3$, obtained by removing zero or more columns (depending on $|V_2(H)|$: one column is removed for each vertex in $V_2(H)$). But, by row independence, F must be $(2, 3)$-sparse, a contradiction. Hence, the result holds. $\square$

In the proof of Proposition 5.1.1, we noted that if $(\tilde{G}, \tilde{p})$ is anti-symmetrically isostatic, then it has no fixed bars. This also reflects on the sparsity of (G, ψ). To see this, recall that a fixed edge of (G, ψ) is either an edge $e = (u, v)$ between two fixed vertices $u, v \in V_2(G)$ or a loop at a free vertex $w \in V_1(G)$. If (G, ψ) is $(2, 1, 3, 2)$-gain sparse, then G contains no edges whose end-points are both fixed, and it contains no loops.

5.1.2 2-fold rotation

Now, let (G, ψ, p) be a $\mathcal{C}_2$-gain framework. Recall from Subsection 4.5.2, that $O_0(G, \psi, p)$ is an $|E(G)| \times 2|V_1(G)|$ matrix and $O_1(G, \psi, p)$ is an $|E_1(G)| \times 2|V(G)|$ matrix. A similar proof as that for Proposition 5.1.1 shows the following.

Proposition 5.1.2. Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_2$-symmetric framework with $\mathcal{C}_2$-gain framework (G, ψ, p). The following hold:

(1) If $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic, then (G, ψ) is $(2, 0, 3, 1)$-gain tight.

(2) If $(\tilde{G}, \tilde{p})$ is anti-symmetrically isostatic, then (G, ψ) is $(2, 2, 3, 2)$-gain tight.

Proof. If $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic, then $\operatorname{null} O_0(G, \psi, p) = 1$, by Lemma 4.5.8. Since $O_0(G, \psi, p)$ is an $|E(G)| \times 2|V_1(G)|$ matrix, by the Rank-Nullity Theorem, we deduce that $|E(G)| = 2|V_1(G)| - 1$. Moreover, there is no subgraph $\left(H, \psi|_{E(H)}\right)$ of (G, ψ) such that $|E(H)| > 2|V_1(H)| - 1$ as this would imply a row dependency in the ρ_0-orbit rigidity matrix of (G, ψ, p). Similarly, if $(\tilde{G}, \tilde{p})$ is anti-symmetrically isostatic, then $|E_1(G)| = 2|V(G)| - 2$ and $|E_1(H)| \leq 2|V(H)| - 2$, for all subgraphs $\left(H, \psi|_{E(H)}\right)$ of (G, ψ) with $E(H) \neq \emptyset$. Moreover, if $(\tilde{G}, \tilde{p})$ is anti-symmetrically isostatic, then all of its bars are free, since removing fixed bars from $(\tilde{G}, \tilde{p})$ does not change the rank of $O_1(G, \psi, p)$. Hence, $|E(G)| = 2|V(G)| - 2$ and $|E(H)| \leq 2|V(H)| - 2$, for all subgraphs $\left(H, \psi|_{E(H)}\right)$ of (G, ψ) with $E(H) \neq \emptyset$.

Now, let $j = 0, 1$ and suppose for a contradiction that $(\tilde{G}, \tilde{p})$ is ρ_j-symmetrically isostatic and there is a balanced subgraph $\left(H, \psi|_{E(H)}\right)$ of (G, ψ) with non-empty edge set which satisfies $|E(H)| > 2|V(H)| - 3$. Let M be the submatrix of $O_j\left(G, \psi, p\right)$ obtained by removing all columns corresponding to the elements of $V(G) \setminus V(H)$, together with the rows corresponding to their incident edges. By Proposition 3.2.6, Lemma 3.2.11, Corollary 4.5.4 and Proposition 4.5.5, we can assume that $\psi(e) = \mathrm{id}$ for all $e \in E(H)$. M is a submatrix of a standard rigidity matrix for a graph F with $|E(F)| > 2|V(F)| - 3$, obtained by removing zero or two columns. (If $|V_2(H)| = 1$ and $j = 1$, two column are removed from $O_1(G, \psi, p)$; if $j = 0$ or $V_2(H) = \emptyset$, no columns are removed.) By row independence, F must be $(2,3)$-sparse, a contradiction. Hence, the result holds. $\square$

Note that the proof for (1) does not use the fact that the rotation group which acts on the framework has order 2: for all $k \geq 2$, the ρ_0-orbit matrix of a $\mathcal{C}_k$-gain framework is an $|E(G)| \times 2|V_1(G)|$ matrix, and the only trivial infinitesimal motions which maintain symmetry of the framework are infinitesimal rotations around the origin. In fact, the exact same proof used for Proposition 5.1.1 shows the following.

Lemma 5.1.3. Let $k \geq 2$ be an integer and $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_k$-symmetric framework with $\mathcal{C}_k$-gain framework (G, ψ, p). If $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic, then (G, ψ) is $(2, 0, 3, 1)$-gain tight.

Note also that if (G, ψ) is $(2,2,3,2)$-gain tight, then G has no loops at a vertex. Since $|V_2(G)| \leq 1$, the only fixed bars of a $\mathcal{C}_2$-symmetric framework are the bars connecting two joints in the same orbit. It follows that, if (G, ψ) is $(2,2,3,2)$-gain tight, then all the bars of $(\tilde{G}, \tilde{p})$ are free.

5.1.3 Higher order rotation

Let $k \geq 3$. Lemma 5.1.3 gives us necessary conditions for the ρ_0-rigidity of $\mathcal{C}_k$-symmetric frameworks. For all $1 \leq j \leq k-1$, we provide necessary conditions that a $\mathcal{C}_k$-symmetric framework must satisfy in order to be ρ_j-isostatic. We start with the cases where $j = 1$ or $j = k-1$. Recall from Subsection 4.5.3 that for such j, the ρ_j-orbit rigidity $O_j(G, \psi, p)$ of a $\mathcal{C}_k$-gain framework has $2|V_1(G)| + |V_k(G)|$ columns and that $O_j(G, \psi, p)$ has $|E(G)|$ rows if j is even, and it has $|E_1(G)|$ rows if j is odd.

Lemma 5.1.4. For $k \geq 3$, let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_k$-symmetric framework with $\mathcal{C}_k$-gain framework (G, ψ, p). Suppose that $(\tilde{G}, \tilde{p})$ is ρ_j-isostatic for some $j \in \{1, k-1\}$. Then, (G, ψ) is $(2,1,3,1)$-gain tight. In addition, if $k \geq 4$ is even, then (G, ψ) is $(2,1,3,1)'$-gain tight.

Proof. Fix some $j \in \{1, k-1\}$ and suppose that $(\tilde{G}, \tilde{p})$ is ρ_j-rigid. If $O_j(G, \psi, p)$ has $|E_1(G)|$ rows, then $|E(G)| = |E_1(G)|$, since removing any fixed edge would not change $O_j(G, \psi, p)$. By Lemma 4.5.9 and the Rank-Nullity Theorem, it follows that $|E(G)| = 2|V_1(G)| + |V_k(G)| - 1$ and for all subgraphs H of G with $E(H) \neq \emptyset$, $|E(H)| \leq 2|V_1(H)| + |V_k(H)| - 1$.

Assume, for a contradiction, that there is a balanced subgraph $\big(H, \psi|_{E(H)}\big)$ of (G, ψ) with non-empty edge set such that $|E(H)| > 2|V(H)| - 3$. Let M be the submatrix of $O_j\,(G, \psi, p)$ obtained by removing all the columns representing the vertices that are not in $V(H)$, together with the rows corresponding to their incident edges. By Proposition 3.2.6, Lemma 3.2.11, Corollary 4.5.4 and Proposition 4.5.5, we can assume that $\psi(e) = \mathrm{id}$ for all $e \in E(H)$. If H has no fixed vertex, M is a standard rigidity matrix for a graph F with $|E(F)| > 2|V(F)| - 3$, contradicting the row independence of $O_j(G,\psi,p)$. So, we may assume that H has exactly one fixed vertex v_0. Let $v_1,\dots,v_t$ be the vertices that are incident with v_0 in H and, for $1\le i\le t$, let $p_i := p(v_i) = (x_i \quad y_i)^T$. Then, M has the form

$$\left(\begin{array}{c|ccc|ccc}
-x_1+\sqrt{-1}y_1 & p_1 & \dots & 0 & 0 & \dots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
-x_t+\sqrt{-1}y_t & 0 & \dots & p_t & 0 & \dots & 0\\
\hline
0 & & \vdots & & & \vdots & \\
\vdots & \dots & p_i-p_j & \dots & & p_j-p_i & \dots\\
0 & & \vdots & & & \vdots &
\end{array}\right)$$

if $j=1$. (If $j=k-1$, it has the exact same form, except that all entries of the form $-x_i+y_i$ in the first column are replaced by $-x_i-y_i$.) Let M' be the matrix obtained from M by replacing the first column with the following two columns:

$$\begin{pmatrix}
x_1 & y_1\\
\vdots & \vdots\\
x_t & y_t\\
0 & 0\\
\vdots & \vdots\\
0 & 0
\end{pmatrix}.$$

Since M is row independent, so is M' (both when $j=1$ and when $j=k-1$). But M' is a standard rigidity matrix for a graph F with $|E(F)|>2|V(F)|-3$, contradicting the row independence of $O_1(G,\psi,p)$. This proves the result when $k\ge 3$ is odd.

Therefore, let $k\ge 4$ be even and assume, for a contradiction, that there is a subgraph $(H,\psi_{E(H)})$ of (G,ψ) such that $\langle H\rangle \simeq \mathbb{Z}_2$ and $|E(H)|>2|V(H)|-2$. Since (G,ψ,p) is a $\mathcal{C}_k$-gain framework, (G,ψ) is a Γ-gain graph for some cyclic group $\Gamma=\langle\gamma\rangle$ of order k, where $\Gamma\simeq\mathbb{Z}_k$ through the isomorphism which maps γ to 1. So $\langle H\rangle$ is the group Γ' of order 2 generated by $\gamma^{k/2}$.

Let ρ_1' be the non-trivial irreducible representation of Γ', and let $\tau':\Gamma'\to\mathcal{C}_2$ be the homomorphism which maps $\gamma^{k/2}$ to the rotation C_2. Let $e=(u,v)\in E(H)$.

By Proposition 3.2.6, Lemma 3.2.11, Corollary 4.5.4 and Proposition 4.5.5, we can assume that $\psi(e) \in \{\mathrm{id}, \gamma^{k/2}\}$. Therefore, since $j = 1, k-1$ is odd, we have

$$\rho_j(\psi(e)) = \exp\left(\frac{2\pi\sqrt{-1}j}{k}\frac{k}{2}\right) = \exp(\pi\sqrt{-1}j) = -1 = \rho_1'(\psi(e)).$$

Note that $\tau(\psi(e)) = \tau'(\psi(e))$. It follows that $O_j(H, \psi|_{E(H)}, p|_{V(H)})$ is the ρ_1-orbit matrix of a $\mathcal{C}_2$-symmetric framework. Then, by Proposition 5.1.2(2), we have a contradiction. $\square$

In the statement of Lemma 5.1.4 (and with the same notation as in the proof), we do not allow any loop at a free vertex to have gain $\gamma^{k/2}$ whenever k is even. Such edges lift to semi-free edges. Therefore, since $1, k-1$ are odd, the ρ_1 and ρ_{k-1}-orbit matrices of (G, ψ, p) do not have rows corresponding to all such edges, as expected.

Lemmas 5.1.3 and 5.1.4 give the necessary conditions for the infinitesimal rigidity of $\mathcal{C}_3$-symmetric frameworks (see Proposition 5.1.5).

Proposition 5.1.5. Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_3$-symmetric framework with $\mathcal{C}_3$-gain framework (G, ψ, p). The following hold:

(1) If $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic, then (G, ψ) is $(2, 0, 3, 1)$-gain tight.

(2) If $(\tilde{G}, \tilde{p})$ is ρ_1-isostatic or ρ_2-isostatic, then (G, ψ) is $(2, 1, 3, 1)$-gain tight.

If $k \geq 4$, we must also consider the ρ_j-rigidity of a $\mathcal{C}_k$-symmetric framework, where $2 \leq j \leq k-2$. For all $2 \leq j \leq k-2$, we show that the underlying gain graph of a ρ_j-isostatic $\mathcal{C}_k$-gain framework is $\mathbb{Z}_k^j$-gain sparse. Recall that $\mathbb{Z}_k^j$-gain sparsity relies on the notion of near-balancedness, which is only defined for gain graphs with no fixed vertices. The following result was given in [27]. (See Lemma A.0.8 in Appendix A for a proof.)

Lemma 5.1.6 ([27], Lemma 5.5)**.** Let $k := |\Gamma| \geq 4, 2 \leq j \leq k-2$, $\tau : \Gamma \to \mathcal{C}_k$ be an injective homomorphism, (G, ψ) be a Γ-gain graph, and $p : V(G) \to \mathbb{R}^2$. If $O_j(G, \psi, p)$ is row independent, $|E(H)| \leq 2|V(H)| - 1$ for any near-balanced subgraph H of G with non-empty edge set.

For $k \geq 4$, let (G, ψ, p) be a $\mathcal{C}_k$-gain framework. For the following result, recall that for all $2 \leq j \leq k-2$, $O_j(G, \psi, p)$ has $2|V_1(G)|$ columns and it has $|E(G)|$ rows if j is even, and $|E_1(G)|$ rows if j is odd.

Lemma 5.1.7. For $k \geq 4, 2 \leq j \leq k-2$, let $(\tilde{G}, \tilde{p})$ be a ρ_j-isostatic $\mathcal{C}_k$-symmetric framework with $\mathcal{C}_k$-gain framework (G, ψ, p). Then, (G, ψ) is $\mathbb{Z}_k^j$-gain sparse.

Proof. Similarly as in the proof of Lemma 5.1.4, we may assume that G has no fixed edges if j is odd. Hence, by the Rank-Nullity Theorem and Lemma 4.5.9, $|E(G)| = 2|V_1(G)|$ and $|E(H)| \leq 2|V_1(H)|$ for all subgraphs H of G with $E(H) \neq \emptyset$.

Now, suppose for a contradiction that there is a balanced subgraph $\left(H, \psi|_{E(H)}\right)$ of (G, ψ) with non-empty edge set which satisfies $|E(H)| > 2|V(H)| - 3$. Let M be the submatrix of $O_j\left(G, \psi, p\right)$ obtained by removing all columns corresponding to the elements of $V(G) \setminus V(H)$, together with the rows corresponding to their incident edges. By Proposition 3.2.6, Lemma 3.2.11, Corollary 4.5.4 and Proposition 4.5.5, we can assume that $\psi(e) = \mathrm{id}$ for all $e \in E(H)$. M is a submatrix of a standard rigidity matrix for a graph F with $|E(F)| > 2|V(F)| - 3$, obtained by removing zero or two columns. (If $|V_2(H)| = 1$, two column are removed; otherwise, no columns are removed.) By row independence, F must be $(2,3)$-sparse, a contradiction. Hence, all balanced subgraphs of (G, ψ) with non-empty edge set are $(2,3)$-sparse. By Lemma 5.1.6, all near-balanced subgraphs of G with non-empty edge set are $(2,1)$-sparse. So we only need to consider the subgraphs of G which are $S(k,j)$ and, in the case where j is odd, the subgraphs H of G with $\langle H \rangle \simeq \mathbb{Z}_2$.

So, suppose that H is a subgraph of G with non-empty edge set such that $\langle H \rangle \simeq \mathbb{Z}_n$ for some $n \in S_0(k,j) \cup S_{-1}(k,j) \cup S_1(k,j) \cup \{2\}$, where $n = 2$ only if j is odd. Since (G, ψ, p) is a $\mathcal{C}_k$-gain framework, (G, ψ) is a Γ-gain graph for some cyclic group $\Gamma = \langle \gamma \rangle$ of order k. Note that $\mathbb{Z}_k \simeq \Gamma$ with the isomorphism mapping 1 to γ. So the group $\langle H \rangle$ is the group Γ' of order n generated by $\gamma^{k/n}$. Moreover, $j \equiv i (\mathrm{mod}\, n)$, where $i = 0$ if $n \in S_0(k,j)$ and $i = \pm 1$ otherwise. Hence, there is some integer $m \geq 1$ such that $j = i + mn$.

Let ρ_i' be the irreducible representation of Γ' which sends the generator $\gamma^{k/n}$ to

$\exp\left(2\pi i\sqrt{-1}/n\right)$, and let $\tau' : \Gamma' \to \mathcal{C}_n$ be the homomorphism which sends $\gamma^{k/n}$ to the rotation C_n. Let $e = (u, v) \in E(H)$. Then, $\psi(e) = \gamma^{sk/n}$ for some $0 \leq s \leq n-1$. Since $j = i + mn$, we have

$$\begin{aligned}\rho_j(\psi(e)) &= \exp\Big(\frac{2\pi(i+mn)\sqrt{-1}}{k}\frac{sk}{n}\Big)\\ &= \exp\Big(\frac{2\pi i\sqrt{-1}}{n}s\Big)\exp\Big(2\pi ms\sqrt{-1}\Big)\\ &= \exp\Big(\frac{2\pi i\sqrt{-1}}{n}s\Big) = \rho'_i(\psi(e)).\end{aligned}$$

Thus, we have

$$p_u - \tau(\psi(e))p_v = p_u - \tau'(\psi(e))p_v$$

and

$$\rho_j(\psi(e))(p_v - \tau(\psi(e))^{-1}p_u) = \rho'_i(\psi(e))(p_v - \tau'(\psi(e))^{-1}p_u).$$

(See also the proofs of [[27], Lemma 5.4] and [[56], Lemma 6.13] for the free action case.) Hence, $O_j(H, \psi|_{E(H)}, p|_{V(H)})$ is the ρ'_i-orbit matrix of a $\mathcal{C}_n$-symmetric framework. If $i \equiv 0 \mod n$, this implies that H must satisfy $|E(H)| \leq 2|V_1(H)| - 1$ by Lemma 5.1.3. If $i \equiv \pm 1 \mod n$ and $n = 2$, this implies that H must satisfy $|E(H)| \leq 2|V(H)| - 2$ by Proposition 5.1.2(2). If $i \equiv \pm 1 \mod n$ and $n \geq 3$, this implies that H must satisfy $|E(H)| \leq 2|V_1(H)| + |V_k(H)| - 1$ by Lemma 5.1.4. This gives the result. $\square$

We conclude the section by combining Lemmas 5.1.3, 5.1.4 and 5.1.7 in order to obtain the necessary conditions for the infinitesimal rigidity of $\mathcal{C}_k$-symmetric frameworks, where $k \geq 4$ (see Proposition 5.1.8).

Proposition 5.1.8. Let $k \geq 4$ be an integer and $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_k$-symmetric framework with $\mathcal{C}_k$-gain framework (G, ψ, p). The following hold:

(1) If $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic, then (G, ψ) is $(2, 0, 3, 1)$-gain tight.

(2) If $(\tilde{G}, \tilde{p})$ is ρ_1-isostatic or ρ_{k-1}-isostatic, then (G, ψ) is $(2, 1, 3, 1)'$-gain tight.

(3) If $(\tilde{G}, \tilde{p})$ is ρ_j-isostatic for some $2 \leq j \leq k-2$, then (G, ψ) is $\mathbb{Z}_k^j$-gain tight.

5.2 Necessary conditions for dihedral symmetry

We now consider dihedral groups. Since the dihedral group of order 4 is the only dihedral group for which we defined all phase-symmetric orbit matrices in Chapter 4, we start by considering $\mathcal{C}_{2v}$.

5.2.1 Dihedral group of order 4

Recall from Section 4.6 that, for a $\mathcal{C}_{2v}$-gain framework (G, φ, ψ, p), $O_1(G, \varphi, \psi, p)$ and $O_2(G, \varphi, \psi, p)$ have $2|V_1(G)| + |V_2(G)| + |V_4(G)|$ columns, whereas $O_0(G, \varphi, \psi, p)$ and $O_3(G, \varphi, \psi, p)$ have $2|V_1(G)| + |V_2(G)|$ columns.

Lemma 5.2.1. Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_{2v}$-symmetric framework with $\mathcal{C}_{2v}$-gain framework (G, φ, ψ, p). Suppose that $(\tilde{G}, \tilde{p})$ is ρ_j-isostatic for some $0 \leq j \leq 3$.

(1) If $j = 0$, then $|E(G)| = 2|V_1(G)| + |V_2(G)|$ and for all subgraphs H of G with $E(H) \neq \emptyset$ we have $|E(H)| \leq 2|V_1(H)| + |V_2(H)|$.

(2) If $j = 1, 2$, then $|E(G)| = 2|V_1(G)| + |V_2(G)| + |V_4(G)| - 1$ and for all subgraphs H of G with $E(H) \neq \emptyset$ we have $|E(H)| \leq 2|V_1(H)| + |V_2(H)| + |V_4(H)| - 1$.

(3) If $j = 3$, then $|E(G)| = 2|V_1(G)| + |V_2(G)| - 1$ and for all subgraphs H of G with $E(H) \neq \emptyset$ we have $|E(H)| \leq 2|V_1(H)| + |V_2(H)| - 1$.

(4) All balanced subgraphs of G with non-empty edge set are $(2, 3)$-sparse.

Proof. Fix some $0 \leq j \leq 3$ and suppose that $(\tilde{G}, \tilde{p})$ is ρ_j-isostatic. Since $(\tilde{G}, \tilde{p})$ is ρ_j-isostatic, $E(G)$ does not contain any edge which does not have a row in $O_j(G, \varphi, \psi, p)$. Hence, by Lemma 4.6.2 and the Rank-Nullity Theorem, (1),(2) and (3) hold. (See, e.g., the proof of Proposition 5.1.2.) We prove (4).

So, assume for a contradiction that there is a balanced subgraph H of G with non-empty edge set such that $|E(H)| > 2|V(H)| - 3$. By Propositions 3.3.6 and 4.6.5, and by Lemma 3.3.12, we may assume that $\psi(e) = \mathrm{id}$ for all $e \in E(H)$. Let M be the matrix obtained from $O_j(G, \varphi, \psi, p)$ by removing all vertices in G which are not in H, together with the rows corresponding to their incident edges. M is a submatrix of a standard rigidity matrix for a graph F with $|E(F)| > 2|V(F)| - 3$, obtained by removing zero or more columns. (Depending on $|V_2(H)|, |V_4(H)|$: one column is removed for each vertex in $V_2(H)$; and one column is removed if $j = 1, 2$ and $V_4(H) \neq \emptyset$.) By row independence, F must be $(2,3)$-sparse, a contradiction. Hence, the result holds. $\square$

Recall the notions of rotational and reflectional subgraphs given in Chapter 3. With the same notation as above, given a rotational subgraph H of G, $\big(H, \varphi|_{V(H)}, \psi|_{E(H)}, p|_{V(H)}\big)$ can also be seen as a $\mathcal{C}_2$-gain framework. Since the conditions for $\mathcal{C}_2$-symmetric frameworks to be infinitesimally rigid can be stronger that the conditions for $\mathcal{C}_{2v}$-symmetric frameworks to be infinitesimally rigid, it is important to consider rotational subgraphs of G. Similarly, given a reflectional subgraph H of G with reflection s (or sr), $\big(H, \varphi|_{V(H)}, \psi|_{E(H)}, p|_{V(H)}\big)$ can be seen as a $\mathcal{C}_s$-gain framework. Note, in this case, every vertex in $V_s(H)$ (or $V_{sr}(H)$) is seen as a fixed vertex of a $\mathcal{C}_s$-symmetric framework, and all other vertices are seen as free.

Figure 5.1 shows a $\mathcal{C}_{2v}$-symmetric framework $(\tilde{G}, \tilde{p})$ and its $\mathcal{C}_{2v}$-gain framework (G, φ, ψ, p). The graph (G, φ, ψ) is reflectional with reflection s, and $(\tilde{G}, \tilde{p})$ can be seen as a $\mathcal{C}_s$-symmetric framework with symmetry line $x = 0$. Note that $u_1^\star, ru_1^\star$ do not lie on this symmetry line. So, they are treated as free vertices of the $\mathcal{C}_s$-framework $(\tilde{G}, \tilde{p})$.

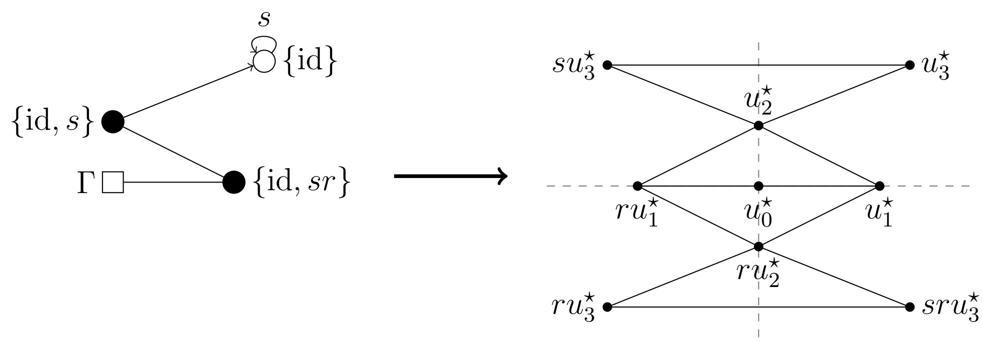


Figure 5.1: $\mathbb{D}_4$-gain graph and its corresponding $\mathbb{D}_4$-lifting. Here, the unlabelled edges have identity gain.

Lemma 5.2.2. Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_{2v}$-symmetric framework with $\mathcal{C}_{2v}$-gain framework (G, φ, ψ, p). If $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic, then all rotational subgroups H of G with $E(H) \neq \emptyset$ satisfy $|E(H)| \leq 2|V_1(H)| + 2|V_2(H)| - 1$.

Proof. Assume, for a contradiction, that $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic and that $|E(H)| > 2|V_1(H)| + 2|V_2(H)| - 1$ for some rotational subgraph H of G with non-empty edge set. Note that $V_2(H) = \emptyset$, as otherwise $|E(H)| > 2|V_1(H)| + |V_2(H)|$, contradicting Lemma 5.2.1(1). Let ψ' be the edge-gain map equivalent to ψ such that $\langle H \rangle_{\psi'} = \{\mathrm{id}, r\}$. By Propositions 3.3.6 and 4.6.5, and by Lemma 3.3.12, we may assume that $\psi'(e) \in \{\mathrm{id}, r\}$ for all $e \in E(H)$. Let M be the submatrix of $O_0(G, \varphi, \psi, p)$ obtained by removing the columns corresponding to the vertices in $V(G) \setminus V(H)$, together with the rows corresponding to their incident edges. Then, M is the ρ_0-orbit rigidity matrix for a $\mathcal{C}_2$-symmetric framework whose underlying graph F satisfies $|E(F)| > 2|V_1(F)| - 1$. By Proposition 5.1.2(1), this contradicts the row independence of $O_0(G, \varphi, \psi, p)$. Hence, the result holds. $\square$

Lemma 5.2.3. Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_{2v}$-symmetric framework with $\mathcal{C}_{2v}$-gain framework (G, φ, ψ, p). Suppose that $(\tilde{G}, \tilde{p})$ is ρ_j-isostatic for some $0 \leq j \leq 3$. For some $i = 0, 1$, let H be a reflectional subgraph of G with reflection sr^i and with non-empty edge set. The following hold:

(1) If $j = 0$, then $|E(H)| \leq 2|V(H - V_{sr^i}(H))| + |V_{sr^i}(H)| - 1$.

(2) If $j = 2, 3$ and $i = 0$, then $|E(H)| \leq 2|V(H - V_s(H))| + |V_s(H)| - 2$.

(3) If $j = 1, 3$ and $i = 1$, then $|E(H)| \leq 2|V(H - V_{sr}(H))| + |V_{sr}(H)| - 2$.

Proof. First, suppose that $i = 0$, and let j be one of $0, 2, 3$. By Propositions 3.3.6 and 4.6.5, and by Lemma 3.3.12, we may assume that $\psi(e) \in \{\mathrm{id}, s\}$ for all $e \in E(H)$. Moreover, we know that $\varphi(v)$ is either $\{\mathrm{id}, s\}$ or $\mathbb{D}_4$ for all $v \in V_s(H)$. Hence, $M_v^j = (0 \;\; 1)^T$ for all $v \in V_2(H) \cap V_s(H)$. Let M be the submatrix of $O_j(G, \varphi, \psi, p)$ obtained by removing the columns corresponding to the vertices in $V(G) \setminus V(H)$ and rows corresponding to their incident edges. If $j = 0$, then $\rho_j(s) = (1)$, and M is a submatrix of the ρ_0-orbit rigidity matrix for a $\mathcal{C}_s$-gain framework with $|V(H - V_s(H))|$ free vertices and $|V_s(H)|$ fixed vertices obtained by removing zero or more columns. (For each $v \in V_1(H - V_s(H))$ a column is removed; if $v \in V_s(H)$ is fixed, a column is removed.)

Otherwise, $\rho_j(s) = (-1)$, and M is a submatrix of the ρ_1-orbit rigidity matrix for a $\mathcal{C}_s$-gain framework with $|V(H - V_s(H))|$ free vertices and $|V_s(H)|$ fixed vertices obtained by removing zero or more columns. (For each $v \in V_1(H - V_s(H))$ a column is removed; if $v \in V_s(H)$ is fixed, a column is removed.) Since $O_j(G, \varphi, \psi, p)$ has no row dependence, it follows that H has at most $2|V(H - V_s(H))| + |V_s(H)| - 1$ edges if $j = 0$, and it has at most $2|V(H - V_s(H))| + |V_s(H)| - 2$ edges if $j = 2, 3$.

Now, suppose that $i = 1$, and let j be one of $0, 1, 3$. By Propositions 3.3.6 and 4.6.5, and by Lemma 3.3.12, we may assume that $\psi(e) \in \{\mathrm{id}, sr\}$ for all $e \in E(H)$. Moreover, we know that $\varphi(v)$ is either $\{\mathrm{id}, sr\}$ or $\mathbb{D}_4$ for all $v \in V_s(H)$. Apply a clockwise rotation of the Euclidean plane around the origin with angle $\pi/2$. Then, M_v^j becomes $(0 \quad 1)^T$ and $\tau(sr)$ becomes $\mathrm{diag}(-1, 1)$. Let M be the submatrix of $O_j(G, \varphi, \psi, p)$ obtained by removing the columns corresponding to the vertices in $V(G) \setminus V(H)$ and the rows corresponding to their incident edges. If $j = 0$, then $\rho_j(s) = (1)$, and M is a submatrix of the ρ_0-orbit rigidity matrix for a $\mathcal{C}_s$-gain framework with $|V(H - V_{sr}(H))|$ free vertices and $|V_{sr}(H)|$ fixed vertices obtained by removing zero or more columns. (For each $v \in V_1(H - V_{sr}(H))$ a column is removed; if $v \in V_{sr}(H)$ is fixed, two columns are removed.)

Otherwise, $\rho_j(sr) = (-1)$, and M is a submatrix of the ρ_1-orbit rigidity matrix for a $\mathcal{C}_s$-gain framework with $|V(H - V_{sr}(H))|$ free vertices and $|V_{sr}(H)|$ fixed vertices obtained by removing zero or more columns. (For each $v \in V_1(H - V_{sr}(H))$ a column is removed; if $v \in V_{sr}(H)$ is fixed, a column is removed.) Since $O_j(G, \varphi, \psi, p)$ has no row dependence, H has at most $2|V(H - V_{sr}(H))| + |V_{sr}(H)| - 1$ edges if $j = 0$, and it has at most $2|V(H - V_{sr}(H))| + |V_{sr}(H)| - 2$ edges if $j = 1, 3$. $\square$

Notice that in the proofs of Lemma 5.2.1(1) and of Lemma 5.2.2, we did not use the fact that the symmetry group has order 4. In fact the exact same proofs show the following.

Lemma 5.2.4. Let $k \geq 2$ be an integer and $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_{kv}$-symmetric framework with $\mathcal{C}_{kv}$-gain framework (G, φ, ψ, p). If $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic, then the following hold:

(1) $|E(G)| = 2|V_1(G)| + |V_2(G)|$ and for all subgraphs H of G with $E(H) \neq \emptyset$ we have $|E(H)| \leq 2|V_1(H)| + |V_2(H)|$.

(2) $|E(H)| \leq 2|V_1(H)| + 2|V_2(H)| - 1$ for all rotational subgraphs H of G with non-empty edge set.

5.2.2 Dihedral groups of higher order

We conclude the section by considering $\mathcal{C}_{kv}$, where $k \geq 3$. For all such groups, we have not defined phase-symmetric orbit rigidity matrices. However, [61] provided the ρ_0-orbit rigidity matrix, which can be used to study the forced symmetric rigidity of $\mathcal{C}_{kv}$-symmetric frameworks (recall Section 4.1). Therefore, we provide necessary conditions for the infinitesimal rigidity of a $\mathcal{C}_{kv}$-symmetric frameworks. First, we need the following result.

Lemma 5.2.5. Let $k \geq 3$, and $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_{kv}$-symmetric framework with $\mathcal{C}_{kv}$-gain framework (G, φ, ψ, p). Let H be a reflectional subgraph of G with reflection sr^j (where $0 \leq j \leq k-1$). Suppose that $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic. Then, $|E(H)| \leq 2|V(H - V_{sr^j}(H))| + |V_{sr^j}(H)| - 1$.

Proof. By Propositions 3.3.6, 4.1.4 and 4.1.5, and by Lemma 3.3.12, we may assume that $\psi(e) \in \{\text{id}, sr^j\}$ for all $e \in E(H)$. Without loss of generality, we may assume that $j = 0$. (If $j \neq 0$, we can apply a rotation of the plane around the origin with the angle $2\pi j/k$.) Let M be the matrix obtained from $O_0(G, \varphi, \psi, p)$ by removing the columns corresponding to the vertices in $V(G) \backslash V(H)$ and the rows corresponding to their incident edges. If $V_2(H) = \emptyset$, then M is the submatrix of the ρ_0-orbit rigidity matrix of a $\mathcal{C}_s$-gain framework with $|V(H - V_{2k}(H))|$ free vertices and $|V_{2k}(H)|$ fixed vertices obtained by removing zero or one columns. (If there is a fixed vertex in G, a column is removed; otherwise no column is removed.) Then, the result follows by Proposition 5.1.1(1). Therefore we may assume that $V_2(H) \neq \emptyset$.

Let $v \in V_2(H)$. Then, $p(v)$ lies on a reflection line, and so there are some $0 \leq j \leq k-1, a \in \mathbb{R}$ such that $p(v) = a(-\sin\alpha_j \quad \cos\alpha_j)^T$, where $\alpha_j := 2\pi j/k$. Hence, we may choose $\mathcal{B}_v^0 = \{M_v^0 = (-\sin\alpha_j \quad \cos\alpha_j)^T\}$ as a basis for $U(p(v))$. Choose such a basis to construct $O_0(G, \varphi, \psi, p)$. Let $v_1, \ldots, v_t$ be the vertices adjacent to v in H and, for $1 \leq i \leq t$, let $p(v_i) = (x_i \quad y_i)^T$. We may assume that all edges incident to v are directed from v. So, for $1 \leq i \leq t$, let $e_i := (v, v_i)$. For each $1 \leq i \leq t$, the row in M corresponding to e_i has the form

$$\begin{aligned} e_i &= \Big(\ldots \quad [p(v) - \tau(\psi(e_i))p(v_i)]^T M_v^0 \quad \ldots \quad [p(v_i) - \tau(\psi(e_i))p(v)]^T M_{v_i}^0 \quad \ldots\Big) \\ &= \Big(\ldots \quad a - \tau(\psi(e_i))p(v_i)^T M_v^0 \quad \ldots \quad [p(v_i) - \tau(\psi(e_i))p(v)]^T M_{v_i}^0 \quad \ldots\Big) \\ &= \Big(\ldots \quad a \pm \sin\alpha_j x_i - \cos\alpha_j y_i \quad \ldots \quad \Big(x_i \pm a\sin\alpha_j \quad y_i - a\cos\alpha_j\Big) M_{v_i}^0 \quad \ldots\Big). \end{aligned}$$

(Here, $\pm$ depends on $\psi(e_i)$: if $\psi(e_i) = \mathrm{id}$, we choose $+$; otherwise, we choose $-$.) Order the rows of M such that, for all $1 \leq i \leq t$, the i^{th} row in M is the row corresponding to e_i. If $j \neq 0$, substitute the columns corresponding to v in M with the following two columns:

$$\begin{pmatrix} -a\sin\alpha_j \mp x_1 & a\cos\alpha_j - y_1 \\ \vdots & \vdots \\ -a\sin\alpha_j \mp x_t & a\cos\alpha_j - y_t \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} [p(v) - \tau(\psi(e_1))p(v_1)]^T \\ \vdots \\ [p(v) - \tau(\psi(e_t))p(v_t)]^T \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(Here, $\mp$ depends on $\psi(e_i)$: if $\psi(e_i) = \mathrm{id}$, we choose $-$; otherwise, we choose $+$.) Let M' be the matrix obtained by applying this change. We show that a row dependency in M' implies a row dependency in M. Since we are only changing the columns corresponding to v, it suffices to show that for all $a_1, \ldots, a_t$ such that

$\sum_{i=0}^{t} a_i(-a\sin\alpha_j \mp x_i) = \sum_{i=0}^{t} a_i(a\cos\alpha_j - y_i) = 0$, we have

$$A := \sum_{i=0}^{t} a_i[a \pm \sin\alpha_j x_i - \cos\alpha_j y_i] = 0.$$

So, let $a_1, \dots, a_t$ be such integers. Since $\sum_{i=0}^{t} a_i(-a\sin\alpha_j \mp x_i) = 0$, we have

$$\pm\sin\alpha_j \sum_{i=0}^{t} a_i x_i = -a\sin^2\alpha_j \sum_{i=0}^{t} a_i \tag{5.1}$$

Similarly, since $\sum_{i=0}^{t} a_i(a\cos\alpha_j - y_i) = 0$, we have

$$\cos\alpha_j \sum_{i=0}^{t} a_i y_i = a\cos^2\alpha_j \sum_{i=0}^{t} a_i. \tag{5.2}$$

Then, by Equations (5.1) and (5.2), we have

$$\begin{aligned} A &= \sum_{i=0}^{t} a_i[a \pm \sin\alpha_j x_i - \cos\alpha_j y_i] = a\sum_{i=0}^{t} a_i \pm \sin\alpha_j \sum_{i=0}^{t} a_i x_i - \cos\alpha_j \sum_{i=0}^{t} a_i y_i \\ &= a\sum_{i=0}^{t} a_i - a\sin^2\alpha_j \sum_{i=0}^{t} a_i - a\cos^2\alpha_j \sum_{i=0}^{t} a_i = a\sum_{i=1}^{t} a_i[1 - \sin^2\alpha_j - \cos^2\alpha_j] = 0. \end{aligned}$$

Hence, a row dependence in M' implies a row dependence in M. Since M is row independent by assumption, it follows that M' is row independent. Apply this process to all semi-free vertices of $H - V_s(H)$. The resulting matrix is the submatrix of the ρ_0-orbit rigidity matrix of a $\mathcal{C}_s$-gain framework with $|V(H - V_s(H))|$ free vertices and $|V_s(H)|$ fixed vertices obtained by removing zero or one columns. (If $V_{2k}(H) = \emptyset$, no columns are removed; otherwise, one column is removed.) The result follows by Proposition 5.1.1(1). $\square$

Proposition 5.2.6. Let $k \geq 3$, and $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_{kv}$-symmetric framework with $\mathcal{C}_{kv}$-gain framework (G, φ, ψ, p). If $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic, then (G, φ, ψ) is $\mathbb{D}_{2k}$-gain tight.

Proof. By Lemmas 5.2.4 and 5.2.5, we need only show that all balanced subgraphs H of G with non-empty edge set are $(2,3)$-sparse whenever $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic. So assume, for a contradiction, that $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic and that G has a balanced subgraph H with non-empty edge set with $|E(H)| > 2|V(H)| - 3$. By Propositions 3.3.6, 4.1.4 and 4.1.5, and by Lemma 3.3.12, we may assume that $\psi(e) = \mathrm{id}$ for all $e \in E(H)$.

Let M be the matrix obtained from $O_0(G, \varphi, \psi, p)$ by removing the columns which correspond to vertices in $V(G) \setminus V(H)$, together with the rows corresponding to their incident edges. If $V_2(H) = \emptyset$, then M is the submatrix of a standard rigidity matrix for a graph F with $|E(F)| > 2|V(F)| - 3$, obtained by removing zero or two columns. (If $V_{2k}(H) = \emptyset$, no columns are removed; if $|V_{2k}(H)| = 1$, two columns are removed.) By row independence, this implies that H is $(2, 3)$-sparse, a contradiction. Hence, we may assume that $V_2(H) \neq \emptyset$.

For $v \in V_2(H)$, let $0 \leq j \leq k-1, a \in \mathbb{R}$ be such that $p(v) = a(-\sin\alpha_j \quad \cos\alpha_j)^T$, where $\alpha_j := 2\pi j/k$. Choose $\mathcal{B}_v^0 = \{M_v^0 = (-\sin\alpha_j \quad \cos\alpha_j)^T\}$ as a basis for $U(p(v))$ when constructing $O_0(G, \varphi, \psi, p)$. Let $v_1, \ldots, v_t$ be the vertices incident to v in H and, for $1 \leq i \leq t$, let $p(v_i) = (x_i \quad y_i)^T$. For $1 \leq i \leq t$, let $e_i := (v, v_i)$. For each $1 \leq i \leq t$, the row in M corresponding to e_i has the form

$$e_i = \Big(\ldots \quad a + \sin\alpha_j x_i - \cos\alpha_j y_i \quad \ldots \quad \Big(x_i + a\sin\alpha_j \quad y_i - a\cos\alpha_j\Big) M_{v_i}^0 \quad \ldots\Big).$$

Order the rows of M such that, for all $1 \leq i \leq t$, the i^{th} row in M is the row corresponding to e_i. Then, substitute the columns corresponding to v in M with the following two columns:

$$\begin{pmatrix} -a\sin\alpha_j - x_1 & a\cos\alpha_j - y_1 \\ \vdots & \vdots \\ -a\sin\alpha_j - x_t & a\cos\alpha_j - y_t \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} [p(v) - p(v_1)]^T \\ \vdots \\ [p(v) - p(v_t)]^T \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Let M' be the matrix obtained by applying this change. Apply this change to all semi-free vertices of M. Similarly as in the proof of Lemma 5.2.5, we can see that the matrix obtained has full rank. However, the resulting matrix is the submatrix of the standard rigidity matrix for a graph F with $|E(F)| > 2|V(F)| - 3$, obtained by removing zero or two columns. (If $V_{2k}(H) = \emptyset$, no columns are removed; if $V_{2k}(H) \neq \emptyset$, two columns are removed.) This is a contradiction, so the result holds. □

5.3 Gain graph extensions

We conclude the chapter by presenting some gain graph operations known as *extensions*, which will be used in Chapters 6 and 7 to prove the sufficiency of the conditions given in Subsections 5.1.1, 5.1.2 and, in some cases, 5.1.3. As the name suggests, extensions add vertices to the gain graph. Each extension has (at least) one inverse operation, called a *reduction*. Throughout this section, we let Γ be a cyclic group and (G, ψ) be a Γ-gain graph. We will construct a Γ-gain graph (G', ψ') by applying an extension to (G, ψ). Depending on the extension we are working with, we may apply restrictions on the order of Γ, in which case we will specify it.

5.3.1 Adding a vertex of degree 1

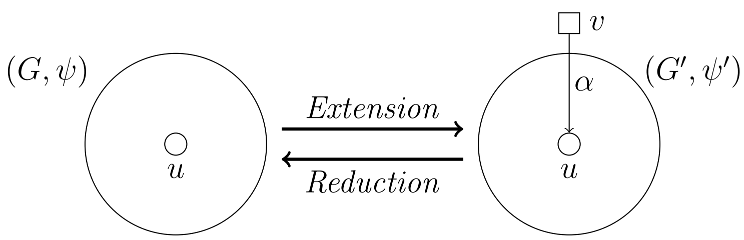


Figure 5.2: Example of a fix-0-extension, where u is free and α is an arbitrary gain.

The following move will only be used to study the infinitesimal rigidity of frameworks which are symmetric with respect to the reflection group $\mathcal{C}_s$. Hence, in this subsection, we let $|\Gamma| = 2$.

Definition 5.3.1. A *fix-0-extension* chooses a vertex $u \in V(G)$, adds a new fixed vertex v to $V_2(G)$, and connects it to u with a new edge e. We label e arbitrarily, unless $u \in V_2(G)$, in which case $\psi'(e) = \mathrm{id}$, and we let $\psi'(f) = \psi(f)$ for all $f \in E(G)$. The inverse operation of a fix-0-extension is called a *fix-0-reduction*. See Figure 5.2 for an illustration.

Lemma 5.3.2. For $0 \le j \le 1$, let (G, ψ, p) be a ρ_j-isostatic $\mathcal{C}_s$-gain framework. Suppose (G', ψ') is obtained by applying a fix-0-extension to (G, ψ). Suppose further that, whenever $j = 1$, the fix-0-extension from which we obtain (G', ψ') connects the new fixed vertex to a free vertex. Then there is a map $p' : V(G') \to \mathbb{R}^2$ such that (G', ψ', p') is a ρ_j-isostatic $\mathcal{C}_s$-gain framework.

Proof. With the same notation as in Definition 5.3.1, we define $p' : V(G') \to \mathbb{R}^2$ so that $p'_w = p_w$ for all $w \in V(G)$, p'_v lies on the y-axis, and the y-coordinates of p'_v, p_u differ. Let $p'_v = (0 \quad y_v)^T$ and $p'_u = (x_u \quad y_u)^T$. If $j = 1$, then $x_u \neq 0$ since $u \in V_1(G)$. We have

$$O_0\left(G', \psi', p'\right) = \left(\begin{array}{c|c} y_v - y_u & 0 \\ \hline 0 & O_0\left(G, \psi, p\right) \end{array}\right)$$

and

$$O_1\left(G', \varphi', \psi', p'\right) = \left(\begin{array}{c|c} -\,x_u & 0 \\ \hline 0 & O_1\left(G, \varphi, \psi, p\right) \end{array}\right).$$

For $j = 0, 1$, we have added one row and one column to $O_j(G, \psi, p)$. Hence, it suffices to show that the rows of the new matrices are independent. This follows from the fact that $y_u \neq y_v$ and $x_u \neq 0$. $\square$

Note that Lemma 5.3.2 does not take into consideration the case where $j = 1$ and $u \in V_2(G)$. This is because, by Proposition 5.1.1 (2), if $(\tilde{G}, \tilde{p})$ is a ρ_1-symmetrically isostatic $\mathcal{C}_s$-symmetric framework, then its Γ-gain graph (G, ψ) is $(2, 1, 3, 2)$-gain tight. In particular, any two vertices in $V_2(G)$ cannot be joined by an edge. Hence, when proving the sufficiency of the sparsity conditions for this case, if we apply a fix-0-reduction at a fixed vertex $v \in V_2(G)$, we may always assume that the vertex v adjacent to it is free.

5.3.2 Adding a vertex of degree 2

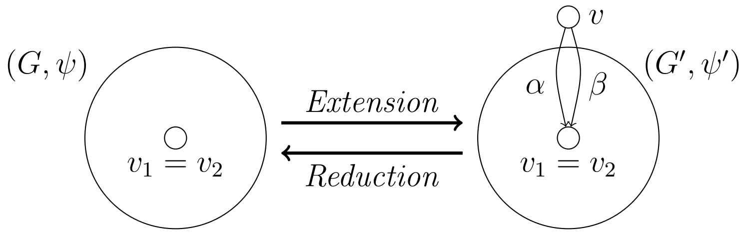


Figure 5.3: Example of a 0-extension where v_1 and v_2 coincide. Here we must have $\alpha \neq \beta$.

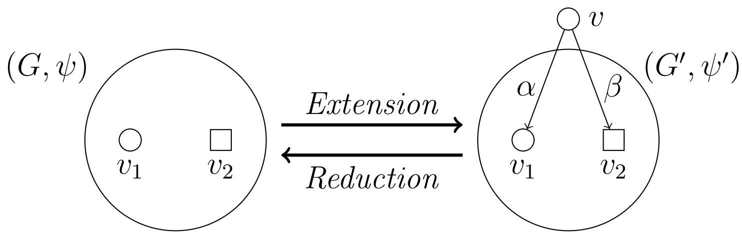


Figure 5.4: Example of a 0-extension where v_1 is free and v_2 is fixed. Here the gains α and β are arbitrary. (The cases where v_1 and v_2 are both free or both fixed are also allowed.)

Definition 5.3.3. A *0-extension* chooses two vertices $v_1, v_2 \in V(G)$ (we may choose $v_1 = v_2$ provided that $v_1 \in V_1(G)$) and adds a free vertex v, together with two edges $e_1 = (v, v_1), e_2 = (v, v_2)$. We let $\psi'(e) = \psi(e)$ for all $e \in E(G)$. If v_1, v_2 coincide, we choose ψ' such that $\psi'(e_1) \neq \psi'(e_2)$. In all other cases, we label e_1, e_2 freely. The inverse operation of a 0-extension is called a 0*-reduction*. See Figures 5.3 and 5.4 for illustrations.

Lemma 5.3.4. Given an irreducible representation ρ of Γ and an injective homomorphism $\tau : \Gamma \to O(\mathbb{R}^2)$, let (G, ψ, p) be a ρ-isostatic $\tau(\Gamma)$-gain framework. If (G', φ', ψ') is obtained by applying a 0-extension to (G, ψ), then there is a map $p' : V(G') \to \mathbb{R}^2$ such that (G', ψ', p') is a ρ-isostatic $\tau(\Gamma)$-gain framework.

Proof. With the same notation as in Definition 5.3.3, we define $p' : V(G') \to \mathbb{R}^2$ such that $p'_w = p_w$ for all $w \in V(G)$ for all $w \in V(G)$ and p'_v does not lie on the line through $\tau(\psi(e_1))p(v_1)$ and $\tau(\psi(e_2))p(v_2)$. Then, the ρ-orbit matrix of (G', ψ', p') is the matrix

$$M' = \left(\begin{array}{c|c} [p'_v - \tau(\psi(e_1))p(v_1)]^T M_v^j & \star \\ [p'_v - \tau(\psi(e_2))p(v_2)]^T M_v^j & \star \\ \hline 0 & M \end{array}\right),$$

where M denotes the ρ-orbit matrix of (G, ψ, p). Since M is row independent by assumption, it suffices to show that the first two rows in M are row independent. This follows from the fact that p'_v does not lie on the line between $\tau(\psi(e_1))p(v_1)$ and $\tau(\psi(e_1))p(v_1)$. $\square$

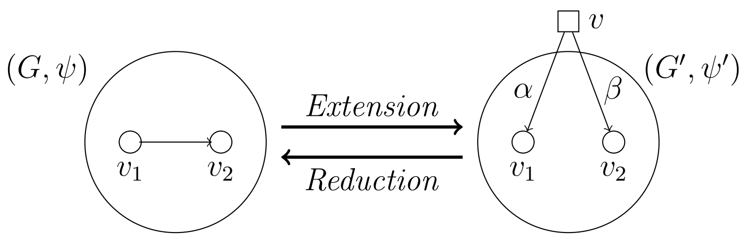


Figure 5.5: Example of a fix-1-extension, where α and β are arbitrary gains. The vertices v_1, v_2 are allowed to be fixed, although for a ρ_1-isostatic framework, there is no edge joining fixed vertices.

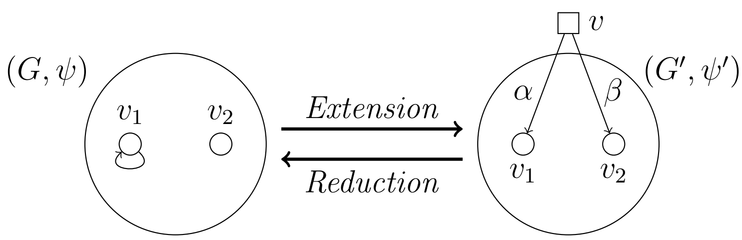


Figure 5.6: Example of a fix-1-extension, where α and β are arbitrary gains. The vertex v_2 is allowed to be fixed.

The following extension will only be used to study the infinitesimal rigidity of $\mathcal{C}_s$-symmetric frameworks.

Definition 5.3.5. A *fix-1-extension* chooses two distinct vertices $u_1, u_2 \in E(G)$ and an edge $e \in E(G)$ which can either be (v_1, v_2) or, if v_1 (respectively, v_2) is free, a loop at v_1 (respectively, v_2). It removes e, and adds a fixed vertex v, together with the edges $e_1 = (v, v_1), e_2 = (v, v_2)$. We label e_1 and e_2 freely, and we let $\psi'(f) = \psi(f)$ for all $f \in E(G)$. The inverse operation of a fix-1-extension is called a *fix-1-reduction.* See Figures 5.5 and 5.6 for illustrations.

Lemma 5.3.6. Let $\Gamma = \langle \gamma \rangle$ be the cyclic group of order 2, $0 \leq j \leq 1$, and let (G, ψ, p) be a ρ_j-symmetrically isostatic $\mathcal{C}_s$-gain framework. Let (G', ψ') be obtained by applying a fix-1-extension to (G, ψ). With the same notation as in Definition 5.3.5, assume that if $e = (v_1, v_2)$, then the line through $p(v_1)$ and $\tau(\psi(e))p(v_2)$ and the line through $\sigma p(v_1)$ and $\sigma\tau(\psi(e))p(v_2)$ meet in at least one point. Assume further that if e is a loop, then $p(v_1), p(v_2)$ do not share the same y-coordinate. Then there is a map $p' : V(G') \to \mathbb{R}^2$ such that (G', ψ', p') is a ρ_j-symmetrically isostatic $\mathcal{C}_s$-gain framework.

Proof. Throughout the proof, we use the same notation as that in Definition 5.3.5 and, for $1 \leq i \leq 2$, we let x_i and y_i be, respectively, the x-coordinate and y-coordinate of $p(v_i)$. We let H be the subgraph obtained from G by removing e. Since v is fixed, we may assume that $\psi(e_1) = \psi(e_2) = \text{id}$.

We first show the result holds when e is a loop. Assume, without loss of generality, that v_1 is free and that e is a loop at v_1, and notice that $\psi(e) = \gamma$. By assumption, $y_1 - y_2 \neq 0$. Moreover, since (G, ψ) is ρ_j-symmetrically isostatic and G contains a loop edge, we can use Proposition 5.1.1(2) to deduce that $j = 0$. Define $p' : V(G') \to \mathbb{R}^2$ such that $p'_w = p_w$ for all $w \in V(G)$ and p'_v be the mid-point of the line segment between $p(v_1)$ and $\sigma p(v_1)$. Then, p'_v lies on the y-axis and has y-coordinate y_1, so that

$$O_0(G', \psi', p') = \left(\begin{array}{c|ccc} 0 & x_1 & 0 & 0 \\ y_1 - y_2 & 0 & 0 & [p(u_2) - p'_v]^T M^0_{v_2} \\ \hline 0 & \multicolumn{3}{c}{O_0\left(H, \psi|_{E(H)}, p|_{V(H)}\right)} \end{array}\right).$$

Multiplying the first row by 4, we obtain the row corresponding to e which, added to the bottom right block, forms $O_0(G,\psi,p)$. Since $O_0(G',\psi',p')$ is obtained by adding one row and one column to $O_0(G,\psi,p)$, it suffices to show that the additional row does not add a dependence. This follows from the fact that $y_1-y_2\neq 0$. Hence, the result holds whenever e is a loop.

Now, assume that $e=(v_1,v_2)$. Let $t:=0$ if $\psi(e)=\gamma$ and $t:=1$ if $\psi(e)=\mathrm{id}$. Since the line through $p(v_1)$ and $\tau(\psi(e))p(v_2)$ and the line through $\sigma p(v_1)$ and $\sigma\tau(\psi(e))p(v_2)$ meet, they must meet in a point P that lies on the y-axis. Simple calculations show that the y-coordinate of P is

$$y=-\frac{y_1-y_2}{x_1+(-1)^t x_2}x_1+y_1=(-1)^{t+1}\frac{y_2-y_1}{x_1+(-1)^t x_2}x_2+y_2. \tag{5.3}$$

Define $p':V(G')\to\mathbb{R}^2$ such that $p'_w=p_w$ for all $w\in V(G)$ and $p'_v=P$. Then, we have

$$O_j(G',\psi',p')=\left(\begin{array}{c|cc}\begin{pmatrix}-x_1 & y-y_1\end{pmatrix}M_v^j & [p(v_1)-P]^T M_{v_1}^j & 0\\ \begin{pmatrix}-x_2 & y-y_2\end{pmatrix}M_v^j & 0 & [p(v_2)-P]^T M_{v_2}^j\\ \hline 0 & \multicolumn{2}{c}{O_j\left(H,\psi|_{E(H)},p|_{V(H)}\right)}\end{array}\right).$$

So, multiplying the row corresponding to e_i by $(x_1+(-1)^t x_2)/x_i$ for $1\leq i\leq 2$, and using (5.3), we see that $O_j(G',\psi',p')$ is

$$\left(\begin{array}{c|cccc} a & x_1+(-1)^t x_2 & y_1-y_2 & 0 & 0\\ b & 0 & 0 & x_1+(-1)^t x_2 & (-1)^t(y_2-y_1)\\ \hline 0 & \multicolumn{4}{c}{O_j\left(H,\psi|_{E(H)},p|_{V(H)}\right)}\end{array}\right),$$

where the first column corresponding to v_1 (respectively, v_2) in $O_0(G',\psi',p')$ vanishes if v_1 (respectively, v_2) is fixed, and the second column corresponding to v_1 (respectively, v_2) in $O_1(G',\psi',p')$ vanishes if v_1 (respectively, v_2) is fixed, and where $a=\begin{pmatrix}-x_1+(-1)^{t+1}x_2 & y_2-y_1\end{pmatrix}M_v^j$, $b=\begin{pmatrix}-x_1+(-1)^{t+1}x_2 & (-1)^{t+1}(y_2-y_1)\end{pmatrix}M_v^j$. Apply the following row operations: if $j=t=0$, add the second row to the first; in all other cases, subtract the second row from the first. Then, we obtain the row corresponding to e which, added to the bottom right block, forms $O_j(G,\psi,p)$.

Similarly as in the case where e is a loop, it suffices to show that the second row does not add a dependence to $O_j(G, \psi, p)$. This follows from the fact that the line through $p(v_1)$ and $\tau(\psi(e))p(v_2)$ and the line through $\sigma p(v_1)$ and $\sigma\tau(\psi(e))p(v_2)$ meet at a point, which implies that the entry in the leftmost column is not zero. $\square$

5.3.3 Adding a vertex of degree 3

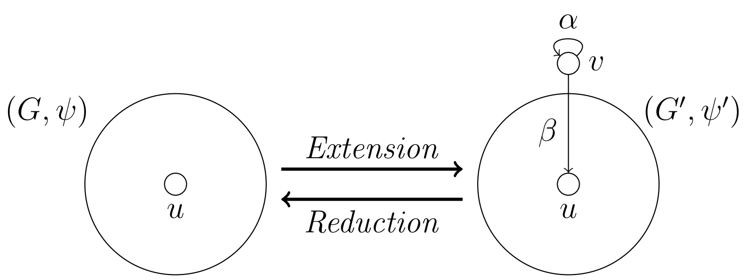


Figure 5.7: Example of a loop-1-extension, where α is a non-identity gain, and β is an arbitrary gain. (The case where u is not free is also allowed.)

Definition 5.3.7. A *loop-1-extension* adds a free vertex v to $V(G)$ together with an edge $e = (v, u)$ for some $u \in V(G)$ and a loop $e_L = (v, v)$. We let $\psi'(f) = \psi(f)$ for all $f \in E(G)$. We let $\psi'(e_L)$ be any non-identity element of Γ, and $\psi'(e)$ can be chosen freely. The inverse operation of a loop-1-extension is called a *loop-1-reduction*. See Figure 5.7 for an illustration.

Lemma 5.3.8. Let Γ be a cyclic group of order $k \geq 2$, $\tau : \Gamma \to O(\mathbb{R}^2)$ be an injective homomorphism and (G, ψ, p) be a ρ_j-isostatic $\tau(\Gamma)$-gain framework for some $0 \leq j \leq k-1$ such that $j = 0$ when $k = 2$. Let $\gamma \in \Gamma$ correspond to the k-fold rotation (or the reflection) under τ. Let (G', ψ') be obtained from (G, ψ) by applying a loop-1-extension. With the same notation as Definition 5.3.7, let $g := \psi'(e_L)$ and $h := \psi'(e)$. Assume the following hold:

(i) If k is even and j is odd, then $g \neq \gamma^{k/2}$.

(ii) If $\tau(\Gamma) = \mathcal{C}_k$ and $j = 0$, then u is free.

(iii) If $k \geq 4, 2 \leq j \leq k-2$ and u is fixed, then there is no $n \in S_0(k,j)$ such that $\langle g \rangle \simeq \mathbb{Z}_n$.

Then there is a map $p' : V(G') \to \mathbb{R}^2$ such that (G', ψ', p') is a ρ_j-isostatic $\tau(\Gamma)$-gain framework.

Proof. With the same notation as in Definition 5.3.7, let $p' : V(G') \to \mathbb{R}^2$ be defined such that $p'_w = p_w$ for all $w \in V(G)$. We have

$$O_j(G', \psi', p') = \left(\begin{array}{c|c} [I_2 + \rho_j(g)I_2 - \tau(g) - \rho_j(g)\tau(g^{-1})](p'_v)^T & 0 \\ [p'_v - \tau(h)p_u]^T & \star \\ \hline 0 & O_j\left(G, \psi, p\right) \end{array}\right).$$

So $O_j(G', \psi', p')$ is obtained from $O_j(G, \psi, p)$ by adding two rows and two columns. Since $O_j(G, \psi, p)$ has full rank by assumption, it is enough to show that the first two rows of $O_j(G', \psi', p')$ are linearly independent for some choice of p'_v. Let A be the matrix $I_2 + \rho_j(g)I_2 - \tau(g) - \rho_j(g)\tau(g^{-1})$. If $\tau(\Gamma) = \mathcal{C}_s$, then $j = 0$ and $\tau(g) = \sigma$ by assumption, so A is the 2×2 matrix whose only non-zero entry is $(A)_{1,1} = 4$. If we choose p'_v such that it does not share the same y-coordinate as p'_u, it is then easy to see that the first two rows of the matrix are linearly independent. Hence, $O_j(G', \psi', p')$ has full rank, as required.

So, we may assume that $\tau(\Gamma) = \mathcal{C}_k$. Let $g = \gamma^t$ for some $1 \leq t \leq k-1$, $\alpha = 2\pi t/k$, and $\omega = \exp(2\pi\sqrt{-1}/k)$. Then,

$$A = \begin{pmatrix} (1-\cos(\alpha))(1+\omega^{jt}) & \sin(\alpha)(1-\omega^{jt}) \\ -\sin(\alpha)(1-\omega^{jt}) & (1-\cos(\alpha))(1+\omega^{jt}) \end{pmatrix}.$$

We show that A is not the zero matrix. Assume, for a contradiction, that A is the zero matrix. Since $1 \leq t \leq k-1$, we know $\cos(\alpha) \neq 1$ and so $\omega^{jt} = -1$, i.e. there is some odd integer m such that $2\pi jt/k = m\pi$. Moreover, $\sin(\alpha)(1-\omega^{jt}) = 2\sin(\alpha) = 0$. Since $1 \leq t \leq k-1$, $\alpha = \pi$, and so $t = k/2$. It follows that $j = m$, and so j is odd. This contradicts (i), so, as claimed, A is not the zero matrix.

If u is free, then, by the injectivity of p, the vector p_u, and hence also the vector $\tau(h)p_u$, cannot be zero. So unless q is a multiple of $\tau(h)p_u$, the affine map

$q \mapsto q - \tau(h)p_u$ applied to λq gives vectors of different directions for each scalar λ. The linear map A applied to λq, however, only produces vectors that are multiples of the vector Aq. This implies that $\{Aq, q-\tau(h)p_u\}$ is linearly independent for some q, and so we may choose p'_v to be such q. Then $O_j(G', \psi', p')$ is linearly independent, as required.

So, assume that u is fixed. By (ii), we may also assume that $1 \leq j \leq k-1$. In particular, this implies, by assumption, that $k \neq 2$. Assume, for a contradiction, that there is no choice of p'_v such that the first two rows of $O_j(G', \psi', p')$ are linearly independent. This implies that A is a scalar multiple of I_2. This happens exactly when $\sin(\alpha)(1-\omega^{jt}) = 0$. If $\sin(\alpha) = 0$, then $\alpha = \pi$, and $t = k/2$. Hence, $\omega^{jt} = \exp(\pi i j)$. By (i), j must be even, and so $\omega^{jt} = 1$. If $\sin(\alpha) \neq 0$, then clearly $\omega^{jt} = 1$. Hence, in both cases we have $\omega^{jt} = 1$, i.e. $jt = mk$ for some integer m. If $j = 1$, this implies that t is a multiple of k, contradicting the fact that $1 \leq t \leq k-1$. Hence, $j \neq 1$. Similarly, $j \neq k-1$: if $j = k-1$, then $\omega^{jt} = \omega^{-t}$. Since ω^{jt} is real, this equals $\omega^t = 1$, and so $t = k$, a contradiction. Hence, $k \geq 4$ and $2 \leq j \leq k-2$. We show that there is an integer $n \in S_0(k, j)$ such that $\langle g \rangle \simeq \mathbb{Z}_n$, contradicting (iii).

Let $n = k/\gcd(k,t) = \mathrm{lcm}\,(k,t)/t$. Then, we know from group theory (see e.g. [17]) that $\langle g \rangle = \langle \gamma^t \rangle \simeq \mathbb{Z}_n$, and that $m' = mk/\mathrm{lcm}\,(k,t)$ is an integer (since mk is a multiple of both k and t), and so, since $j = mk/t = nm'$, we have $j \equiv 0 \mod n$. Moreover, $k = n\gcd(k,t)$, so $n|k$. Hence, $n \in S_0(k,j)$, as required. This contradicts (iii). Thus, there is a choice of p'_v such that the first two rows of $O_j(G', \psi', p')$ are linearly independent. It follows that $O_j(G', \psi', p')$ has full rank. □

Remark 5.3.9. It was shown in [[39], Section 4.3] that for the Γ-gain graph of a rotationally symmetric framework in the plane, an edge joining the fixed vertex u with a free vertex v gives the same constraint in the fully-symmetric orbit rigidity matrix as a loop edge on v which corresponds to a regular $|\Gamma|$-polygon in the covering framework. Hence we have condition (ii) in Lemma 5.3.8. This is clear geometrically, because both of these edges force the vertices in the orbit of v to keep their distance to the origin in any symmetry-preserving motion. Thus, for analysing fully-symmetric infinitesimal rigidity, one may always reduce the problem to the case when the group acts freely on the vertices. However, Lemma 5.3.8 shows that this simple reduction is not possible for the reflection group nor for analysing "incidentally symmetric" infinitesimal rigidity for any rotational group.

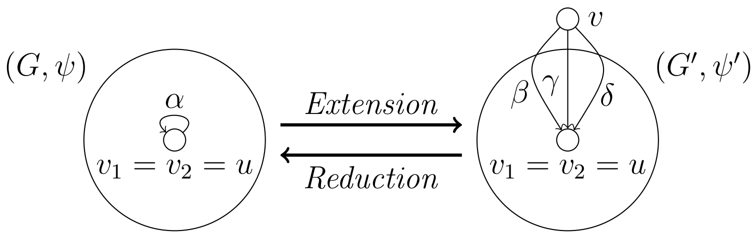


Figure 5.8: Example of a 1-extension, where $\alpha = \beta\gamma^{-1}$ and δ is an arbitrary gain. In this example, we can see that v_1, v_2, u are allowed to coincide.

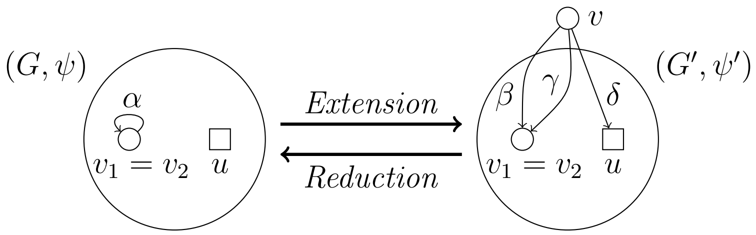


Figure 5.9: Example of a 1-extension, where $\alpha = \beta\gamma^{-1}$ and δ is an arbitrary gain. In this example, we can see that v_1 and v_2 are allowed to coincide. We also allow u to be free.

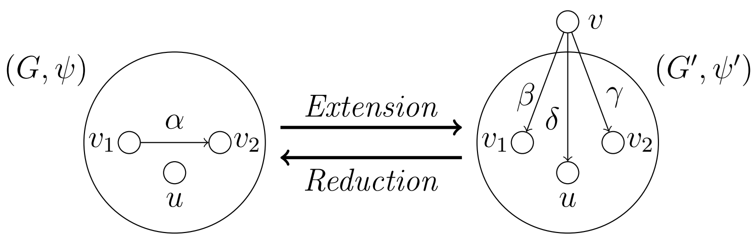


Figure 5.10: Example of a 1-extension, where $\alpha = \beta\gamma^{-1}$ and δ is an arbitrary gain. Any vertex in $\{v_1, v_2, v_3\}$ is allowed to be fixed.

Definition 5.3.10. A *1-extension* chooses a vertex $u \in V(G)$ and an edge $e = (v_1, v_2) \in E(G)$ (any pair of free vertices in $\{v_1, v_2, u\}$ are allowed to coincide; further, v_1, v_2, u are all allowed to coincide, provided they are not fixed and $|\Gamma| \geq 3$), removes e and adds a new free vertex v to $V(G)$, together with three edges $e_1 = (v, v_1), e_2 = (v, v_2), e_3 = (v, u)$. We let $\psi'(f) = \psi(f)$ for all $f \in E(G)$. The edges e_1, e_2 are labelled such that $\psi'(e_1)^{-1}\psi'(e_2) = \psi(e)$. The label of e_3 is chosen such that it is locally unbalanced, i.e. every two-cycle $e_i e_j^{-1}$, if it exists, is unbalanced. The inverse operation of a 1-extension is called a 1-*reduction*. See Figures 5.8, 5.9 and 5.10 for illustrations.

Lemma 5.3.11. Let Γ be a cyclic group of order k, $\tau : \Gamma \to O(\mathbb{R}^2)$ be an injective homomorphism, and (G, ψ, p) be a ρ_j-symmetrically isostatic $\tau(\Gamma)$-gain framework for some $0 \leq j \leq k-1$. With the same notation as in Definition 5.3.10, assume that the points $\tau(\psi(e_1))p(v_1), \tau(\psi(e_2))p(v_2)$ and $\tau(\psi(e_3))p(u)$ do not lie on the same line. If (G', ψ') is obtained from (G, ψ) by applying a 1-extension, then there is a map $p' : V(G') \to \mathbb{R}^2$ such that (G', ψ', p') is a ρ_j-isostatic $\tau(\Gamma)$-gain framework.

Proof. With the same notation as that of Definition 5.3.10, let H be the subgraph of G obtained by removing e. If v_1, v_2 are free, then an analogous proof to that of [[56], Lemma 6.1] gives the result. So, without loss of generality, assume v_1 is fixed. In particular, v_1 cannot coincide with either v_2 or u, and we may assume $\psi(e_1) = \psi(e_2) = \psi(e) = \mathrm{id}$. Let $\psi(e_3) = \delta$.

Define $p' : V(G') \to \mathbb{R}^2$ such that $p'_w = p_w$ for all $w \in V(G)$ and p'_v lies on the midpoint of the line through $p(v_1)$ and $p(v_2)$. Then, $O_j(G', \psi', p')$ is the matrix

$$\left(\begin{array}{c|cccc}
\rho_j(\delta)[p'_v - \tau(\delta)p_u]^T & 0 & \star & \star \\
1/2[p(v_2) - p(v_1)]^T & 1/2[p(v_1) - p(v_2)]^T M^j_{v_1} & 0 & 0 \\
1/2[p(v_1) - p(v_2)]^T & 0 & 1/2[p(v_2) - p(v_1)]^T M^j_{v_2} & 0 \\
\hline
0 & \multicolumn{4}{c}{O_j\left(H, \psi_{E(H)}, p|_{V(H)}\right).}
\end{array}\right).$$

Adding the second row to the third, and multiplying the result by 2, we obtain the row representing e in $O_j\left(H, \psi_{E(H)}, p|_{V(H)}\right)$. Now, $O_j(G', \psi', p')$ is obtained by adding two rows and two columns to $O_j(G, \psi, p)$, so it suffices to show that the first two entries of the two added rows are independent. Since $p(v_1), p(v_2)$ and $\tau(\delta)p_u$ do not lie on the same line, the line through p'_v and $\tau(\delta)p_u$ is not parallel to the line through $p(v_1)$ and $p(v_2)$. Hence, the upper left 2×2 matrix has full rank, and $O_j(G', \psi', p')$ has full rank. If $\tau(\Gamma) = \mathcal{C}_s$, and $p(v_1), p(v_2)$ both lie on the symmetry line, then p'_v also lies on the symmetry line. In such a case, we may perturb p'_v slightly without changing the rank of $O_j(G', \psi', p')$, in order to avoid placing the free vertex v on the symmetry line. $\square$

5.3.4 Adding two vertices of degree 3

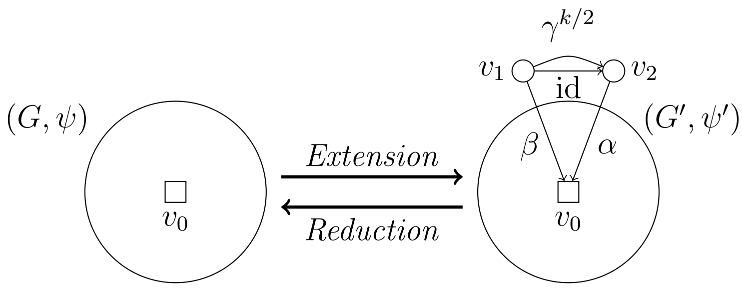


Figure 5.11: Example of a 2-vertex extension, where α, β are arbitrary gains.

The following extension is defined on Γ-gain graphs with $|V_{|\Gamma|}(G)| = 1$ and with $|\Gamma| \geq 2$ even. Recall that, for $k := |\Gamma|$, the cyclic group $\Gamma = \langle \gamma \rangle$ is isomorphic to $\mathbb{Z}_k$, through the isomorphism which maps γ to 1.

Definition 5.3.12. A *2-vertex-extension* adds two free vertices v_1, v_2 and connects them to the fixed vertex. Then, it adds two parallel edges $e_1, e_2 = (v_1, v_2)$ between v_1 and v_2. We define ψ' such that $\psi'(e) = \psi(e)$ for all $e \in E(G)$, the new edges incident with the fixed vertex are labelled arbitrarily, and $\psi'(e_1) = \mathrm{id}, \psi'(e_1) = \gamma^{k/2}$. The inverse operation of a 2-vertex-extension is called a *2-vertex-reduction*. See Figure 5.11 for an illustration.

Lemma 5.3.13. Let $k \geq 2$ be even and $1 \leq j \leq k-1$. Let (G, ψ, p) be a ρ_j-symmetrically isostatic $\mathcal{C}_k$-gain framework with $V_2(G) = \{v_0\}$. If (G', ψ') is obtained by applying a 2-vertex-extension to (G, ψ), then there is a map $p' : V(G') \to \mathbb{R}^2$ such that (G', ψ', p') is a ρ_j-symmetrically isostatic $\mathcal{C}_k$-gain framework.

Proof. With the same notation as that of Definition 5.3.12, let $p' : V(G') \to \mathbb{R}^2$ be defined by letting $p'_w = p_w$ for all $w \in V(G)$ and such that $p'(v_1)$ and $p'(v_2)$ are not scalar multiples of each other. Note that $\rho_j(k/2) = \exp(\pi\sqrt{-1}j) = (-1)^j$. Moreover, since v_0 is fixed, we may assume that the new edges incident to the fixed vertex are labelled id. Then, $O_j(G', \psi', p')$ is

$$\left(\begin{array}{cc|c} [p'(v_1) - p'(v_2)]^T & [p'(v_2) - p'(v_1)]^T & 0 \\ [p'(v_1) + p'(v_2)]^T & (-1)^j[p'(v_1) + p'(v_2)]^T & 0 \\ [p'(v_1)]^T & 0 & \star \\ [p'(v_2)]^T & 0 & \star \\ \hline \multicolumn{2}{c|}{0} & O_j\left(G, \psi, p\right) \end{array}\right),$$

where the first two columns correspond to v_1 and the second two columns correspond to v_2. Since $O_j(G, \psi, p)$ is row independent, it suffices to show that the rows of the new matrix are independent. This follows from the fact that $p'(v_1)$ and $p'(v_2)$ are not scalar multiples of each other. □

Chapter 6

Sufficient conditions for groups of order 2 and 3

In this chapter we establish the characterisations of symmetry-generic infinitesimally rigid bar-joint frameworks, where the symmetry group is cyclic and has order 2 or 3. Namely, given a $\tau(\Gamma)$-generic framework $(\tilde{G}, \tilde{p})$ where $\tau(\Gamma)$ is one of $\mathcal{C}_s, \mathcal{C}_2$ and $\mathcal{C}_3$, we show that the conditions of Section 5.1 which the Γ-gain graph (G, ψ) of $\tilde{G}$ must satisfy in order for $(\tilde{G}, \tilde{p})$ to be infinitesimally rigid are also sufficient (recall Propositions 5.1.1, 5.1.2 and 5.1.5). As mentioned in Chapter 5, we adopt a proof by induction on the order of (G, ψ), for which we employ reduction operations, i.e. inverse operations as the extensions given in Section 5.3.

We structure the chapter as follows. In Section 6.1, we give some preliminary results which we need for a full characterisation of infinitesimally rigid $\mathcal{C}_s$-generic, $\mathcal{C}_2$-generic and $\mathcal{C}_3$-generic frameworks. In Section 6.2 we show that, under certain conditions, a free vertex of (G, ψ) can always be reduced in a way that the sparsity conditions of (G, ψ) do not break. This will be crucial for the inductive arguments which we use to prove the main results of the chapter. The final combinatorial results are given in Sections 6.3, 6.4 and 6.5, in which we will consider the groups $\mathcal{C}_s, \mathcal{C}_2$ and $\mathcal{C}_3$, respectively. As we will see, the notion of $(2, m, 3, l)$-gain sparsity suffices to combinatorially characterise infinitesimally rigid $\mathcal{C}_s$-generic frameworks, $\mathcal{C}_2$-generic frameworks and $\mathcal{C}_3$-generic frameworks. However, we often work with the slightly stronger notion of $(2, m, 3, l)'$-gain sparsity, since the results concerning $(2, m, 3, l)$-gain tight graphs in this chapter easily generalise to $(2, m, 3, l)'$-gain tight graphs. This slight generalisation will be useful in Chapter 7 when we combinatorially characterise infinitesimally rigid $\mathcal{C}_4$-generic frameworks and $\mathcal{C}_6$-generic frameworks.

6.1 General combinatorial results

Lemma 6.1.1. Let $0 \leq m, l \leq 2$ be such that $0 \leq l - m \leq 1$. Let Γ be a cyclic group of order $k \geq 2$ and (G, ψ) be a Γ-gain graph with at least one free vertex, and let $s, t \in \mathbb{N}$ be the number of free vertices in G of degree 2 and 3, respectively. Assume (G, ψ) is $(2, m, 3, l)$-gain tight. The following hold:

(i) Each free vertex has degree at least 2, each fixed vertex has degree at least m.

(ii) Suppose that there is some $d \geq 0$ such that $\deg(v) \geq d$ for all $v \in V_k(G)$. Then $2s + t \geq |V_k(G)|(d - 2m) + 2l$.

Proof. For (i), let $v \in V(G)$. By the sparsity of (G, ψ), the subgraph H obtained from G by removing v satisfies

$$|E(H)| \leq \begin{cases} 2|V_1(G)| + m|V_k(G)| - l - 2 & \text{if } v \text{ is free} \\ 2|V_1(G)| + m|V_k(G)| - l - m & \text{if } v \text{ is fixed.} \end{cases}$$

But $|E(G)| = 2|V_1(G)| + m|V_k(G)| - l$. So there are at least 2 edges in G that are not in H when v is free, and there are at least m edges in G that are not in H when v is fixed. (i) follows.

For (ii), the average degree of G is

$$\hat{\rho} = \frac{2|E(G)|}{|V(G)|} = \frac{4|V_1(G)| + 2m|V_k(G)| - 2l}{|V(G)|}.$$

The minimum average degree ρ_{min} of G is attained when all free vertices, which are not the s and t vertices of degree 2 and 3, have degree 4, and all fixed vertices have degree d. So

$$\rho_{min} = \frac{2s + 3t + d\,|V_k\,(G)| + 4(|V_1(G)| - s - t)}{|V(G)|}.$$

By minimality, $\rho_{min} \leq \hat{\rho}$, and (ii) follows. $\square$

Proposition 6.1.2. Let $0 \leq m \leq 2, 0 \leq l \leq 3$. Let Γ be a cyclic group of order k and let (G, ψ) be a Γ-gain graph. Suppose there is some $v \in V_1(G)$ of degree 3 with no incident loops. If G is $(2, m, l)$-sparse, then there is no $(2, m, l)$-tight subgraph of $G - v$ which contains all neighbours of v (the neighbours of v need not be distinct).

Proof. Suppose such a subgraph H exists. Then the subgraph H' of G obtained from H by adding v and its incident edges satisfies

$$|E(H')| = |E(H)|+3 = 2|V_1(H)|+m|V_k(H)|-l+3 = 2|V_1(H')|+m|V_k(H')|-l+1,$$

a contradiction. Therefore, the result holds. $\square$

It is straightforward to check that all except two of the reductions are admissible, i.e. they maintain the relevant sparsity counts. However, when applying a 1-reduction or a fix-1-reduction, we add an edge. This edge might give rise to a subgraph that violates the sparsity count.

Definition 6.1.3. Let Γ be a cyclic group and (G, ψ) be a Γ-gain graph. Let $0 \leq m \leq l \leq 3$ be such that $m \leq 2$ and suppose that (G, ψ) is $(2, m, 3, l)$-gain tight (respectively, $(2, m, 3, l)'$-gain tight). We say a Γ-gain graph (G', ψ') obtained from (G, ψ) is *admissible* if (G', ψ') is $(2, m, 3, l)$-gain tight (respectively, $(2, m, 3, l)'$-gain tight). Equivalently, we say (G, ψ) *admits a reduction*.

Definition 6.1.4. Let Γ be a cyclic group and (G, ψ) be a Γ-gain graph. Let $0 \leq m \leq l \leq 3$ be such that $m \leq 2$ and suppose (G, ψ) is $(2, m, 3, l)$-gain tight. Let $v \in V(G)$ be a free vertex of degree 3, or a fixed vertex of degree 2. Let (G', ψ') be obtained from (G, ψ) by applying a 1-reduction or a fix-1-reduction at v, and let $e = (v_1, v_2)$ be the edge we add when we apply such reduction. Let H be a subgraph of $G - v$ with $v_1, v_2 \in E(H)$ and $E(H) \neq \emptyset$.

(i) We say H is a *general-count blocker* of v_1, v_2 (equivalently, of (G', ψ')) if $H + e$ is connected and H is $(2, m, l)$-tight.

(ii) We say H is a *balanced blocker* of e (equivalently, of (G', ψ')) if H is $(2, 3)$-tight and $H + e$ is balanced.

(iii) If (G, ψ) is also $(2, m, 3, l)'$-gain tight, we say H is a $\mathbb{Z}_2$*-blocker* of e (equivalently, of (G', ψ')) if H is $(2, 2)$-tight and $\langle H + e \rangle \simeq \mathbb{Z}_2$.

General-count blockers, balanced blockers and $\mathbb{Z}_2$-blockers are simply referred to as *blockers of* (G', ψ').

The following result states that, given two blockers H_1, H_2 with $E(H_1 \cap H_2) \neq \emptyset$, their union $H_1 \cup H_2$ can also be seen as a blocker. It will be used in Section 6.2 to show that a vertex of degree 3 always admits a 1-reduction, except for special cases (see Theorem 6.2.1).

Lemma 6.1.5. Let $0 \leq m \leq l \leq 2$ be such that $l \geq 1$, let Γ be a cyclic group of order k, and (G, ψ) be a $(2, m, 3, l)$-gain tight or $(2, m, 3, l)'$-gain tight Γ-gain graph. Suppose there is some $v \in V_1(G)$ of degree 3 with no incident loops. Let $(G_1, \psi_1), (G_2, \psi_2)$ be obtained from (G, ψ) by applying two different 1-reductions at v, which add the edges f_1 and f_2, respectively. Assume that, for $i = 1, 2$, (G_i, ψ_i) has a blocker H_i, and that $E(H_1 \cap H_2) \neq \emptyset$. Suppose that if $|V_k(G)| \geq 1$ then $m = 1$ and H_1, H_2 are not $\mathbb{Z}_2$-blockers. Let $H := H_1 \cup H_2$. The following hold:

(i) The blockers H_1, H_2 are not general-count blockers.

(ii) Either $H + f_1 + f_2$ is balanced and H is $(2, 3)$-tight or (G, ψ) is $(2, m, 3, l)'$-gain tight, $\langle H \rangle \simeq \mathbb{Z}_2$ and H is $(2, 2)$-tight.

Proof. Notice that $H_1 \cup H_2$ always contains all neighbours of v. To see this, we consider $|N_G(v)|$. If $|N_G(v)| = 1$ this is clear. If $|N_G(v)| = 2$, let v_1, v_2 be the neighbours of v and $e_1 = (v, v_1), e_1' = (v, v_1), e_2 = (v, v_2) \in E(G)$. By Propositions 3.2.6 and 3.2.9, we may assume that $\psi(e_1) = \psi(e_2) = \text{id}$ and that

$\psi(e_1') \neq \mathrm{id}$. Then, at most one 1-reduction at v adds a loop at v_1 (with gain $\psi(e_1')$) and no 1-reduction at v adds a loop at v_2. It follows that one of H_1, H_2 contains v_1 and v_2, and so $v_1, v_2 \in V(H_1 \cup H_2)$. Finally, let $|N_G(v)| = 3$. For $1 \leq i \leq 3$, let $e_i = (v, v_i) \in E(G)$. By Propositions 3.2.6 and 3.2.9, we may assume that $\psi(e_i) = \mathrm{id}$ for all $1 \leq i \leq 3$. Then, for each pair $1 \leq i \neq j \leq 3$, there is at most one 1-reduction at v which adds an edge between v_i and v_j (with gain id). It follows that $v_1, v_2, v_3 \in V(H_1 \cup H_2)$.

Throughout the proof, we let $H' = H_1 \cap H_2$ and we let $H_1', \dots, H_c'$ be the connected components of H'. Let $c_0 \leq c - 1$ be the number of isolated vertices of H', so that $H_1', \dots, H_{c_0}'$ are the isolated vertices of H', and $H_{c_0+1}', \dots, H_c'$ are the connected components of H' with non-empty edge set.

We first prove (i). Assume, for a contradiction, that one of H_1, H_2 is a general-count blocker. Without loss of generality, let it be H_1. If H_2 is also a general-count blocker then, since $|E(H')| \leq 2|V_1(H')| + m|V_k(H')| - l$, it is easy to check that

$$|E(H)| = |E(H_1)| + |E(H_2)| - |E(H')| \geq 2|V_1(H)| + m|V_k(H)| - l.$$

By Proposition 6.1.2, this is a contradiction. Hence, we may assume that H_2 is either a balanced blocker or, in the case where (G, ψ) is $(2, m, 3, l)'$-gain tight, a $\mathbb{Z}_2$-blocker. So, let $2 \leq l_2 \leq 3$ be the integer such that H is $(2, l_2)$-tight. Since H' is a subgraph of H_2, for each $c_0 + 1 \leq i \leq c$, H_i' must be $(2, l_2)$-sparse, and so

$$\begin{aligned}|E(H')| &= \sum_{i=1}^{c} |E(H_i')| \leq \sum_{i=1}^{c_0} [2|V(H_i')| - 2] + \sum_{i=c_0+1}^{c} [2|V(H_i')| - l_2] \\ &= 2|V(H')| - (2c_0 + l_2(c - c_0)).\end{aligned}$$

Therefore, letting $g = 2c_0 + l_2(c - c_0)$ and recalling that $V(G) = V_1(G) \,\dot\cup\, V_k(G)$,

$$\begin{aligned}|E(H)| &= |E(H_1)| + |E(H_2)| - |E(H')| \\ &\geq (2|V_1(H_1)| + m|V_k(H_1)| - l) + (2|V(H_2)| - l_2) - (2|V(H')| - g) \\ &= 2|V_1(H)| + m|V_k(H)| - l + (2 - m)(|V_k(H_2)| - |V_k(H')|) + g - l_2 \\ &\geq 2|V_1(H)| + m|V_k(H)| - l,\end{aligned}$$

where the last inequality holds because $0 \leq c_0 \leq c-1, m \leq 2$ and $V(H') \subseteq V(H_2)$. By Proposition 6.1.2, this is a contradiction. So H_1, H_2 cannot be general count blockers, as required.

We now prove (ii). If H_1, H_2 are $\mathbb{Z}_2$-blockers, then $|E(H_i')| \leq 2|V(H_i')|-2$ for all $1 \leq i \leq c$, since each H_i' is a subgraph of H_1, H_2. Hence, $|E(H')| \leq 2|V(H')|-c$ and so $|E(H)| \geq (2|V(H_1)|-2)+(2|V(H_2)|-2)-(2|V(H')|-2c) = 2|V(H)|+2(c-2)$. If $c \geq 2$ or $|V_k(H)| = 1$, then $|E(H)| \geq 2|V(H)|-2 \geq 2|V_1(H)|+m|V_k(H)|-l$, since $0 \leq m \leq l \leq 2$ and $l \geq 1$. This contradicts Proposition 6.1.2 or the sparsity of (G, ψ). Therefore, H' is connected and $V_k(H) = \emptyset$, so $\langle H + f_1 + f_2 \rangle \simeq \mathbb{Z}_2$ by Lemma 3.2.15. Hence, $|E(H)| = 2|V(H)|-2$ and (ii) holds.

So, we may assume that at least one of H_1, H_2 is a balanced blocker. Without loss of generality, assume that H_1 is a balanced blocker. Since each H_i' is a subgraph of H_1, $|E(H')| \leq 2|V(H')| - (2c_0 + 3(c - c_0))$ (see the proof of (i) for details). Let $2 \leq l_2 \leq 3$ be the integer such that H_2 is $(2, l_2)$-tight. We have

$$\begin{aligned} |E(H)| &= |E(H_1)| + |E(H_2)| - |E(H')| \\ &\geq (2|V(H_1)|-3) + (2|V(H_2)|-l_2) - (2|V(H')| - (2c_0 + 3(c-c_0))) \\ &= 2|V(H)| + 2c_0 + 3(c-c_0) - 3 - l_2 \geq 2|V(H)| + 3c - c_0 - 3 - l_2. \quad (6.1) \end{aligned}$$

Since $c_0 \leq c-1, 3c-c_0-3 \geq 2c-2$ and so $|E(H)| \geq 2|V(H)|+2c-2-l_2$. Since $m \leq 2, 1 \leq l$ and $l_2 \leq 3$, $|E(H)| \geq 2|V_1(H)|+m|V_k(H)|-l$ whenever $c \geq 2$. By Proposition 6.1.2 and the sparsity of (G, ψ), this is a contradiction. Therefore, H' is connected and $|E(H)| \geq 2|V(H)| - l_2$.

We now show that H' has at most one fixed vertex. Assume, for a contradiction, that $|V_k(H')| \geq 2$. By assumption, this implies that $m \neq 2$ and H_2 is a balanced blocker, so $l_2 = 3$. By Equation (6.1), $|E(H)| \geq 2|V(H)|-3 \geq 2|V_1(H)|+1$. If $m = 0$, this contradicts the sparsity of (G, ψ). Hence, $m = 1$. By Equation (6.1),

$$\begin{aligned} |E(H)| &\geq 2|V(H)|-3 = 2|V_1(H)| + |V_k(H)| + (|V_k(H)|-3) \\ &\geq 2|V_1(H)| + |V_k(H)| - 1 \geq 2|V_1(H)| + |V_k(H)| - l, \end{aligned}$$

where the last inequality holds because $m \leq l$ and $m = 1$. This contradicts Proposition 6.1.2 or the sparsity of (G, ψ). Hence, H' has at most one fixed vertex. If $V_k(H') = \emptyset$, then by Lemma 3.2.14, $\langle H + f_1 + f_2 \rangle \simeq \langle H_2 + f_2 \rangle$, and so (ii) holds by the sparsity of (G, ψ) and Equation (6.1). So we may assume that $V_k(H') = \{v_0\}$. Then, by Equation (6.1),

$$|E(H)| \geq 2|V(H)| - l_2 = 2|V_1(H)| - l_2 + 2 \geq 2|V_1(H)| - l,$$

since $1 \leq l$ and $l_2 \leq 3$. If $m = 0$, this contradicts Proposition 6.1.2 or the sparsity of (G, ψ), so $m \neq 0$. Similarly, by Equation (6.1) and the fact that $l_2 \leq 3$, we have $|E(H)| \geq 2|V(H)| - 3 \geq 2|V_1(H)| + |V_k(H)| - 2$. If $(m, l) = (1, 2)$, this contradicts Proposition 6.1.2, so $(m, l) \neq (1, 2)$. Hence, (m, l) is one of $(1, 1)$ and $(2, 2)$. In both cases, we can see that H' is $(2, 3)$-tight, as otherwise we can see $|E(H)| \geq 2|V(H)| - l_2 + 1 \geq 2|V(H)| - 2 = 2|V_1(H)| + |V_k(H)| - 1$, contradicting Proposition 6.1.2 or the sparsity of (G, ψ). By Lemma 3.2.30, v_0 is not a cut vertex of H'. Then, $\langle H + f_1 + f_2 \rangle = \langle H + f_2 \rangle$ by Lemma 3.2.14. By the sparsity of (G, ψ) and Equation (6.1), (ii) holds. □

6.2 A gain tight graph admits a reduction

The following result is crucial for our combinatorial characterisations of symmetry-generic infinitesimal rigidity. It states that, except for the specific cases in Figure 6.1, there is always an admissible reduction at a vertex v of degree 3 of a $(2, m, 3, l)$-gain tight or $(2, m, 3, l)'$-gain tight graph, where $(m, l) = (0, 2), (1, 1), (1, 2), (2, 2)$.

Theorem 6.2.1. Let Γ be a cyclic group of order k and (G, ψ) be a Γ-gain graph with a free vertex v of degree 3 which has no loop. Suppose that (G, ψ) is $(2, m, 3, l)$-gain tight or $(2, m, 3, l)'$-gain tight, where (m, l) is one of the pairs $(0, 1), (1, 1), (1, 2)$ or $(2, 2)$. Suppose further that if $|V_k(G)| \geq 2$, then $m = 1$ and (G, ψ) is not $(2, m, 3, l)'$-gain tight. If there is not an admissible 1-reduction at v, then exactly one of the following holds.

(i) (G, ψ) is $(2, 2, 3, 2)$-gain tight or $(2, m, 3, l)'$-gain tight, and v has exactly one free neighbour v_1 and exactly one fixed neighbour v_2 (see Figure 6.1 (a)). If (G, ψ) is $(2, m, 3, l)'$-gain tight, then the graph H spanned by v and v_1 satisfies $\langle H \rangle \simeq \mathbb{Z}_2$ (see Figure 6.1 (b)).

(ii) (G, ψ) is $(2, 1, 3, 2)$-gain tight and v has three neighbours, all of which are fixed (see Figure 6.1 (c)).

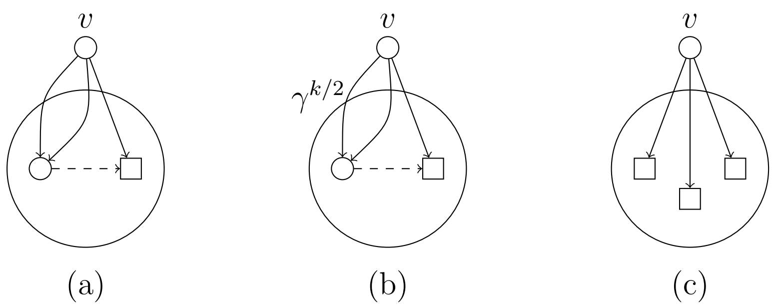


Figure 6.1: Three instances of a vertex v of degree 3 of a Γ-gain graph. In (a,b), v has two neighbours, one of which is fixed, and there may or may not be an edge between the neighbours of v. In (b), $\Gamma = \langle \gamma \rangle \simeq \mathbb{Z}_k$ for some even $k \geq 2$ through the isomorphism which maps γ to 1, and all the unlabelled edges have identity gain. In (c), v has three neighbours, all of which are fixed.

In the proof of Theorem 6.2.1, we consider the three cases where $|N_G(v)| = 1, 2, 3$ separately. Therefore, the proof becomes lengthy. In order to ease the flow, we split the theorem into three different propositions, which we prove separately in Subsections 6.2.1, 6.2.2 and 6.2.3.

6.2.1 v has exactly one neighbour

Proposition 6.2.2. Let Γ be a cyclic group of order k and (G, ψ) be a Γ-gain graph with a free vertex v of degree 3. Suppose that v has no loop, and exactly one neighbour u. Suppose further that (G, ψ) is $(2, m, 3, l)$-gain tight or $(2, m, 3, l)'$-gain tight, where (m, l) is one of the pairs $(0, 1), (1, 1), (1, 2)$ or $(2, 2)$. Then, there is an admissible 1-reduction at v.

Proof. Let e_1, e_2, e_3 be the edges incident to u and v. By Propositions 3.2.6 and 3.2.9, we may assume that $\psi(e_1) = \mathrm{id}$. Moreover, by the definition of gain graph, $\psi(e_2)\psi(e_3)^{-1}, \psi(e_2), \psi(e_3) \neq \mathrm{id}$. Let $(G_1, \psi_1), (G_2, \psi_2)$ be obtained from $G - v$ by adding the loops f_1, f_2 at u with gains $\psi(e_2), \psi(e_2)$, respectively. Suppose, for a contradiction, that for $i = 1, 2$, H_i is a blocker of (G_i, ψ_i). Since both $H_1 + f_1$ and $H_2 + f_2$ contain an unbalanced loop, H_1, H_2 are not balanced blockers. Then, by Proposition 6.1.2, (G, ψ) is $(2, m, 3, l)'$-gain tight and H_1, H_2 are $\mathbb{Z}_2$-blockers. Since $\langle H_1 + f_1 \rangle, \langle H_2 + f_2 \rangle \simeq \mathbb{Z}_2$, we have $\psi(e_2) = \psi(e_3)$, a contradiction. Therefore, the result holds. □

6.2.2 v has exactly two neighbours

Proposition 6.2.3. Let Γ be a cyclic group of order k and (G, ψ) be a Γ-gain graph with a free vertex v of degree 3. Suppose that v has no loop, and exactly two neighbours v_1, v_2. Suppose further that (G, ψ) is $(2, m, 3, l)$-gain tight or $(2, m, 3, l)'$-gain tight, where (m, l) is one of the pairs $(0, 1), (1, 1), (1, 2)$ or $(2, 2)$. If there is no admissible 1-reduction at v, then one of the following holds:

(i) (G, ψ) is $(2, 2, 3, 2)$-gain tight, v has one free and one fixed neighbour.

(ii) (G, ψ) is $(2, m, 3, l)'$-gain tight, v has one free neighbour v_1 and one fixed neighbour v_2. The graph H spanned by v, v_1 satisfies $\langle H \rangle \simeq \mathbb{Z}_2$.

Proof. Let $e_1, e_1' := (v, v_1)$ and $e_2 := (v, v_2)$, and let $g = \psi(e_1')$. By Propositions 3.2.6 and 3.2.9, we may assume that $\psi(e_1) = \psi(e_2) = \mathrm{id}$ and $g \neq \mathrm{id}$. We look at the cases where v_2 is free and fixed separately.

Case 1: *v_2 is free.*

Let $(G_1, \psi_1), (G_2, \psi_2), (G_3, \psi_3)$ be obtained from $G - v$ by adding, respectively, the edge $f_1 = (v_1, v_2)$ with gain id, the edge $f_2 = (v_1, v_2)$ with gain g, and a loop f_3 at

v_1 with gain g. Assume, for a contradiction, that H_1, H_2 and H_3 are blockers for $(G_1, \psi_1), (G_2, \psi_2)$ and (G_3, ψ_3), respectively. Let H and H' denote $H_1 \cup H_2 \cup H_3$ and $H_1 \cap H_2 \cap H_3$, respectively. By Proposition 6.1.2(i), H_1, H_2 are not general count blockers. Moreover, by Proposition 6.4 and Lemma 6.1.5(ii), $E(H_1 \cap H_2) = \emptyset$. Since $H_3 + f_3$ contains the loop f_3, H_3 either a general-count blocker or, in the case where (G, ψ) is $(2, m, 3, l)'$-gain tight, a $\mathbb{Z}_2$-blocker.

Either way, we show that $E(H_1 \cap H_3) = E(H_2 \cap H_3) = \emptyset$. If H_3 is a general count blocker, this follows directly from Lemma 6.1.5(i). So suppose H_3 is a $\mathbb{Z}_2$-blocker (so that (G, ψ) is $(2, m, 3, l)'$-gain tight) and assume, for a contradiction, that $E(H_i \cap H_3) \neq \emptyset$ for some $1 \leq i \leq 2$. By Lemma 6.1.5(ii), $H_i \cup H_3$ is $(2, 2)$-tight. Since H_i is either a balanced blocker or a $\mathbb{Z}_2$-blocker, every $v_1 - v_2$-walk in H_i only containing free vertices has gain id or g. Hence, $\langle H_i \cup H_3 + v \rangle \simeq \mathbb{Z}_2$. By Proposition 6.1.2, this is a contradiction. So $E(H_1 \cap H_3) = E(H_2 \cap H_3) = \emptyset$.

Suppose H_3 is a general count blocker. Then, by Proposition 6.1.2, $v_2 \notin V(H_3)$. For $i = 1, 2$, let $2 \leq l_i \leq 3$ be the integer such that H_i is $(2, l_i)$-tight. Since $m \leq 2$, for $i = 1, 2$, we have $|E(H_i)| = 2|V(H_i)| - l_i \geq 2|V_1(H_i)| + m|V_k(H_i)| - l_i$. So letting $S_1 = \sum_{1 \leq i \neq j \leq 3} |V_1(H_i \cap H_j)| - |V_1(H')|$ and $S_k = \sum_{1 \leq i \neq j \leq 3} |V_k(H_i \cap H_j)| - |V_k(H')|$, we have

$$
\begin{aligned}
|E(H)| &\geq 2 \sum_{i=1}^{3} |V_1(H_i)| + m \sum_{i=1}^{3} |V_k(H_i)| - l_1 - l_2 - l \\
&= 2\left(|V_1(H)| + S_1\right) + m\left(|V_k(H)| + S_k\right) - l_1 - l_2 - l \\
&\geq 2\left(|V_1(H)| + S_1\right) + m|V_k(H)| - 6 - l,
\end{aligned}
$$

where the last inequality holds because $l_1 + l_2 \leq 6$ and $V_k(H') \subseteq V_k(H_i \cap H_t)$ for all pairs $1 \leq i \neq t \leq 3$. Since the free vertex v_2 is not contained in H', it is easy to see that $S_1 \geq 3$. Hence, $|E(H)| \geq 2|V_1(H)| + m|V_k(H)| - l$. This contradicts Proposition 6.1.2. So, we may assume that $(2, m, 3, l)'$-gain tight and that H_3 is a $\mathbb{Z}_2$-blocker. If H_i is a $\mathbb{Z}_2$-blocker for some $i = 1, 2$, then

$$
|E(H_i \cup H_3)| = 2|V(H_i)| - 2 + 2|V(H_3)| - 2 = 2|V(H_i \cup H_3)| + 2|V(H_i \cap H_3)| - 4.
$$

By Proposition 6.1.2, $|V(H_i \cap H_3)| = 1$, and so $|E(H_i \cup H_3)| = 2|V(H_i \cup H_3)| - 2$. But then $\langle H_1 \cup H_i + f_1 + f_i\rangle = \langle H_1 \cup H_i + v\rangle \simeq \mathbb{Z}_2$, contradicting Proposition 6.1.2. Therefore, H_1, H_2 are balanced blockers. Since $E(H_1 \cap H_2) = \emptyset$, it is easy to see that $|E(H_1 \cup H_2)| = 2|V(H_1 \cup H_2)| + 2|V(H_1 \cap H_2)| - 6$. By Proposition 6.1.2, $V(H_1 \cap H_2)$ is composed of the two isolated vertices v_1, v_2 and $V_k(H) = \emptyset$.

By Lemma 3.2.21, $H_1 \cup H_2 + f_1 + f_2$ is near-balanced with base vertex v_1 (and with base vertex v_2). Since $H_1 \cup H_2 + f_1 + f_2$ contains the 2-cycle f_1, f_2, it is near-balanced with gain g. So there is a gain ψ' equivalent to ψ such that $\psi'(e) \in \{\mathrm{id}, g, g^{-1}\}$ for all edges e in $E(H_1 \cup H_2)$ incident to v_1, and $\psi'(f) = \mathrm{id}$ for all other edges $f \in E(H_1 \cup H_2)$. In particular, $\langle H_1 \cup H_2 + v\rangle = \langle H_1 \cup H_2 + f_1 + f_2\rangle = \langle g\rangle \simeq \mathbb{Z}_2$. But $|E(H)| \geq 2|V(H)| - 2$, contradicting Proposition 6.1.2. This proves the result for the case where v_2 is free.

Case 2: *v_2 is fixed.*

Assume that (i) and (ii) do not hold. Let $(G_1, \psi_1), (G_2, \psi_2)$ be the graphs obtained from $G - v$ by adding, respectively, an edge $f_1 = (v_1, v_2)$ with gain id, and a loop f_2 at v_1 with gain g (see Figure 6.2). Notice that, if there is already an edge $(v_1, v_2) \in E(G)$, (G_1, ψ_1) is not a well-defined gain graph. We look at the cases where $(v_1, v_2) \in E(G)$ and $(v_1, v_2) \notin E(G)$ separately.

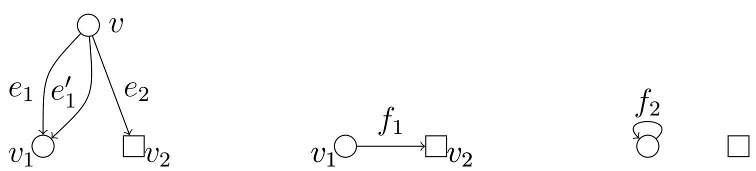


Figure 6.2: Two possible 1-reductions at v.

First, assume $(v_1, v_2) \in E(G)$. Then it is easy to check that the graph induced by v, v_1, v_2 violates both $(2, 0, 3, 1)$-gain sparsity and $(2, 1, 3, 2)$-gain sparsity. Hence, we may assume that (G, ψ) is $(2, 1, 3, 1)$-gain tight (and possibly $(2, 1, 3, 1)'$-gain tight).

Assume, for a contradiction, that (G_2, ψ_2) has a blocker H_2. Since $H_2 + f_2$ contains the loop f_2 with gain g, it is unbalanced and, in the case where (G, ψ)

is $(2,1,3,1)'$-gain tight, $\langle H_2 + f_2\rangle \neq \mathbb{Z}_2$. Hence, H_2 is a general-count blocker. It follows, from Proposition 6.1.2, that $v_2 \notin V(H_2)$. Hence, the graph H obtained from H_2 by adding v, v_2, together with the edges $e_1, e_1', e_2, (v_1, v_2)$, satisfies

$$|E(H)| = |E(H_2)| + 4 = 2|V_1(H_2)| + |V_k(H_2)| + 3 = 2|V_1(H)| + |V_k(H)|,$$

contradicting the sparsity of (G, ψ). Thus, the 1-reduction at v which yields (G_2, ψ_2) is admissible. Now, let $(v_1, v_2) \notin E(G)$. Assume that H_1 and H_2 are blockers for (G_1, ψ_1) and (G_2, ψ_2), respectively. By Proposition 6.1.2, either H_1 is a balanced blocker or (G, ψ) is $(2, m, 3, l)'$-gain tight and H_1 is a $\mathbb{Z}_2$-blocker. So, let $2 \leq l_1 \leq 3$ be the integer such that H_1 is $(2, l_1)$-tight. Since $H_2 + f_2$ contains the loop f_2, it is a general-count blocker. By Proposition 6.1.2, $v_2 \notin V(H_2)$. Moreover, by Lemma 6.1.5(i), $E(H_1 \cap H_2) = \emptyset$. Let $H := H_1 \cup H_2$ and $H' := H_1 \cap H_2$. We have

$$\begin{aligned}|E(H)| &= (2|V(H_1)| - l_1) + (2|V_1(H_2)| + m|V_k(H_2)| - l)\\ &= 2|V_1(H)| + m|V_k(H)| - l + 2|V_1(H')| + m|V_k(H')|\\ &\quad + (2-m)|V_k(H_1)| - l_1 \geq 2|V_1(H)| + m|V_k(H)| - l,\end{aligned}$$

where the last inequality holds because $|V_1(H')| \geq 1$, $|V_k(H_1)| \geq 1, m \leq 1$ and $l_1 \leq 3$. This contradicts Proposition 6.1.2. Hence, there is an admissible 1-reduction at v, as required. □

6.2.3 v has exactly three neighbours

Proposition 6.2.4. Let Γ be a cyclic group of order k and (G, ψ) be a Γ-gain graph with a free vertex v of degree 3. Suppose that v has no loop, and exactly three neighbours v_1, v_2, v_3. Suppose further that (G, ψ) is $(2, m, 3, l)$-gain tight or $(2, m, 3, l)'$-gain tight, where (m, l) is one of the pairs $(0, 1), (1, 1), (1, 2)$ or $(2, 2)$, and that if $|V_k(G)| \geq 2$, then $m = 1$ and (G, ψ) is not $(2, m, 3, l)'$-gain sparse. Either there is an admissible 1-reduction at v, or (G, ψ) is $(2, 1, 3, 2)$-gain tight and v_1, v_2, v_3 are fixed.

Proof. For $i = 1, 2, 3$, let $e_i = (v, v_i)$ be the edges incident with v. Let f_1, f_2 and f_3 denote the edges $(v_1, v_2), (v_2, v_3)$ and (v_3, v_1), respectively. By Propositions 3.2.6

and 3.2.9, we may assume $\psi(e_1) = \psi(e_2) = \psi(e_3) = \text{id}$. For $1 \leq i \leq 3$, let (G_i, ψ_i) be obtained by applying a 1-reduction at v, during which we add the edge f_i with gain id, and assume that (G_i, ψ_i) has a blocker H_i. Let $H := H_1 \cup H_2 \cup H_3$ and $H' := H_1 \cap H_2 \cap H_3$.

We first show that $E(H_i \cap H_j) = \emptyset$ for all pairs $i \neq j$. As a first step, we show that $E(H_i \cap H_j) \neq \emptyset$ for at most one pair $i \neq j$. So assume for a contradiction that this is not the case. Without loss of generality, suppose that $E(H_1 \cap H_2) \neq \emptyset$ and $E(H_1 \cap H_3) \neq \emptyset$. By Lemma 6.1.5(ii), either $H_1 \cup H_2$ is $(2,3)$-tight and $H_1 \cup H_2 + f_1 + f_2$ is balanced or $H_1 \cup H_2$ is $(2,2)$-tight and $\langle H_1 \cup H_2 + f_1 + f_2 \rangle \simeq \mathbb{Z}_2$. Let $2 \leq l' \leq 3$ be the integer such that $H_1 \cup H_2$ is $(2, l')$-tight. Then,

$$|E(H_1 \cup H_2 + v)| = |E(H_1 \cup H_2)| + 3 = 2|V(H_1 \cup H_2)| + 3 - l' = 2|V(H_1 \cup H_2 + v)| - l' + 1.$$

If $\langle H_1 \cup H_2 + v \rangle = \langle H_1 \cup H_2 + f_1 + f_2 \rangle$, this contradicts the sparsity of (G, ψ). So, we may assume that $\langle H_1 \cup H_2 + v \rangle = \langle H_1 \cup H_2 + f_1 + f_2 \rangle$. The group $\langle H_1 \cup H_2 + v \rangle \simeq \langle H_1 \cup H_2 + f_1 + f_2 + f_3 \rangle$ is given by the elements of the group $\langle H_1 \cup H_2 + f_1 + f_2 \rangle$, together with the gains of the walks from v_1 to v_3 which do not contain fixed vertices. Since $\langle H_1 \cup H_2 + v \rangle \neq \langle H_1 \cup H_2 + f_1 + f_2 \rangle$, there must be a path P from v_3 to v_1 in $H_1 \cup H_2$ with gain $g \notin \langle H_1 \cup H_2 + f_1 + f_2 \rangle$, which contains only free vertices. In particular, v_1, v_3 are free. Moreover, v_2 must be fixed, for otherwise, f_1, f_2, P is a closed path in $H_1 \cup H_2 + f_1 + f_2$ with gain $g \notin \langle H_1 \cup H_2 + f_1 + f_2 \rangle$ and with no fixed vertex, a contradiction.

Applying the same argument to $H_1 \cup H_3$, we may conclude that v_1 is fixed and v_2, v_3 are free. But this contradicts the fact that v_1 is free and v_2 is fixed. Hence, $E(H_i \cap H_j) \neq \emptyset$ for at most one pair $1 \leq i \neq j \leq 3$. Without loss of generality, $E(H_1 \cap H_3) = E(H_2 \cap H_3) = \emptyset$.

Assume, for a contradiction, that $E(H_1 \cap H_2) \neq \emptyset$. Then, as shown above, v_2 is fixed and v_1 and v_3 are free. In particular, $H_1 \cup H_2$ cannot be $(2,2)$-tight, as otherwise $|E(H_1 \cup H_2)| = 2|V(H_1 \cup H_2)| - 2 \geq 2|V_1(H_1 \cup H_2)| + m|V_k(H_1 \cup H_2)| - l$, contradicting Proposition 6.1.2 or the sparsity of (G, ψ). (This is true because $v_2 \in V(H_1 \cup H_2)$ is fixed.) Therefore, $H_1 \cup H_2$ is $(2,3)$-tight and $H_1 \cup H_2 + f_1 + f_2$ is balanced.

If H_3 is a balanced blocker or a $\mathbb{Z}_2$-blocker, there is an integer $2 \leq l_3 \leq 3$ such that H_3 is $(2, l_3)$-tight. Then, since H is the union of $H_1 \cup H_2$ and H_3, we have

$$|E(H)| = 2|V(H)| + 2|V((H_1 \cup H_2) \cap H_3)| - 3 - l_3. \tag{6.2}$$

If $|V((H_1 \cup H_2) \cap H_3)| \geq 3$ or if $l_3 = 2$ then, since v_2 is fixed, it is easy to see that $|E(H)| \geq 2|V_1(H)| + m|V_k(H)| - l$, contradicting Proposition 6.1.2 or the sparsity of (G, ψ). Hence, $V((H_1 \cup H_2) \cap H_3) = \{v_1, v_3\}$ and H_3 is a balanced blocker. It follows that H is balanced. (Every closed walk in H is composed of closed walks in $H_1 \cup H_2$, of closed walks in H_3, and of concatenations of walks from v_1 to v_3 in $H_1 \cup H_2$ together with walks from v_3 to v_1 in H_3; all such walks must have identity gain, since $H_1 \cup H_2 + f_1 + f_2, H_3 + f_3$ are balanced.) However, by Equation (6.2), we have $|E(H)| = 2|V(H)| - 2$, contradicting the sparsity of (G, ψ). So, we may assume that H_3 is a general-count blocker. Then, it is easy to see that

$$\begin{aligned}|E(H)| &= (2|V(H_1 \cup H_2)| - 3) + (2|V_1(H_3)| + m|V_k(H_3)| - l) \\ &= 2|V_1(H)| + m|V_k(H)| - l + f - 3\end{aligned}$$

where $f = (2 - m)|V_k(H_1 \cup H_2)| + m|V_k((H_1 \cup H_2) \cap H_3)| + 2|V_1((H_1 \cup H_2) \cap H_3)|$. Since $v_1, v_2 \in V_1((H_1 \cup H_2) \cap H_3)$ and $0 \leq m \leq 2$, we have $f \geq 3$, contradicting the sparsity of (G, ψ). Hence, $E(H_i \cap H_j) = \emptyset$ for all pairs $i \neq j$.

We now show that at least one of H_1, H_2, H_3 is a general count blocker. Assume, for a contradiction, that for $1 \leq i \leq 3$ there is some $2 \leq l_i \leq 3$ such that H_i is $(2, l_i)$-tight. If $|V(H_i \cap H_j)| = 1$ for all $1 \leq i \neq j \leq 3$, then it is easy to see that $\langle H + v \rangle$ is generated by the elements in $\langle H_1 + f_1 \rangle, \langle H_2 + f_2 \rangle, \langle H_3 + f_3 \rangle$. (Since every closed walk in H is composed of closed walks in H_i for $1 \leq i \leq 3$, and the concatenation of walks from v_1 to v_2 in H_1, walks from v_2 to v_3 in H_2, and walks from v_3 to v_1 in

H_3.) Moreover

$$
\begin{aligned}
|E(H)| &= \sum_{i=1}^{3} |E(H_i)| = \sum_{i=1}^{3} [2|V(H_i)| - l_i] \\
&= 2|V(H)| + 2 \sum_{1 \le i \ne j \le 3} [|V(H_i \cap H_j)| - |V(H')|] - \sum_{i=1}^{3} l_i \\
&= 2|V(H)| + 2(1+1+1) - \sum_{i=1}^{3} l_i = 2|V(H)| + 6 - \sum_{i=1}^{3} l_i. \qquad (6.3)
\end{aligned}
$$

If $\sum_{i=1}^{3} l_i = 9$, then H_1, H_2, H_3 are all balanced blockers, and so $H+v$ is balanced. But since $|E(H)| = 2|V(H)| - 9$, this contradicts Proposition 6.1.2. If $\sum_{i=1}^{3} l_i < 9$, then at least one of H_1, H_2, H_3 is a $\mathbb{Z}_2$-blocker and, since H_1, H_2, H_3 are all either balanced blockers or $\mathbb{Z}_2$-blocker, $\langle H+v \rangle \simeq \mathbb{Z}_2$. However, by Equation (6.3), $|E(H)| \ge 2|V(H)| - 2$, contradicting Proposition 6.1.2 or the sparsity of (G, ψ). Hence, $|V(H_i \cap H_j)| \ge 2$ for some $1 \le i \ne j \le 3$. Without loss of generality, let $|V(H_1 \cap H_2)| \ge 2$. Then,

$$
\begin{aligned}
|E(H_1 \cup H_2)| &= 2|V(H_1 \cup H_2)| + 2|V(H_1 \cap H_2)| - l_1 - l_2 \\
&\ge 2|V(H_1 \cup H_2)| - l_1 - l_2 + 4 \\
&\ge 2|V_1(H_1 \cup H_2)| + m|V_k(H_1 \cup H_2)| - l_1 - l_2 + 4,
\end{aligned}
$$

where the last inequality holds because $m \le 2$. If $l = 2$ or if $l_1 + l_2 \le 5$, this contradicts Proposition 6.1.2. Hence, $l = 1$ and $l_1 + l_2 = 6$. It follows that H_1, H_2 are balanced blockers. Moreover, rearranging Equation (6.3) and noting that $l_1 = l_2 = 3$ and $l_3 \le 3$, we know that

$$
\begin{aligned}
|E(H)| &\ge 2|V(H)| - 1 + 2\left(\sum_{1 \le i \ne j \le 3} |V(H_i \cap H_j)| - |V(H')| - 4 \right) \\
&\ge 2|V_1(H)| + m|V_k(H)| - 1 + 2\left(\sum_{1 \le i \ne j \le 3} |V(H_i \cap H_j)| - |V(H')| - 4 \right),
\end{aligned}
$$

since $m \le 2$. If we show that $f := \sum_{1 \le i \ne j \le 3} |V(H_i \cap H_j)| - |V(H')| \ge 4$, this contradicts Proposition 6.1.2. Notice that $|V(H')|$ is at most the minimum of $|V(H_i \cap H_j)|$, where $i \ne j$ run from 1 to 3. Call this number *min*. Hence,

$f \geq \sum_{1 \leq i \neq j \leq 3} |V(H_i \cap H_j)| - min$. If $min = |V(H_1 \cap H_2)|$, then $|V(H_2 \cap H_3)| \geq 2$ and $|V(H_1 \cap H_3)| \geq 2$, and so $f \geq 2 + 2 \geq 4$. So assume, without loss of generality, that $min = |V(H_2 \cap H_3)|$, and hence that $f \geq |V(H_1 \cap H_2)| + |V(H_1 \cap H_3)|$. If $|V(H_1 \cap H_3)| \geq 2$, then $f \geq 4$. So, assume that $V(H_1 \cap H_3) = \{v_1\}$. By minimality, $V(H_2 \cap H_3) = \{v_3\}$. It follows that $V(H') = \emptyset$, and so $f \geq 2 + 1 + 1 = 4$. Since we at least one of H_1, H_2, H_3 is a general count blocker. Assume, without loss of generality, that H_1 is a general-count blocker.

Claim: *For $2 \leq i \leq 3$, we have $|V(H_1 \cap H_i)| = 1$.*

Proof: If H_i is also a general-count blocker, since $|E(H_1 \cup H_i)| = |E(H_1)| + |E(H_i)|$, the graph $H'_i := H_1 \cup H_i$ satisfies

$$\begin{aligned}|E(H'_i)| &= 2|V_1(H'_i)| + m|V_k(H'_i)| - l + (2|V_1(H_1 \cap H_i)| + m|V_k(H_1 \cap H_i)| - l) \\ &\geq 2|V_1(H'_i)| + m|V_k(H'_i)| - l + (2|V_1(H_1 \cap H_i)| + m|V_k(H_1 \cap H_i)| - 2),\end{aligned}$$

since $l \leq 2$. If $|V_1(H_1 \cap H_i)| \geq 1$, or if $|V_k(H_1 \cap H_i)| \geq 2$ (and so $m = 1$ by assumption), it is easy to see that this is at least $2|V_1(H'_i)| + m|V_k(H'_i)| - l$. This contradicts Proposition 6.1.2. Hence, $|V(H_1 \cap H_i)| = |V_k(H_1 \cap H_i)| = 1$, and the claim holds. Whether H_i is a balanced blocker or a $\mathbb{Z}_2$-blocker, $|E(H_i)| \geq 2|V(H_i)| - 3$. Hence,

$$\begin{aligned}|E(H'_i)| &\geq 2|V_1(H'_i)| + m|V_k(H'_i)| - l + (2|V_1(H_1 \cap H_i)| + 2|V_k(H_1 \cap H_i)| - 3) \\ &= 2|V_1(H'_i)| + m|V_k(H'_i)| - l + (2|V(H_1 \cap H_i)| - 3),\end{aligned}$$

where the inequality holds because $V_k(H_1 \cap H_i) \subseteq V_k(H_i)$ and $m \leq 2$. If H_1 and H_i have more than two vertices in common, then this contradicts the sparsity of (G, ψ). Hence, the claim holds. □

By the Claim, $V(H_1 \cap H_2) = \{v_2\}$ and $V(H_1 \cap H_3) = \{v_1\}$. Hence, v_1, v_2, v_3 do not lie in $V(H')$. Let n be the number of free vertices in $\{v_1, v_2, v_3\}$. Since each vertex in $\{v_1, v_2, v_3\}$ lies in $H_i \cap H_j$ for some $0 \leq i \neq j \leq 1$, this implies that

$$S_1 := \sum_{1 \leq i \neq j \leq 3} |V_1(H_i \cap H_j)| - |V_1(H')| \geq n$$

and

$$S_k := \sum_{1 \le i \ne j \le 3} |V_k(H_i \cap H_j)| - |V_k(H')| \ge 3 - n.$$

We look at the following cases separately: H_2, H_3 are not general count blockers; H_2 is a general-count blocker, but H_3 is not; H_2, H_3 are general-count blockers.

Case 1: *H_2, H_3 are not general count blockers.*

Whether H_2, H_3 are balanced blockers or $\mathbb{Z}_2$-blocker, we have $|E(H_i)| \ge 2|V(H_i)| - 3$ for $i = 1, 2$. Hence,

$$\begin{aligned} |E(H)| &\ge (2|V_1(H_1)| + m|V_k(H_1)| - l) + (2|V(H_2)| - 3) + (2|V(H_3)| - 3) \\ &= 2[|V_1(H)| + S_1] + m[|V_k(H)| + S_k] + (2 - m)(|V_k(H_2)| + |V_k(H_3)|) - 6 - l \\ &\ge 2|V_1(H)| + m|V_k(H)| - l + f, \end{aligned}$$

where $f := 2n + m(3 - n) + (2 - m)(|V_k(H_2)| + |V_k(H_3)|) - 6$. If $f \ge 0$, Proposition 6.1.2 leads to a contradiction, and so there is an admissible 1-reduction at v. We will show that indeed $f \ge 0$. This is clear if $n = 3$. Suppose $n = 2$, so that

$$f = m + (2 - m)(|V_k(H_2)| + |V_k(H_3)|) - 2.$$

Since $n = 2$, at least one of v_1, v_2, v_3 is fixed, and so $|V_k(H_2)| + |V_k(H_3)| \ge 1$. Hence, $f \ge m + 2 - m - 2 = 0$. So, we may assume $n \le 1$. Hence, there are at least two fixed vertices in $\{v_1, v_2, v_3\} \subset V(G)$, and so $|V_k(H_2)| + |V_k(H_3)| \ge 2$. By assumption, this implies that $m = 1$ and H_2 is a balanced blocker. Hence, $f = n - 3 + |V_k(H_2)| + |V_k(H_3)| \ge n - 1$. When $n = 1$, $f \ge 0$. So, let $n = 0$. Then $|V_k(H_2)|, |V_k(H_3)| \ge 2$, so $f \ge 1$.

Case 2: *H_2 is a general-count blocker, but H_3 is a not.*

Whether H_3 is a balanced blocker or a $\mathbb{Z}_2$-blocker, we have $|E(H_3)| \ge 2|V(H_3)| - 3$. Therefore,

$$\begin{aligned} |E(H)| &\ge (2|V_1(H_1)| + |V_k(H_1)| - l) + (2|V_1(H_2)| + |V_k(H_2)| - l) + (2|V(H_3)| - 3) \\ &= 2[|V_1(H)| + S_1] + m[|V_k(H)| + S_k] + (2 - m)|V_k(H_3)| - 3 - 2l \\ &\ge 2[|V_1(H)| + n] + m[|V_k(H)| + (3 - n)] + (2 - m)|V_k(H_3)| - 3 - 2l \\ &= 2|V_1(H)| + m|V_k(H)| - l + f, \end{aligned}$$

where $f := 2n + m(3-n) + (2-m)|V_k(H_3)| - 3 - l \geq 0$. If $f \geq 0$, then we obtain a contradiction by Proposition 6.1.2. We show that indeed $f \geq 0$. If $n = 3$, then $f \geq 3 - l > 0$, since $l \leq 2$. If $n = 2$, then $f \geq 1 + m - l \geq 0$, since $l - m \leq 1$. Hence, we may assume that $n \leq 1$. So, at least two of the elements in $\{v_1, v_2, v_3\} \subset V(G)$ are fixed. It follows that $m = 1$ and $f = n - l + |V_k(H_3)|$. If $n = 1$, then $|V_k(H_3)| \geq 1$ and $f = 1 - l + |V_k(H_3)| \geq 2 - l \geq 0$, since $l \leq 2$. If $n = 0$, then $|V_k(H_3)| \geq 2$ and $f \geq 2 - l \geq 0$.

Case 3: *H_2, H_3 are general-count blockers.*

In this case, we have

$$\begin{aligned} |E(H)| = \sum_{i=1}^{3} |E(H_i)| &= 2\sum_{i=1}^{3} |V_1(H_i)| + m\sum_{i=1}^{3} |V_k(H_i)| - 3l \\ &= 2(|V_1(H)| + S_1) + m(|V_k(H)| + S_k) - 3l \\ &\geq 2|V_1(H)| + m|V_k(H)| - l + [2n + m(3-n) - 2l]. \end{aligned}$$

If $f := 2n + m(3-n) - 2l \geq 0$, then we obtain a contradiction by Proposition 6.1.2. Assume that either (G, ψ) is not $(2,1,3,2)$-gain tight or at least one of v_1, v_2, v_3 is free. We will show that $f \geq 0$. If $n = 3$, $f = 6 - 2l = 2(3 - l) > 0$, since $l \leq 2$. If $n = 2$, then $f = 4 + m - 2l = 2(2 - l) + m \geq 0$, since $l \geq 2$ and $m \geq 0$. So, we may assume that $n \leq 1$, which implies that $m = 1$. Hence, $f = n + 3 - 2l$. If $n = 1$, then $f = 2(2 - l) \geq 0$. If $n = 0$, then $f = 3 - 2l$. Moreover, by assumption, $l \geq 1$. But then $f \geq 1$. This proves the result. □

6.3 Main result: reflection

Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_s$-generic framework. Recall that the Γ-gain graph (G, ψ) of $\tilde{G}$ is $(2,1,3,1)$-gain tight whenever $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic, and (G, ψ) is $(2,1,3,2)$-gain tight whenever $(\tilde{G}, \tilde{p})$ is anti-symmetrically isostatic (see Proposition 5.1.1 in Subsection 5.1.1). In this section, we show that the converse statements are also true.

To do so, we employ a proof by induction on $|V(G)|$, which uses the vertex extension and reduction moves described in Section 5.3. Hence, we first need to show that there is an admissible reduction of (G, ψ), whose corresponding extension does not break fully-symmetric or anti-symmetric infinitesimal rigidity.

Let v be a free vertex of degree 3 with no loop. By Theorem 6.2.1, there is always an admissible 1-reduction at v, unless all neighbours of v are fixed and (G, ψ) is $(2, 1, 3, 2)$-gain tight. Lemma 5.3.11 shows that a 1-extension maintains the fully-symmetric and anti-symmetric infinitesimal rigidity of a framework. However, the result assumes that all neighbours of the added vertex do not lie on the same line, and hence they cannot all be fixed. This issue arises both in the fully-symmetric and the anti-symmetric cases. Hence, our proof by induction cannot rely on applying a 1-reduction to a vertex whose neighbours are all fixed.

In the following result we show that, if G has at least two free vertices, and all free vertices of degree 3 in $V(G)$ have three fixed neighbours, then there is another vertex in $V(G)$ at which we may apply an admissible reduction.

Lemma 6.3.1. Let Γ be a cyclic group of order 2. For $1 \leq l \leq 2$, let (G, ψ) be a $(2, 1, 3, l)$-gain tight Γ-gain graph with $|V_1(G)| \geq 2$. Then there is a reduction of (G, ψ) which yields a $(2, 1, 3, l)$-gain tight graph (G', ψ'). The reduction which yields (G', ψ') is one of the following: a fix-0-reduction, a 0-reduction, a loop-1-reduction, a 1-reduction at a vertex with at least one free neighbour, or a fix-1-reduction.

Proof. The case where there are no fixed vertices is known (see e.g., [[56], Theorem 6.3]), so we may assume $V_2(G) \neq \emptyset$. Suppose G has a vertex v which is either a fixed vertex of degree 1, or a free vertex of degree 2, or a free vertex of degree 3 with a loop. (Notice that if v has a loop, then $l = 1$.) Then, we may apply a fix-0-reduction, or a 0-reduction, or a loop-1-reduction at v. All such reductions are clearly admissible. Hence, we may assume that there is no such vertex v. By Theorem 6.2.1, we may also assume that all free vertices of degree 3 in $V(G)$ have three distinct neighbours, all of which are fixed. Let n be the number of vertices of degree 2 in $V_2(G)$.

Claim: *Under the above assumptions, we have $n \geq 3$.*

Proof. To see this, let $v_1, \dots, v_t$ be the free vertices in G of degree 3 and assume that for all $1 \leq i \leq t$, the edges incident with v_i are directed to v_i. Notice that $t \geq 2$, by Lemma 6.1.1. Define the set

$$V' := \{v \in V_2(G) : (v, v_i) \in E(G) \text{ for some } 1 \leq i \leq t\}.$$

Let $n' = |V'|$ and consider the subgraph H of G induced by $\{v_1, \dots, v_t\} \cup V'$. By the sparsity of (G, ψ), $3t \leq |E(H)| \leq 2t + n' - l$ and hence $n' \geq t + l$.

Now, the average degree of G is $\hat{\rho} = (4|V_1(G)| + 2|V_2(G)| - 2l)/|V(G)|$. This average is smallest when all vertices in $V_1(G) \setminus \{v_1, \dots, v_t\}$ have degree 4, and all fixed vertices in $V(G)$ which do not have degree 2, have degree 3. This gives $\hat{\rho} \geq (4|V_1(G)| + 3|V_2(G)| - n - t)/|V(G)|$. Hence,

$$n \geq |V_2(G)| + 2l - t \geq n' + 2l - t \geq (t + l) + 2l - t = 3l \geq 3,$$

as required. $\square$

So, there is a fixed vertex v of degree 2. Let u_1, u_2 be the neighbours of v. Notice that there is no $(2, 1, 3, l)$-gain tight subgraph H of G with $u_1, u_2 \in V(H)$, $v \notin V(H)$, as otherwise the graph $H' := H + v$ satisfies

$$|E(H')| = |E(H)| + 2 = 2|V_1(H)| + |V_2(H)| - l + 2 = 2|V_1(H')| + |V_2(H')| - l + 1, \quad (6.4)$$

contradicting the sparsity of (G, ψ). We show that there is an admissible fix-1-reduction at v.

First, suppose that $l = 2$. By the sparsity of (G, ψ), u_1, u_2 are free. Let $(G_1, \psi_1), (G_2, \psi_2)$ be obtained from (G, ψ) by removing v and adding the edge $e = (u_1, u_2)$ with gains id and $\gamma \neq \text{id}$, respectively. Assume, for a contradiction, that for $1 \leq i \leq 2$, (G_i, ψ_i) has a blocker H_i. By Equation (6.4), H_1, H_2 are balanced blockers. If $E(H_1 \cap H_2) = \emptyset$, then

$$\begin{aligned} |E(H_1 \cup H_2)| &= 2|V(H_1)| - 3 + 2|V(H_2)| - 3 \\ &= 2|V(H_1 \cup H_2)| + 2|V(H_1 \cap H_2)| - 6 \geq 2|V(H_1 \cup H_2)| - 2, \end{aligned}$$

where the inequality holds because $u_1, u_2 \in V(H_1 \cap H_2)$. But then $H' := H_1 \cup H_2 + v$ satisfies

$$|E(H')| = 2|V(H')| - 2 = 2|V_1(H')| + |V_2(H')| - 2 + |V_2(H')| \geq 2|V_1(H')| + |V_2(H')| - 1,$$

where the inequality holds because $v \in V_2(H')$. This contradicts the sparsity of (G, ψ). So $E(H_1 \cap H_2) \neq \emptyset$. Since H_1, H_2 are balanced blockers, all paths from u_1 to u_2 in H_1 have gain id and all paths from u_1 to u_2 in H_2 have gain γ. By the sparsity count, $H_1 \cap H_2$ is connected (see, e.g., the proof of Lemma 6.1.5(ii)). So, there is a path from u_1 to u_2 in $H_1 \cap H_2$ with two different gains, a contradiction. Hence, at least one of $(G_1, \psi_1), (G_2, \psi_2)$ is $(2, 1, 3, 2)$-gain tight.

Now, let $l = 1$. Let (G_1, ψ_1) be obtained from (G, ψ) by removing v and adding the edge $e_1 = (u_1, u_2)$ with gain id. Assume that (G_1, ψ_1) has a blocker H_1. By Equation (6.4), H_1 is a balanced blocker. Hence, $H_1 + v$ satisfies

$$|E(H_1 + v)| = 2|V(H_1 + v)| - 3 = (2|V_1(H_1 + v)| + |V_2(H_1 + v)| - 3) + |V_2(H_1 + v)|.$$

If $H_1 + v$ contains three fixed vertices, this contradicts the sparsity of (G, ψ). Since v is fixed, this implies that at most one of u_1, u_2 is fixed. Assume, without loss of generality, that u_1 is free.

Let (G_2, ψ_2) be obtained from (G, ψ) by removing v and adding a loop e_2 at u_1 with gain $\gamma \neq$ id. Assume that (G_2, ψ_2) has a blocker H_2. Since $H_2 + e_2$ contains the unbalanced loop e_2, H_2 is a general-count blocker. Hence, by Equation (6.4), $u_2 \notin V(H_2)$. If $E(H_1 \cap H_2) \neq \emptyset$, then $|E(H_1 \cap H_2)| \leq 2|V(H_1 \cap H_2)| - 3$ and so $H_{12} := H_1 \cup H_2$ satisfies

$$\begin{aligned}
|E(H_{12})| &\geq (2|V(H_1)| - 3) + (2|V_1(H_2)| + |V_2(H_2)| - 1) - (2|V(H_1 \cap H_2)| - 3) \\
&= 2|V_1(H_{12})| + |V_2(H_{12})| - 1 + (|V_2(H_1)| - |V_2(H_1 \cap H_2)|) \\
&\geq 2|V_1(H_{12})| + |V_2(H_{12})| - 1.
\end{aligned}$$

By Equation (6.4), this contradicts the sparsity of (G, ψ). So, $E(H_1 \cap H_2) = \emptyset$.

Hence,

$$\begin{aligned}|E(H_{12})| &= (2|V(H_1)| - 3) + (2|V_1(H_2)| + |V_k(H_2)| - 1)\\ &= 2|V_1(H_{12})| + |V_2(H_{12})| - 4 + |V_2(H_1)| + 2|V_1(H_1 \cap H_2)| + |V_2(H_1 \cap H_2)|\\ &\geq 2|V_1(H_{12})| + |V_2(H_{12})| - 2 + (|V_2(H_1)| + |V_2(H_1 \cap H_2)|).\end{aligned}$$

where the inequality holds because $u_1 \in V_1(H_1 \cap H_2)$. If $|V_2(H_1)| \geq 1$, then H_{12} is $(2,1,3,1)$-gain tight which, by Equation (6.4), contradicts the sparsity of (G,ψ). Hence, $V_2(H_1) = \emptyset$. In particular, u_2 is free.

Let (G_3, ψ_3) be obtained from (G,ψ) by removing v and adding a loop e_3 at u_2 with gain $\gamma \neq \mathrm{id}$. Assume that (G_3,ψ_3) has a blocker H_3. Similarly as we did with H_2, it is easy to see that H_3 is a general-count blocker, that $u_1 \notin V(H_3)$ and that $E(H_1 \cap H_3) = \emptyset$. Moreover, $E(H_2 \cap H_3) = \emptyset$, as otherwise $H_2 \cup H_3$ is $(2,1,3,1)$-gain tight which, by Equation (6.4), contradicts the sparsity of (G,ψ). Let

$$S_1 = \sum_{1 \leq i \neq j \leq 3} |V_1(H_i \cap H_j)| - |V_1(H_1 \cap H_2 \cap H_3)|$$

and

$$S_2 = \sum_{1 \leq i \neq j \leq 3} |V_2(H_i \cap H_j)| - |V_2(H_1 \cap H_2 \cap H_3)|.$$

Since $u_1, u_2 \notin V_1(H_1 \cap H_2 \cap H_3)$, we have $S_1 \geq 2$. So the graph $H := H_1 \cup H_2 \cup H_3$ satisfies

$$\begin{aligned}|E(H)| &= (2|V(H_1)| - 3) + (2|V_1(H_2)| + |V_2(H_2)| - 1) + (2|V_1(H_3)| + |V_2(H_3)| - 1)\\ &= 2|V_1(H)| + |V_2(H)| - 5 + (|V_2(H_1)| + 2S_1 + S_2)\\ &\geq 2|V_1(H)| + |V_2(H)| - 1.\end{aligned}$$

By Equation (6.4), this contradicts the sparsity of (G,ψ). Hence, there is an admissible fix-1-reduction at v. □

Theorem 6.3.2. Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_s$-generic framework with $\mathcal{C}_s$-gain framework (G,ψ,p). The following hold:

(1) If (G,ψ) is $(2,1,3,1)$-gain tight, then $(\tilde{G},\tilde{p})$ is fully-symmetrically isostatic.

(2) If (G, ψ) is $(2,1,3,2)$-gain tight, then $(\tilde{G}, \tilde{p})$ is anti-symmetrically isostatic.

Proof. We use a proof by induction on $|V(G)|$. First, assume that $V(G)$ has no free vertex. If (G, ψ) is $(2,1,3,1)$-gain tight, then G is a tree. The base case consists of exactly one single vertex and no edge, which is clearly fully-symmetrically isostatic. Assume that the statement is true for all graphs on m vertices and let G be a graph on $m+1$ vertices. Since G is a tree, it has a vertex v of degree 1. Thus, we may apply a fix-0-reduction at v to obtain a $(2,1,3,1)$-gain tight graph (G', ψ') on m vertices. By the inductive hypothesis, all $\mathcal{C}_s$-generic realisations of $\tilde{G}'$ are fully-symmetrically isostatic. Choose a $\mathcal{C}_s$-generic realisation $(\tilde{G}', \tilde{q}')$ of $\tilde{G}'$. By Lemma 5.3.6, there is a $\mathcal{C}_s$-symmetric realisation $(\tilde{G}, \tilde{q})$ of $\tilde{G}$ which is fully-symmetrically isostatic. By $\mathcal{C}_s$-genericity, $(\tilde{G}, \tilde{p})$ is also fully-symmetrically isostatic.

If (G, ψ) is $(2,1,3,2)$-gain tight, then G consists of exactly two isolated vertices u, v, with no edges, since any edge would violate the sparsity count. In this case, any anti-symmetric motion $\tilde{m}$ of $(\tilde{G}, \tilde{p})$ assigns vectors $(0 \;\; m_1)^T$ and $(0 \;\; m_2)^T$ to u and v, respectively. Since u, v are fixed, $\tilde{p}(u) = (a \;\; 0)^T$ and $\tilde{p}(v) = (b \;\; 0)^T$ for some $a, b \in \mathbb{R}$. Therefore, $\langle \tilde{m}(u) - \tilde{m}(v), \tilde{p}(u) - \tilde{p}(v) \rangle = 0$, and $\tilde{m}$ is trivial. It follows that $(\tilde{G}, \tilde{p})$ is anti-symmetrically isostatic.

Hence, we may assume $|V_1(G)| \geq 1$. If $|V(G)| = 1$, then G is a single vertex with exactly one loop when (G, ψ) is $(2,1,3,1)$-gain tight, and it is a single vertex with no loop when (G, ψ) is $(2,1,3,2)$-gain tight. In the former case, $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic: it is rigid since it is a single edge, and the ρ_0-orbit rigidity matrix of (G, ψ, p) is a non-zero row. In the latter, $(\tilde{G}, \tilde{p})$ is anti-symmetrically isostatic: the only joints $\tilde{p}(u)$ and $\tilde{p}(v)$ of $(\tilde{G}, \tilde{p})$ have coordinates $(x \;\; y)^T$ and $\sigma\tilde{p}(u) = (-x \;\; y)^T$, respectively; By Equation (4.3), any anti-symmetric infinitesimal motion $\tilde{m}$ of $(\tilde{G}, \tilde{p})$ which assigns $(m_1 \;\; m_2)^T$ to u satisfies $\tilde{m}(v) = (m_1 \;\; -m_2)^T$, and so $\langle \tilde{m}(u) - \tilde{m}(v), \tilde{p}(u) - \tilde{p}(v) \rangle = 0$. All base graphs are given in Figure 6.3.

For the inductive step, assume the result holds whenever $|V(G)| = m$ for some $m \in \mathbb{N}$. Let $1 \leq l \leq 2$ and suppose (G, ψ) is $(2,1,3,l)$-gain tight with $|V(G)| = m+1$. If G has a fixed vertex v of degree 1, then we may apply a fix-0-reduction at v to

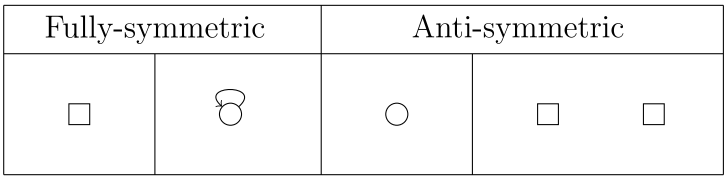


Figure 6.3: Base graphs for reflection.

obtain a $(2,1,3,l)$-gain tight graph (G', ψ') on m vertices. By induction, all $\mathcal{C}_s$-generic realisations of $\tilde{G}'$ are fully-symmetrically isostatic if $l = 1$, and they are anti-symmetrically isostatic if $l = 2$. Then, our result follows from Lemma 5.3.2. So, assume that all fixed vertices of G have degree at least 2.

Suppose that $V_1(G) = \{u\}$, and let $V_2(G) = \{v_1, \ldots, v_t\}$ for some $t \geq 1$. The average degree of G, denoted $\hat{\rho}$, satisfies $\hat{\rho} = 2|E(G)|/|V(G)| = (4+2t-2l)/|V(G)|$. The average degree of G is smallest when all vertices in $V_2(G)$ have degree 2, and so $2t + \deg(u) \leq 4 + 2t - 2l$. Hence $\deg(u) \leq 4 - 2l$. By Lemma 6.1.1(i), $l = 1$ and $\deg(u) = 2$. Then we may apply a 0-reduction at u to obtain a $(2,1,3,1)$-gain tight graph (G', ψ') on m vertices. By induction, all $\mathcal{C}_s$-generic realisations of $\tilde{G}'$ are fully-symmetrically isostatic. Then the result holds by Lemma 5.3.4. So, assume $|V_1(G)| \geq 2$.

By Lemma 6.3.1, (G, ψ) admits a reduction using one of the moves listed in the statement of the lemma. Let (G', ψ') be a $(2,1,3,l)$-gain tight graph obtained by applying such a reduction to (G, ψ). By induction, all $\mathcal{C}_s$-generic realisations of $\tilde{G}'$ are fully-symmetrically isostatic if $l = 1$ and anti-symmetrically isostatic if $l = 2$. Let $\tilde{q}'$ be a $\mathcal{C}_s$-generic configuration of $\tilde{G}'$ which also satisfies the conditions of Lemma 5.3.11 (respectively, Lemma 5.3.6) if $\tilde{G}'$ is obtained from $\tilde{G}$ by applying a 1-reduction (respectively, a fix-1-reduction). Such a configuration exists: if necessary, we may apply a small symmetry-preserving perturbation to the points of a $\mathcal{C}_s$-generic framework, which will maintain $\mathcal{C}_s$-genericity. By Lemmas 5.3.2, 5.3.4, 5.3.6, 5.3.8 and 5.3.11, there is a realisation $(\tilde{G}, \tilde{q})$ of $\tilde{G}$ which is fully-symmetrically isostatic if $l = 1$ and anti-symmetrically isostatic if $l = 2$. Since $\tilde{p}$ is $\mathcal{C}_s$-generic, the result follows. $\square$

The following main result for $\mathcal{C}_s$ is now a consequence of Proposition 5.1.1 and Theorem 6.3.2. Note that it is a generalisation of Theorem 3.1.7, which was given in [56].

Theorem 6.3.3. Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_s$-generic framework, and let (G, ψ) be the $\mathbb{Z}_2$-gain graph of $\tilde{G}$. $(\tilde{G}, \tilde{p})$ is infinitesimally rigid if and only if (G, ψ) has a $(2, 1, 3, 1)$-gain tight spanning subgraph and a $(2, 1, 3, 2)$-gain tight spanning subgraph.

6.4 Main result: 2-fold rotation

Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_2$-generic framework with $\mathcal{C}_2$-gain framework (G, ψ, p). Recall that (G, ψ) is $(2, 0, 3, 1)$-gain tight whenever $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic, and (G, ψ) is $(2, 2, 3, 2)$-gain tight whenever $(\tilde{G}, \tilde{p})$ is anti-symmetrically isostatic (see Proposition 5.1.2 in Subsection 5.1.2). In this section, we show that the converse statements are also true.

We do so by induction on $|V_1(G)|$, using the vertex reduction moves shown in Section 5.3. Hence, we first need to show that there is an admissible reduction of (G, ψ). Let $v \in V(G)$ be a free vertex of degree 3. By Theorem 6.2.1, there is always an admissible 1-reduction at v, unless (G, ψ) is $(2, 2, 3, 2)$-gain tight, v has exactly one free neighbour and exactly one fixed neighbour. In the following Lemma, we take care of this remaining case.

Lemma 6.4.1. Let Γ be a cyclic group of order 2. Let (G, ψ) be a $(2, 2, 3, 2)$-gain tight Γ-gain graph with $|V_2(G)| \leq 1$ and $2 \leq |V(G)|$. Then there is a reduction of (G, ψ) which yields a $(2, 2, 3, 2)$-gain tight graph (G', ψ'). The reduction which yields (G', ψ') is one of the following: a 0-reduction, a 1-reduction or a 2-vertex reduction.

Proof. The case where there is no fixed vertex is already known (see, e.g., [[56], Theorem 6.8]). Hence, we may assume $V_2(G) = \{v_0\}$. By Lemma 6.1.1, there is a free vertex in $V(G)$ of degree 2 or 3. By the sparsity of (G, ψ), no vertex of G has a loop. We may assume that G has no free vertex of degree 2. Otherwise, we may apply a 0-reduction to (G, ψ). (Clearly, any 0-reduction is admissible.) Further, we may assume that all free vertices of degree 3 have exactly 2 distinct neighbours, one of which is v_0: otherwise, we may apply a 1-reduction to (G, ψ), by Theorem 6.2.1.

So let $v_1, \dots, v_t$ be the free vertices in G of degree 3. For $1 \leq i \leq t$ let u_i be the free neighbour of v_i, and $e_i := (u_i, v_0)$. By Lemma 6.1.1(ii), $\deg(v_0) \leq t$. So, if the edge e_i is present for some $1 \leq i \leq t$, then u_i must be a vertex of degree 3. Hence, we can apply a 2-vertex reduction at u_i, v_i. So, we may assume that $e_i \notin E(G)$ for all $1 \leq i \leq t$.

For $1 \leq i \leq t$, let (G_i, ψ_i) be obtained from (G, ψ) by removing v_i and adding e_i with gain id. We will show that, for some $1 \leq i \leq t$, (G_i, ψ_i) is an admissible 1-reduction. Assume, for a contradiction, that for all $1 \leq i \leq t$ there is a blocker H_i for (G_i, ψ_i). By Proposition 6.1.2, each H_i is a balanced blocker.

Moreover, for each $1 \leq i \neq j \leq t$, $v_j \notin V(H_i)$. To see this, suppose, for a contradiction, that $v_j \in V(H_i)$. Since H_i is $(2,3)$-tight, all of its vertices have degree at least 2, by Lemma 6.1.1(i). Hence, two of the edges incident to v_j lie in $E(H_i)$. Moreover, since H_i is balanced, it cannot contain parallel edges. Hence, H_i contains exactly 2 of the edges incident to v_j. Let e be the edge incident to v_j such that $e \notin E(H_i)$. Then

$$|E(H_i + v_i + e)| = |E(H_i)| + 4 = 2|V(H_i)| + 1 = 2|V(H_i + v_i + e)| - 1,$$

contradicting the sparsity of (G, ψ). So $v_j \notin V(H_i)$ for all $1 \leq i \neq j \leq t$.

Claim: *$E(H_i \cap H_j) = \emptyset$ and $V(H_i \cap H_j) = \{v_0\}$ for all $1 \leq i \neq j \leq t$.*

Proof. Choose some $1 \leq i \neq j \leq t$. Suppose that $E(H_i \cap H_j) \neq \emptyset$. We show that this leads to a contradiction. In a similar way as we did in the proof of Lemma 6.1.5(ii), we can see that $|E(H_i \cup H_j)| \geq 2|V(H_i \cup H_j)| + 3c - c_0 - 6$, where c, c_0 are, respectively, the number of connected components and isolated vertices of $H_i \cap H_j$. Notice that $c_0 \leq c - 1$. (Since all isolated vertices of H' are also connected components of H', and H' has at least one connected component with non-empty edge set.) Therefore, $|E(H_i \cup H_j)| \geq 2|V(H_i \cup H_j)| + 2c - 5$. By the sparsity of (G, ψ), $c = 1$ and

$|E(H_i \cup H_j)| = 2|V(H_i \cup H_j)| - 3$. But the graph H obtained from $H_i \cup H_j$ by adding v_i, v_j and its incident edges satisfies $|E(H)| = 2|V(H)| - 1$, contradicting the sparsity of (G, ψ). Thus, $E(H_i \cap H_j) = \emptyset$ for all $1 \leq i \neq j \leq t$.

Now, if $V(H_i \cap H_j) \neq \{v_0\}$, then

$$\begin{aligned}|E(H_i \cup H_j)| &= |E(H_i)| + |E(H_j)| \\ &= 2|V(H_i \cup H_j)| + 2|V(H_i \cap H_j)| - 6 \geq 2|V(H_i \cup H_j)| - 2.\end{aligned}$$

But then the graph H obtained from $H_i \cup H_j$ by adding v_i, v_j and its incident edges satisfies $|E(H)| = 2|V(H)|$, contradicting the sparsity of (G, ψ). So $V(H_i \cap H_j)$ is the singleton $\{v_0\}$. Since i, j were arbitrary, the claim holds. $\square$

Let $H := \bigcup_{i=1}^{t} H_i$. By the Claim,

$$|E(H)| = \sum_{i=1}^{t} |E(H_i)| = 2\sum_{i=1}^{t} |V(H_i)| - 3t = 2(|V(H)| + (t-1)) - 3t = 2|V(H)| - t - 2.$$

So for the graph G' obtained from H by adding the vertices v_i, $i = 1, \ldots, t$, and their incident edges, we have $|E(G')| = 2|V(G')| - 2$. This implies that there is no edge $e \in E(G) \setminus E(H)$ that joins two vertices in $V(H)$.

Next we show that there is no non-empty subgraph H' of G such that $V(G)$ is the disjoint union of $V(G')$ and $V(H')$. Assume, for a contradiction, that such a graph H' exists. By assumption, all vertices of H' have degree at least 4 in G. Let $d(G', H')$ be the number of edges joining a vertex in G' with one in H'. We know $|E(H')| = 2|V(H')| - \alpha$ for some $\alpha \geq 2$. We have that

$$4|V(H')| \leq \sum_{v \in V(H')} \deg_G(v) = 2|E(H')| + d(G', H') = 4|V(H')| - 2\alpha + d(G', H'),$$

and so $d(G', H') \geq 2\alpha$. Hence,

$$\begin{aligned}|E(G)| &= |E(G')| + |E(H')| + d(G', H') \geq 2|V(G')| - 2 + 2|V(H')| - \alpha + 2\alpha \\ &= 2|V(G)| - 2 + \alpha,\end{aligned}$$

which contradicts the sparsity of (G, ψ), since $\alpha \geq 2$. So, H' does not exist, and $G = G'$.

Finally, fix some $1 \leq i \leq t$ and let n, m be the vertices of H which have degree 2 and 3 in H_i. The average degree of H_i is $\hat{\rho} = 2|E(G)|/|V(H_i)| = (4|V(H_i)| - 6)/|V(H_i)|$. The minimum average degree of H_i is $(4|V(H_i)| - 2n - m)/|V(H_i)|$. Hence, $2n + m \geq 6$. In particular, there are at least 3 vertices of degree 2 or 3 in $V(H_i)$, and so there is a free vertex v of degree 2 or 3 that is not v_0 or u_i. This means that v has degree 2 or 3 in $G = G'$. But this is not possible, since we assumed there are no free vertices of degree 2 in G, and that all free vertices of degree 3 are $v_1, \ldots, v_t$. The result follows. $\square$

The following results will be proved in a very similar way to Theorem 6.3.2. However, we now work with the half-turn group. So $|V_2(G)| \leq 1$.

Theorem 6.4.2. Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_2$-generic framework with $\mathcal{C}_2$-gain framework (G, ψ, p). The following hold:

(1) If (G, ψ) is $(2, 0, 3, 1)$-gain tight, then $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic.

(1) If (G, ψ) is $(2, 2, 3, 2)$-gain tight, then $(\tilde{G}, \tilde{p})$ is anti-symmetrically isostatic.

Proof. First, notice that if there is no free vertex, then $\tilde{G}$ is a single fixed vertex. In this case $\tilde{G}$ is not $(2, 0, 3, 1)$-gain tight. It is $(2, 2, 3, 2)$-gain tight and clearly also anti-symmetrically isostatic.

Hence, we may assume $|V_1(G)| \geq 1$. We prove the result by induction on $|V_1(G)|$. Assume $|V_1(G)| = 1$. If (G, ψ) is $(2, 0, 3, 1)$-gain tight, G is either composed of a free vertex and a loop, or a free vertex, a fixed vertex, and an edge connecting them. In either case, $O_0(G, \psi, p)$ is a non-zero row with one-dimensional kernel, and so $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic. If (G, ψ) is $(2, 2, 3, 2)$-gain tight, G must be a single free vertex. Any anti-symmetric motion of any realisation $(\tilde{G}, \tilde{p})$ of $\tilde{G}$ must be a translation of the whole framework, and so $(\tilde{G}, \tilde{p})$ is anti-symmetrically isostatic. The base cases for the fully-symmetric and anti-symmetric case are given in Figure 6.4.

Assume the result holds whenever $|V_1(G)| \leq m$ for some $m \in \mathbb{N}$ and consider the case where $|V_1(G)| = m + 1$.

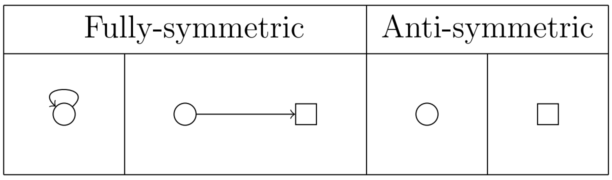


Figure 6.4: Base graphs for 2-fold rotation.

If (G, ψ) is $(2, 0, 3, 1)$-gain tight, G has a free vertex v of degree 2 or 3, by Lemma 6.1.1. If v has degree 2, or if it has degree 3 with a loop, then we may apply a 0-reduction or loop-1-reduction at v to obtain a $(2, 0, 3, 1)$-gain tight graph (G', ψ'), since 0-reductions and loop-1-reductions are always admissible. Moreover, if v has degree 3 with a loop, then it is not incident to a fixed vertex, by the sparsity of (G, ψ). By the inductive hypothesis, all $\mathcal{C}_2$-generic realisations of $\tilde{G}'$ are fully-symmetrically isostatic. Then, our result follows from Lemmas 5.3.4 and 5.3.8.

So, assume that v has degree 3 and no loops. By Lemma 6.2.1, there is a $(2, 0, 3, 1)$-gain tight graph (G', ψ') obtained from (G, ψ) by applying a 1-reduction at v. By induction, all $\mathcal{C}_2$-generic realisations of $\tilde{G}'$ are fully-symmetrically isostatic, so take a $\mathcal{C}_2$-generic realisation $(\tilde{G}', \tilde{q}')$ of $\tilde{G}'$ such that the conditions in Lemma 5.3.11 are satisfied. Then, our result holds by Lemma 5.3.11.

If (G, ψ) is $(2, 2, 3, 2)$-gain tight then, by Lemma 6.4.1, there is a $(2, 2, 3, 2)$-gain tight graph (G', ψ') on at most m vertices (exactly m if we apply a 0-reduction or 1-reduction, and exactly $m - 1$ if we apply a 2-vertex reduction) obtained by applying a reduction to (G, ψ).

By the inductive hypothesis, all $\mathcal{C}_2$-generic realisations of $\tilde{G}'$ are ρ_1-isostatic. Let $\tilde{q}'$ be a $\mathcal{C}_2$-generic configuration of $\tilde{G}'$, which also satisfies the conditions of Lemma 5.3.11 if $\tilde{G}'$ is obtained from $\tilde{G}$ by applying a 1-reduction. By Lemmas 5.3.4, 5.3.11 and 5.3.13, our result holds. $\square$

From Proposition 5.1.2 and Theorem 6.4.2, we obtain the following main result for $\mathcal{C}_2$. Note that this is a generalisation of Theorem 3.1.8, which was given in [56].

Theorem 6.4.3. Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_2$-generic framework, and let (G, ψ) be the $\mathbb{Z}_2$-gain graph of $\tilde{G}$. $(\tilde{G}, \tilde{p})$ is infinitesimally rigid if and only if (G, ψ) has a $(2,0,3,1)$-gain tight spanning subgraph and a $(2,2,3,2)$-gain tight spanning subgraph.

6.5 Main result: 3-fold rotation

Let $k \geq 3$ be odd, and $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_k$-generic framework with $\mathcal{C}_3$-gain framework (G, ψ, p). Recall that (G, ψ) is $(2,0,3,1)$-gain tight whenever $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic, and (G, ψ) is $(2,1,3,1)$-gain tight whenever $(\tilde{G}, \tilde{p})$ is ρ_1-symmetrically isostatic and ρ_{k-1}-symmetrically isostatic (recall Lemmas 5.1.3 and 5.1.4 in Subsections 5.1.2 and 5.1.3, respectively). Here, we prove that the converse is also true, which will give us the desired characterisation for $\mathcal{C}_3$-generic frameworks.

Theorem 6.5.1. For some $k \geq 3$, let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_k$-generic framework with $\mathcal{C}_k$-gain framework (G, ψ, p). The following hold:

(1) If (G, ψ) is $(2,0,3,1)$-gain tight, then $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic.

(2) If (G, ψ) is $(2,1,3,1)$-gain tight and k is odd, then $(\tilde{G}, \tilde{p})$ is ρ_1-isostatic and ρ_{k-1}-isostatic.

Proof. We prove the result by induction on $|V_1(G)|$, with the base cases given in Figure 6.5. It is easy to check that, if G is a single vertex v with a loop, then $O_0(G, \psi, p)$ has full rank and nullity 1 (since it is a non-zero multiple of $p(v)$), and $O_1(G, \psi, p)$ and $O_{k-1}(G, \psi, p)$ have full rank and nullity 1. (See, e.g., the proof of Lemma 5.3.8: using the same notation, $O_1(G, \psi, p) = Ap(v)^T$.) Moreover, in the second case of Figure 6.5, $O_0(G, \psi, p)$ has full rank and nullity 1, and in the fourth case of Figure 6.5, $O_1(G, \psi, p)$ and $O_{k-1}(G, \psi, p)$ have full rank and nullity 1.

For the inductive step, assume the result holds when $|V_1(G)| = t$ for some $t \geq 1$, and let (G, ψ) be a $(2, m, 3, 1)$-gain tight graph with $|V_1(G)| = t + 1$, for some $0 \leq t \leq 1$. Suppose that $m = 0$, and that $V(G)$ has an isolated fixed vertex. Then, we may remove it to obtain a $(2,0,3,1)$-gain tight graph (G', ψ') on t vertices.

By the inductive hypothesis, all $\mathcal{C}_k$-generic realisations of $\tilde{G}'$ are fully-symmetrically isostatic. Let $\tilde{q}'$ be a $\mathcal{C}_k$-generic configuration of $\tilde{G}'$. For any extension $\tilde{q} : V(G) \to \mathbb{R}^2$ of $\tilde{q}'$, we have $O_0(G, \psi, q) = O_0(G', \psi', q')$. So, $(\tilde{G}, \tilde{p})$ is also fully-symmetrically isostatic. By $\mathcal{C}_k$-genericity of $(\tilde{G}, \tilde{p})$, the result follows. So, we may assume that each fixed vertex of (G, ψ) has degree at least 1.

By Lemma 6.1.1, G has a free vertex v of degree 2 or 3 (both when $m = 0$ and when $m = 1$). If v has degree 2, or if it has degree 3 with a loop, then we may apply a 0-reduction or loop-1-reduction at v to obtain a $(2, m, 3, 1)$-gain tight graph (G', ψ') on t vertices. By the inductive hypothesis, all $\mathcal{C}_k$-generic realisations of $\tilde{G}$ are fully-symmetrically isostatic when $m = 0$, and ρ_1-symmetrically isostatic, ρ_{k-1}-symmetrically isostatic when $m = 1$. Moreover, when $m = 0$, the vertex incident to v is free, by the sparsity of (G, ψ). Then, by Lemmas 5.3.4 and 5.3.8, the result holds. So, assume that v has degree 3 and no loop. By Theorem 6.2.1, there is $(2, m, 3, 1)$-gain tight graph (G', ψ') obtained from (G, ψ) by applying a 1-reduction at v.

By the inductive hypothesis, all $\mathcal{C}_k$-generic realisations of $\tilde{G}'$ are ρ_0-isostatic when $m = 0$, and ρ_1-isostatic when $m = 1$. Let $(\tilde{G}, \tilde{q}')$ be any $\mathcal{C}_k$-generic realisation of $\tilde{G}'$ which satisfies the conditions of Lemma 5.3.11. Then, our result holds by Lemma 5.3.11. □

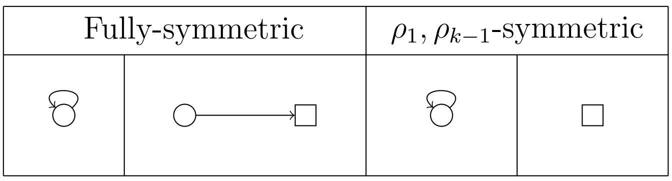


Figure 6.5: Base graphs for 3-fold rotation (and k-fold rotation for ρ_0, ρ_1 and ρ_{k-1}).

We finally have our main combinatorial characterisation for $\mathcal{C}_3$, which is a direct result of Proposition 5.1.5 and Theorem 6.5.1. Note that this is a generalisation of Theorem 3.1.9, which was given in [56].

Theorem 6.5.2. Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_3$-generic framework, and let (G, ψ) be the $\mathbb{Z}_3$-gain graph of $\tilde{G}$. $(\tilde{G}, \tilde{p})$ is infinitesimally rigid if and only if (G, ψ) has a $(2, 0, 3, 1)$-gain tight spanning subgraph and a $(2, 1, 3, 1)$-gain tight spanning subgraph.

Chapter 7

Sufficient conditions for cyclic groups of higher order

In this chapter we consider a $\mathcal{C}_k$-generic framework $(\tilde{G}, \tilde{p})$, where $k \geq 4$. The aim of the chapter is to show that the conditions given in Section 5.1 which the Γ-gain graph (G, ψ) of $\tilde{G}$ must satisfy in order for $(\tilde{G}, \tilde{p})$ to be infinitesimally rigid are also sufficient, and hence fully characterise the infinitesimal rigidity of $\mathcal{C}_k$-generic frameworks. Similarly as in Chapter 6, we adopt a proof by induction on the order of (G, ψ).

We structure the chapter as follows. In Section 7.1 we generalise the notion of admissibility and blocker given in Chapter 6, and we give some results on blockers which will be useful to prove some of the main results of the chapter. In Section 7.2 we show that, under certain conditions, there is always an admissible reduction at a free vertex of (G, ψ). In Section 7.3 we show that, if (G, ψ) satisfies the conditions given in Section 5.1, then $(\tilde{G}, \tilde{p})$ is infinitesimally rigid, provided $5 \leq k \leq 1000$ is odd or $k = 4, 6$. In Section 7.4 we explain the restrictions on the order of the symmetry group. Specifically, we give counterexamples to show that the analogous results do not hold if $k \geq 8$ is even.

7.1 Blockers of a reduction

Let $k \geq 4, 2 \leq j \leq k-2$, and let (G, ψ) be a $\mathbb{Z}_k^j$-gain tight Γ-gain graph for a cyclic group Γ of order k. We say a reduction of (G, ψ) is *admissible* if the Γ-gain graph (G', ψ') which it yields is also $\mathbb{Z}_k^j$-gain tight.

Definition 7.1.1. Let Γ be a cyclic group of order $k \geq 4$. For $2 \leq j \leq k-2$, let (G, ψ) be a $\mathbb{Z}_k^j$-gain tight Γ-gain graph with a free vertex v of degree 3 which has no loop. Let (G', ψ') be a Γ-gain graph obtained from (G, ψ) by applying a 1-reduction at v, and let $e = (v_1, v_2)$ be the edge we add when we apply such reduction. We say a subgraph H of $G - v$ with $v_1, v_2 \in V(H)$ and $E(H) \neq \emptyset$ is a *blocker* of e (equivalently, of (G', ψ')) if $H + e$ is connected and $|E(H)| = 2|V(H)| - 3 + \alpha_k^j(H + e)$, where α_k^j is as defined in Subsection 3.2.5.

If $\alpha_k^j(H + e) = 3 - 2|V_k(H)|$, we say H is a *general-count blocker*. If $H + e$ is balanced, we say H is a *balanced blocker*. If j is odd and $\langle H \rangle \simeq \mathbb{Z}_2$, we say H is a $\mathbb{Z}_2$*-blocker*.

A blocker is defined such that, when joined with the edge added through the 1-reduction, it is connected. However, disconnected graphs may also lead to a break of the sparsity count when applying a 1-reduction. With the same notation as that in Definition 7.1.1, let H' be a disconnected $\mathbb{Z}_k^j$-gain tight subgraph of $G - v$ with no isolated vertices, such that $v_1, v_2 \in V(H')$ and $E(H') \neq \emptyset$. Let $H_1, \dots, H_c$ be the connected components of H'. By Lemma 3.2.30, H' is $(2, 0, 0)$-tight.

Moreover, each connected component of H' is also $(2, 0, 0)$-tight: if, say $|E(H_1)| \leq 2|V_1(H_1)| - 1$, then some other connected component H_i must satisfy $|E(H_i)| \geq 2|V_1(H_i)| + 1$, contradicting the sparsity of (G, ψ). For some (not necessarily distinct) $1 \leq s, t \leq c$, we have $v_1 \in V(H_s), v_2 \in V(H_t)$. Then, $H_s \cup H_t$ is a blocker, as given in Definition 7.1.1.

Lemma 7.1.2. Let Γ be a cyclic group of order $k \geq 4$. For $2 \leq j \leq k-2$, let (G, ψ) be a $\mathbb{Z}_k^j$-gain tight Γ-gain graph with a free vertex v of degree 3 which has no loop. Let $(G_1, \psi_1), (G_2, \psi_2)$ be obtained from (G, ψ) by applying two different

1-reductions at v, which add the edges f_1, f_2, respectively. Let H_1, H_2 be blockers for $(G_1, \psi_1), (G_2, \psi_2)$, respectively, and use H to denote $H_1 \cup H_2$. If $|N_G(v)| = 3$, assume that f_1 and f_2 do not share a fixed vertex. Then, $\langle H + v \rangle \simeq \langle H + f_1 + f_2 \rangle$.

Proof. Since $\langle H \rangle$ is a subgroup of a cyclic group, we know that there is some integer $n \leq k$ such that $\langle H \rangle = \langle h \rangle \simeq \mathbb{Z}_n$ through an isomorphism which maps h to 1. We look at the cases where $|N_G(v)|$ is 1,2,3, separately.

Case 1: $|N_G(v)| = 1$.
Let u be the neighbour of v, let e_1, e_2, e_2 be the edges incident to u and v, and let $\psi(e_i) = g_i$ for $1 \leq i \leq 3$. By Propositions 3.2.6 and 3.2.9, we may assume that $g_1 = \text{id}$. Moreover, by the definition of gain graph, we know that $g_2, g_3, g_2 g_3^{-1} \neq \text{id}$.

By the definition of 1-reduction and the fact that $(G_1, \psi_1), (G_2, \psi_2)$ are obtained by applying two different 1-reductions, we may assume without loss of generality that $\psi_1(f_1) \neq \psi_2(f_2)$ lie in $\{g_2, g_3, g_2 g_3^{-1}\}$. It follows that $\langle \{f_1, f_2\} \rangle = \langle g_2, g_3 \rangle$, since $(g_2 g_3^{-1}) g_3 = g_2$ and $(g_2 g_3^{-1})^{-1} g_2 = g_3$. Similarly, $\langle \{e_1, e_2, e_3\} \rangle = \langle g_2, g_3 \rangle$. Then, $\langle H + v \rangle = \langle H + f_1 + f_2 \rangle = \langle h, g_2, g_3 \rangle$, as required.

Case 2: $|N_G(v)| = 2$.
Let v_1, v_2 be the neighbours of v, let e_1, e_1' be the edges incident to v and v_1, and let e_2 be the edge incident to v and v_2. By Propositions 3.2.6 and 3.2.9, we may assume that $\psi(e_1) = \psi(e_2) = \text{id}$, and by the definition of gain graph, $g := \psi(e_1') \neq \text{id}$.

By the definition of 1-reduction and the fact that $(G_1, \psi_1), (G_2, \psi_2)$ are obtained by applying two different 1-reductions, we know that at most one of $\psi_1(f_1), \psi_2(f_2)$ is id, and we may assume without loss of generality that $\psi_i(f_i) \in \{\text{id}, g\}$ for $1 \leq i \leq 2$.

If v_2 is fixed, it follows that $\langle H + v \rangle = \langle H + f_1 + f_2 \rangle = \langle h, g \rangle$. So, assume that v_2 is free. Let $\mathcal{W}$ be the set of walks from v_1 to v_2 in H with no fixed vertex and notice that, for all $W \in \mathcal{W}$, $g^{-1}(g\psi(W)) = \psi(W)$. Then, $\langle H + v \rangle$ is the group $\langle h, g, \psi(W), g\psi(W) : W \in \mathcal{W} \rangle = \langle h, g, \psi(W) : W \in \mathcal{W} \rangle$. Similarly,

$$\langle H + f_1 + f_2 \rangle = \langle h, g, \psi(W) : W \in \mathcal{W} \rangle = \langle H + v \rangle ,$$

as required.

Case 3: $|N_G(v)| = 3$.

Let v_1, v_2, v_3 be the neighbours of v and, for $1 \le i \le 3$, let $e_i = (v, v_i)$. By Propositions 3.2.6 and 3.2.9, we may assume that $\psi(e_i) = \text{id}$ for $1 \le i \le 3$. Then, by the definition of 1-reduction, $\psi_1(f_1) = \psi_2(f_2) = \text{id}$. Assume, without loss of generality, that $f_1 = (v_1, v_2)$ and that $f_2 = (v_2, v_3)$. By assumption, v_2 is free. For $1 \le s \ne t \le 3$, let $\mathcal{W}_{s,t}$ denote the set of walks from v_s to v_t in H which do not contain a fixed vertex. If v_1, v_2 are free, then $\langle H + f_1 + f_2 \rangle, \langle H + v \rangle$ are both $\langle h, \psi(W_{12}), \psi(W_{23}), \psi(W_{13}) : W_{12} \in \mathcal{W}_{1,2}, W_{23} \in \mathcal{W}_{2,3}, W_{13} \in \mathcal{W}_{1,3} \rangle$. So, we may assume that one of v_1, v_3 is fixed. Assume, without loss of generality, that v_1 is fixed. Then, $\langle H + f_1 + f_2 \rangle = \langle H + v \rangle = \langle h, g, \psi(W) : W \in \mathcal{W}_{2,3} \rangle$. This proves the result. $\square$

Let (G, ψ) be $\mathbb{Z}_k^j$-gain tight Γ-gain graph with a free vertex v of degree 3. It is easy to see that there are at least two possible 1-reductions at v. (It can be seen, for instance, in the proof of Lemma 7.1.2.) Let $(G_1, \psi_1), (G_2, \psi_2)$ be obtained from (G, ψ) by applying two different 1-reductions at v, which add the edges f_1, f_2, respectively. Suppose that neither one of the 1-reductions is admissible, so that $(G_1, \psi_1), (G_2, \psi_2)$ have some blockers H_1, H_2, respectively. Similarly as we did in Chapter 6, we examine $H_1 \cup H_2$ for the case where $E(H_1 \cap H_2) = \emptyset$. (Recall Lemma 6.1.5 in Section 6.1.)

We aim to show that $|E(H_1 \cup H_2)| = 2|V(H_1 \cup H_2)| - 3 + \alpha_k^j(H_1 \cup H_2 + f_1 + f_2)$ whenever $E(H_1 \cap H_2) \ne \emptyset$. Then, if $E(H_1 \cap H_2) \ne \emptyset$, we need only consider the case where $H_1 \cup H_2 + f_1 + f_2$ is proper near-balanced and $H_1 \cup H_2$ is $(2,1)$-tight, and the case where $|N_G(v)| = 3$ and f_1, f_2 share a fixed vertex, by Lemma 7.1.2.

Note that $\mathbb{Z}_k^j$-sparsity is far more refined than the sparsity conditions we needed in Chapter 6: since $0 \le \alpha_j^k(H_1 + f_1), \alpha_j^k(H_2 + f_2) \le 3$, we must consider 10 different cases. If we restrict the values of $\alpha_j^k(H_1 + f_1), \alpha_j^k(H_2 + f_2)$ to lie between 1 and 2, we then only have to consider 3 cases. We therefore split the study of $H_1 \cup H_2$ in different subsections: in Subsection 7.1.1, we show that H_1, H_2 cannot be general-count blockers, so that $\alpha_j^k(H_1 + f_1), \alpha_j^k(H_2 + f_2) \le 2$. (Note, this is the analogous result as Lemma 6.1.5(i) in our new setting.) In Subsection 7.1.2, we show that the desired result holds whenever $\alpha_j^k(H_i + f_i) = 0$ for some $1 \leq i \leq 2$. Then, in Subsection 7.1.3, we prove the full result.

7.1.1 General count blockers

Lemma 7.1.3. Let Γ be a cyclic group of order $k \geq 4$. For $2 \leq j \leq k-2$, let (G,ψ) be a $\mathbb{Z}_k^j$-gain tight Γ-gain graph with a free vertex v of degree 3 which has no loop. Let $(G_1,\psi_1),(G_2,\psi_2)$ be obtained from (G,ψ) by applying two different 1-reductions at v, which add the edges f_1 and f_2, respectively. For $i=1,2$, assume that (G_i,ψ_i) has a blocker H_i. If $E(H_1 \cap H_2) \neq \emptyset$, then H_1, H_2 are not general-count blockers.

Proof. Let $H := H_1 \cup H_2$, $H' := H_1 \cap H_2$, and let $H'_1,\dots,H'_c$ denote the connected components of H'. Let $c_0 \leq c-1$ be the number of isolated vertices of H', so that $H'_1,\dots,H'_{c_0}$ are the isolated vertices of H', and $H'_{c_0+1},\dots,H'_c$ are the connected components of H' with non-empty edge set. Assume, for a contradiction, that $E(H') \neq \emptyset$ and that H_i is a general-count blocker, for some $1 \leq i \leq 2$. Assume, without loss of generality, that H_1 is a general count blocker. We use the abbreviation α to denote $\alpha_k^j(H_2+f_2)$ and, for each $c_0+1 \leq i \leq c$, we use α_i to denote $\alpha_k^j(H'_i)$. By the sparsity of (G,ψ), we have

$$\begin{aligned}|E(H')| &\leq \sum_{i=1}^{c_0}[2|V(H'_i)|-2] + \sum_{i=c_0+1}^{c}[2|V(H'_i)|-3+\alpha_i]\\ &= 2|V(H')| - (2c_0 + 3(c-c_0)) + \sum_{i=c_0+1}^{c}\alpha_i.\end{aligned}$$

Therefore, letting $g = -(2c_0+3(c-c_0)) + \sum_{i=c_0+1}^{c}\alpha_i$, we have

$$\begin{aligned}|E(H)| &\geq 2|V_1(H_1)| + (2|V(H_2)|-3+\alpha) - (2|V(H')|+g)\\ &= 2|V_1(H_1)| + (2|V_1(H_2)| + 2|V_k(H_2)| - 3 + \alpha) - (2|V_1(H')| + 2|V_k(H')| + g)\\ &= 2|V_1(H)| + 2(|V_k(H_2)| - |V_k(H')|) + 2c_0 + 3(c-c_0-1) + (\alpha - \sum_{i=c_0+1}^{c}\alpha_i).\end{aligned}$$

Let $f = 2(|V_k(H_2)| - |V_k(H')|) + 2c_0 + 3(c - c_0 - 1) + (\alpha - \sum_{i=c_0+1}^{c} \alpha_i)$. If we show that $f \geq 0$, then $|E(H)| \geq 2|V_1(H)|$ and so, by Proposition 6.1.2, the result holds by contradiction. We show that indeed $f \geq 0$.

To do so, we first note that, for each $c_0 + 1 \leq i \leq c$, H'_i is a subgraph of $H_2 + f_2$, and so $\alpha_i \leq \alpha$ whenever $V_k(H'_i) = V_k(H_2)$. If $V_k(H') = V_k(H_2) = \emptyset$, it follows that

$$f \geq 2c_0 + 3(c - c_0 - 1) + (\alpha - (c - c_0)\alpha) = 2c_0 + (c - c_0 - 1)(3 - \alpha) \geq 0,$$

where the last inequality holds because $0 \leq c_0 \leq c - 1$ and $\alpha \leq 3$. Hence, we may assume that $V_k(H_2) = \{v_0\}$. By definition, it follows that $\alpha \leq 1$. Moreover, since each connected component of H' is a subgraph of $H_2 + f_2$, we know that $\alpha_i \leq \alpha + 2$ for all $c_0 + 1 \leq i \leq c$. Hence, if $V_k(H') = \emptyset$, it follows that

$$\begin{aligned} f &\geq 2 + 2c_0 + 3(c - c_0 - 1) + (\alpha - (c - c_0)(\alpha + 2)) \\ &= (c - c_0 - 1)(3 - \alpha) + 2(1 - c + 2c_0) \\ &\geq 2(c - c_0 - 1) + 2(1 - c + 2c_0) = 2c_0 \geq 0. \end{aligned}$$

So, we may assume that $V_k(H') = \{v_0\}$. If v_0 is isolated in H', then $c_0 \geq 1$. Hence,

$$f \geq 2c_0 + 3(c - c_0 - 1) + (\alpha - (c - c_0)(\alpha + 2)) \geq 2(c_0 - 1) \geq 0.$$

So assume, without loss of generality, that $v_0 \in V(H'_{c_0+1})$. By definition, $\alpha_{c_0+1} \leq \alpha$. Since $\alpha_i \leq \alpha + 2$ for all $c_0 + 2 \leq i \leq c$, we have

$$\begin{aligned} f &\geq 2c_0 + 3(c - c_0 - 1) + (\alpha - \alpha - (\alpha + 2)(c - c_0 - 1)) \\ &= (c - c_0 - 1)(1 - \alpha) + 2c_0 \geq 0, \end{aligned}$$

where the last inequality holds because $0 \leq c_0 \leq c - 1$ and $\alpha \leq 1$. We always have $f \geq 0$, as required. $\square$

7.1.2 $(2, 3)$-tight blockers

With the same notation as that in Lemma 7.1.3, we now consider the case where $\alpha_k^j(H_1 + f_1) = 0$. By definition, this is equivalent to saying that $H_1 + f_1$ is either balanced or $S_0(k,j)$ with $V_k(H_1) \neq \emptyset$. We consider the two cases separately, in Lemmas 7.1.5 and 7.1.6, respectively. However, in Lemma 7.1.5, we do not assume that $H_1 + f_1$ is balanced. Instead, we make the slightly weaker assumption that the intersection $H_1 \cap H_2$ is balanced. (This weaker assumption will be useful when proving Lemma 7.1.6, as well as Lemma 7.1.7.) In order to prove Lemma 7.1.5, we need the following result.

Lemma 7.1.4. Let $0 \leq m \leq 2, 1 \leq l \leq 3$ be such that $m \leq l$, and let (G, ψ) be a $(2, m, l)$-tight Γ-gain graph for some cyclic group Γ of order $k \geq 4$. Assume that $|V_k(G)| \leq 1$. Then G has no fixed cut-vertex.

Proof. By Lemma 3.2.30, G is connected. If $V_k(G) = \emptyset$, the result clearly holds. Therefore, we may assume that $V_k(G) = \{v_0\}$. Assume, for a contradiction, that v_0 is a cut-vertex. Let $\{G_1, \ldots, G_t\}$ be subgraphs of G such that $G_i \cap G_j$ is v_0 for all $1 \leq i \neq j \leq t$ and $\{E(G_1), \ldots, E(G_t)\}$ forms a partition of $E(G)$. Then, $|E(G_i)| \leq 2|V_1(G_i)| + m|V_k(G_i)| - l = 2|V_1(G_i)| + m - l$ for all $1 \leq i \leq t$. Hence,

$$|E(G)| = \sum_{i=1}^{t} |E(G_i)| \leq 2\sum_{i=1}^{t} |V_1(G_i)| + mt - lt = 2|V_1(G)| + t(m-l).$$

Since $|E(G)| = 2|V_1(G)| + m|V_k(G)| - l = 2|V_1(G)| + m - l$, and since $m - l \leq 0$, it follows that $t \leq 1$. But this contradicts the fact that v_0 is a cut-vertex. Hence, the result holds. □

Lemma 7.1.5. Let Γ be a cyclic group of order $k \geq 4$. For $2 \leq j \leq k-2$, let (G, ψ) be a $\mathbb{Z}_k^j$-gain tight Γ-gain graph with a free vertex v of degree 3 which has no loop. Let $(G_1, \psi_1), (G_2, \psi_2)$ be obtained from (G, ψ) by applying two different 1-reductions at v, which add the edges f_1 and f_2, respectively. For $i = 1, 2$, assume that (G_i, ψ_i) has a blocker H_i, and let $H = H_1 \cup H_2$. If $E(H_1 \cap H_2) \neq \emptyset$ and $H_1 \cap H_2$ is balanced, then $E(H) = 2|V(H)| - 3 + \alpha_k^j(H + f_1 + f_2)$.

Proof. Let $H' := H_1 \cap H_2$, and let $H'_1, \ldots, H'_c$ denote the connected components of H'. Let $c_0 \leq c - 1$ be the number of isolated vertices of H', so that $H'_1, \ldots, H'_{c_0}$ are the isolated vertices of H', and $H'_{c_0+1}, \ldots, H'_c$ are the connected components of H'

with non-empty edge set. For $1 \le i \le 2$, use α_i to denote $\alpha_k^j(H_i + f_i)$. We also use α to denote $\alpha_k^j(H + f_1 + f_2)$. Assume that H' is balanced. Then,

$$|E(H')| \le \sum_{i=1}^{c_0}[2|V(H_i')| - 2] + \sum_{i=c_0+1}^{c}[2|V(H_i')| - 3] = 2|V(H')| - 2c_0 - 3(c - c_0).$$

Therefore,

$$\begin{aligned} |E(H)| &\ge (2|V(H_1)| - 3 + \alpha_1) + (2|V(H_2)| - 3 + \alpha_2) \\ &\quad - (2|V(H')| - 2c_0 - 3(c - c_0)) \\ &= 2|V(H)| - 6 + \alpha_1 + \alpha_2 + 2c_0 + 3(c - c_0). \end{aligned} \tag{7.1}$$

If $c - c_0 \ge 2$, then $|E(H)| \ge 2|V(H)| + \alpha_1 + \alpha_2 + 2c_0 \ge 2|V_1(H)|$, contradicting Proposition 6.1.2 or the sparsity of (G, ψ). Hence, $c - c_0 = 1$ and $|E(H)| \ge 2|V(H)| - 3 + \alpha_1 + \alpha_2 + 2c_0$. If $c_0 \ge 2$, then $|E(H)| \ge 2|V(H)| + 1$, contradicting the sparsity of (G, ψ). Hence, (c_0, c_1) is either $(0, 1)$ or $(1, 2)$.

Suppose that $(c_0, c_1) = (1, 2)$. By Equation (7.1), $|E(H)| \ge 2|V(H)| - 1 + \alpha_1 + \alpha_2$. By Proposition 6.1.2 and the sparsity of (G, ψ), $V_k(H) = \emptyset$ and $\alpha_1 = \alpha_2 = 0$. It follows that H_1, H_2 are balanced blockers. By Lemma 3.2.21, $H + f_1 + f_2$ is proper near-balanced, so $\alpha = 2$. Then, by the sparsity of (G, ψ), $|E(H)| = 2|V(H)| - 1 = 2|V(H)| - 3 + \alpha$. Hence, we may assume that $(c_0, c_1) = (0, 1)$ and so, by Equation (7.1),

$$|E(H)| \ge 2|V(H)| - 3 + \alpha_1 + \alpha_2. \tag{7.2}$$

By Proposition 6.1.2, $\alpha_1 + \alpha_2 \le 2$. We look at the cases where $\alpha_1 + \alpha_2 = 2$, $\alpha_1 + \alpha_2 = 1$ and $\alpha_1 + \alpha_2 = 0$ separately. In all such cases, we show that $|E(H)| = 2|V(H)| - 3 + \alpha$, proving the result.

Case 1: $\alpha_1 + \alpha_2 = 2$

By Equation (7.2), $|E(H)| \ge 2|V(H)| - 1$, so by the sparsity of (G, ψ), $V_k(H) = \emptyset$. Moreover, H' is $(2, 3)$-tight: otherwise, it is easy to see that $|E(H)| \ge 2|V(H)|$, contradicting Proposition 6.1.2 or the sparsity of (G, ψ). Assume, without loss of generality, that (α_1, α_2) is one of $(1, 1)$ and $(0, 2)$. In the former case, j is odd and $\langle H_1 + f_1\rangle = \langle H_2 + f_2\rangle \simeq \mathbb{Z}_2$. Since H' is connected, every closed walk W in $H + f_1 + f_2$ can be decomposed as a concatenation of closed walks in $H_1 + f_1$ and $H_2 + f_2$. It follows, from the fact that $V_k(H) = \emptyset$, that $\langle H + f_1 + f_2\rangle \simeq \mathbb{Z}_2$. Then, by the sparsity of (G, ψ), H is $(2, 1)$-tight, and the result holds. If $(\alpha_1, \alpha_2) = (0, 2)$, then H_1 is a balanced blocker, and $H_2 + f_2$ is either proper near-balanced or $S(k, j)$. In the former case, $H + f_1 + f_2$ is proper near-balanced, by Lemma 3.2.20. In the latter, $H + f_1 + f_2$ is $S(k, j)$, by Lemma 3.2.14. In both cases, $\alpha = 2$ and $|E(H)| = 2|V(H)| - 1 = 2|V(H)| - 3 + \alpha$ by the sparsity of (G, ψ).

Case 2: $\alpha_1 + \alpha_2 = 1$

By Equation 7.2, $|E(H)| \geq 2|V(H)| - 2$. It follows, from Proposition 6.1.2, that $V_k(H) = \emptyset$. Assume, without loss of generality, that $(\alpha_1, \alpha_2) = (1, 0)$. Then, j is odd, $\langle H_1 + f_1\rangle \simeq \mathbb{Z}_2$, and H_2 is a balanced blocker. It follows, from Lemma 3.2.14, that $\langle H + f_1 + f_2\rangle \simeq \mathbb{Z}_2$, and so $\alpha = 1$. By the sparsity of (G, ψ), H satisfies $|E(H)| = 2|V(H)| - 2 = 2|V(H)| - 3 + \alpha$.

Case 3: $\alpha_1 + \alpha_2 = 0$

By Equation 7.2, $|E(H)| \geq 2|V(H)| - 3$. Notice that, if H' is not $(2, 3)$-tight, then $|E(H)| \geq 2|V(H)| - 2$ and so $V_k(H) = \emptyset$ by Proposition 6.1.2. It follows that if H' is not $(2, 3)$-tight, then it does not have a fixed cut-vertex. On the other hand, if H' is $(2, 3)$-tight, then it does not have a fixed cut-vertex by Lemma 7.1.4. So, H' does not have a fixed cut-vertex. For each $1 \leq i \leq 2$, since $\alpha_i = 0$, H_i is either balanced, or it has a fixed vertex and is $S_0(k, j)$. If H_1, H_2 are balanced blocker, then $H + f_1 + f_2$ is balanced by Proposition 3.2.14. If one of $H_1 + f_1, H_2 + f_2$ is balanced, and the other is $S_0(k, j)$, then $H + f_1 + f_2$ is $S_0(k, j)$ by Proposition 3.2.14, and contains the fixed vertex. If $H_1 + f_1, H_2 + f_2$ are both $S_0(k, j)$, then so is $H + f_1 + f_2$ by Lemma 3.2.27(i), and it contains the fixed vertex. In all such cases, $\alpha = 0$, and $|E(H)| = 2|V(H)| - 3 = 2|V(H)| - 3 + \alpha$, as required. □

Lemma 7.1.6. Let Γ be a cyclic group of order $k \geq 4$. For $2 \leq j \leq k - 2$, let (G, ψ) be a $\mathbb{Z}_k^j$-gain tight Γ-gain graph with a free vertex v of degree 3 which has no loop. Let $(G_1, \psi_1), (G_2, \psi_2)$ be obtained from (G, ψ) by applying two different 1-reductions at v, which add the edges f_1 and f_2, respectively. For $i = 1, 2$, assume that (G_i, ψ_i) has a blocker H_i. Assume further that $V_k(H_1) = \{v_0\}$ and that $H_1 + f_1$ is $S_0(k, j)$. If $E(H_1 \cap H_2) \neq \emptyset$, then $H := H_1 \cup H_2$ satisfies $|E(H)| = 2|V(H)| - 3 + \alpha_k^j(H + f_1 + f_2)$.

Proof. Let $H' := H_1 \cap H_2$, and let $H'_1, \dots, H'_c$ denote the connected components of H'. Let $c_0 \leq c - 1$ be the number of isolated vertices of H', so that $H'_1, \dots, H'_{c_0}$ are the isolated vertices of H', and $H'_{c_0+1}, \dots, H'_c$ are the connected components of H' with non-empty edge set. By Lemma 7.1.5, we may assume that H' is unbalanced. In particular, H_2 is not a balanced blocker. Moreover, by Lemma 7.1.3, we may assume that H_2 is not a general-count blocker. Throughout the proof, let α denote $\alpha_k^j(H_2 + f_2)$. We look at the cases where $V_k(H') = \emptyset$ and $V_k(H') = \{v_0\}$ separately.

First, suppose that $V_k(H') = \emptyset$. Since $V_k(H_1) = \{v_0\}$, it follows that $V_k(H_2) = \emptyset$. By assumption, this implies that $1 \leq \alpha \leq 2$. Since each connected component of H' is a subgraph of $H_2 + f_2$ and $V_k(H') = V_k(H_2) = \emptyset$,

$$\begin{aligned}|E(H')| = \sum_{i=1}^{c} |E(H'_i)| &\leq \sum_{i=1}^{c_0} [2|V(H'_i)| - 2] + \sum_{i=1+c_0}^{c} [2|V(H'_i)| - 3 + \alpha] \\ &= 2|V(H')| - 2c_0 + (c - c_0)(\alpha - 3).\end{aligned}$$

Hence, letting $g = -2c_0 + (c - c_0)(\alpha - 3)$, we have

$$\begin{aligned}|E(H)| &\geq (2|V(H_1)| - 3) + (2|V(H_2)| - 3 + \alpha) - (2|V(H')| + g) \\ &= 2|V(H)| - 6 + \alpha + 2c_0 + (c - c_0)(3 - \alpha) \\ &= 2|V_1(H)| - 4 + \alpha + 2c_0 + (c - c_0)(3 - \alpha). \qquad (7.3)\end{aligned}$$

We show that $c_0 = 0$ and $c_1 = 1$. Assume, for a contradiction, that $c - c_0 \geq 2$. Then, by Equation (7.3) and the fact that $\alpha \leq 2$, we have

$$|E(H)| \geq 2|V_1(H)| + 2 - \alpha \geq 2|V_1(H)|.$$

This contradicts Proposition 6.1.2 or the sparsity of (G, ψ). Hence, $c = c_0 + 1$ and, by Equation 7.3, we have $|E(H)| \geq 2|V_1(H)| - 1 + 2c_0$. By Proposition 6.1.2, it follows that $c_0 = 0, c = 1$ and $|E(H)| = 2|V_1(H)| - 1$. If we show that $H + f_1 + f_2$ is

$S_0(k,j)$, it then follows that $|E(H)| = 2|V(H)|-3+\alpha_k^j(H+f_1+f_2)$, as required. We show that $H+f_1+f_2$ is indeed $S_0(k,j)$. Since $1 \leq \alpha \leq 2$ and $V_k(H_2) = \emptyset$, exactly one of the following holds: H_2 is a $\mathbb{Z}_2$-blocker; H_2+f_2 is $S(k,j)$; H_2+f_2 is proper near-balanced. If H_2 is a $\mathbb{Z}_2$-blocker, then $H+f_1+f_2$ is $S_0(k,j)$ by Lemma 3.2.15. If H_2+f_2 is $S(k,j)$, then it is $S_0(k,j)$ by Lemma 3.2.25. Hence, $H+f_1+f_2$ is $S_0(k,j)$ by Lemma 3.2.27(i). If H_2+f_2 is near-balanced, then $H+f_1+f_2$ is $S_0(k,j)$ by Lemma 3.2.27(ii). So, whenever $V_k(H') = \emptyset$, the result holds.

Now, assume that $V_k(H') = \{v_0\}$. This implies that $V_k(H_2) \neq \emptyset$. Hence, $|E(H_2)| = 2|V_1(H_2)| - 1 + \alpha$. If v_0 is isolated in H', then $c_0 \geq 1$. Assume, without loss of generality, that v_0 is H'_1. Since each H'_i is a subgraph of H_1+f_1, we have

$$\begin{aligned}|E(H')| = \sum_{i=1}^{c} |E(H'_i)| &\leq 2|V_1(H'_1)| + \sum_{i=2}^{c_0}[2|V_1(H'_i)|-2] + \sum_{i=c_0+1}^{c}[2|V_1(H'_i)|-1]\\ &= 2|V_1(H')| - 2(c_0-1) - (c-c_0),\end{aligned}$$

and so, letting $g = -2(c_0-1)-(c-c_0)$, we have

$$\begin{aligned}|E(H)| &\geq (2|V_1(H_1)|-1) + (2|V_1(H_2)|-1+\alpha) - (2|V_1(H')|+g)\\ &= 2|V_1(H)| - 2 + \alpha + 2(c_0-1) + (c-c_0).\end{aligned}$$

If $c - c_0 \geq 2$ or if $c_0 \geq 2$, this contradicts Proposition 6.1.2 or the sparsity of (G,ψ). Hence, we may assume that $c_0 = 1, c = 2$. So, $|E(H)| \geq 2|V_1(H)| - 1 + \alpha$. In a similar way, if v_0 is not an isolated vertex of H', we can see that $|E(H)| \geq 2|V_1(H)|-2+\alpha+2c_0+(c-c_0)$. If $c_0 \geq 1$ or $c-c_0 \geq 2$, this contradicts Proposition 6.1.2 or the sparsity of (G,ψ). Hence, $c_0 = 0, c = 1$, and $|E(H)| \geq 2|V_1(H)|-1+\alpha$. Both when v_0 is an isolated vertex of H' and when it is not, Proposition 6.1.2 implies that $\alpha = 0$ and $|E(H)| = 2|V_1(H)| - 1$. Hence, it is enough show that $H+f_1+f_2$ is $S_0(k,j)$. Since $\alpha = 0$ and H_2 is not a balanced blocker, H_2+f_2 is $S_0(k,j)$. Moreover, H' is either connected, or it is composed of two connected components, one of which is the isolated fixed vertex. So, $H+f_1+f_2$ is $S_0(k,j)$ by Lemma 3.2.27(i), and the result holds. □

7.1.3 The union of two blockers

Lemma 7.1.7. Let Γ be a cyclic group of order $k \geq 4$. For $2 \leq j \leq k-2$, let (G, ψ) be a $\mathbb{Z}_k^j$-gain tight Γ-gain graph with a free vertex v of degree 3 which has no loop. Let $(G_1, \psi_1), (G_2, \psi_2)$ be obtained from (G, ψ) by applying two different 1-reductions at v, which add the edges f_1 and f_2, respectively. For $i = 1, 2$, assume that (G_i, ψ_i) has a blocker H_i. If $E(H_1 \cap H_2) \neq \emptyset$, then $|E(H)| = 2|V(H)| - 3 + \alpha_k^j(H + f_1 + f_2)$.

Proof. Throughout the proof, we let $H' = H_1 \cap H_2$ and we let $H'_1, \dots, H'_c$ be the connected components of H'. Let $c_0 \leq c - 1$ be the number of isolated vertices of H', so that $H'_1, \dots, H'_{c_0}$ are the isolated vertices of H', and $H'_{c_0+1}, \dots, H'_c$ are the connected components of H' with non-empty edge set. We abbreviate $\alpha_k^j(H_i + f_i)$ to α_i, for $i = 1, 2$. By Lemma 7.1.5, we may assume that H' is unbalanced. Moreover, by Lemmas 7.1.3 and 7.1.6, we may assume that $1 \leq \alpha_1, \alpha_2 \leq 2$. Without loss of generality, assume that $\alpha_1 \geq \alpha_2$. We look at the following three cases separately: $(\alpha_1, \alpha_2) = (1, 1)$; $(\alpha_1, \alpha_2) = (2, 1)$; and $(\alpha_1, \alpha_2) = (2, 2)$.

Case 1: $\alpha_1 = \alpha_2 = 1$.

If we show that $V_k(H') = \emptyset$ then, by the definition of α_1, α_2, one of H_1, H_2 is a $\mathbb{Z}_2$-blocker. We show that $V_k(H')$ is indeed empty. So assume, for a contradiction, that $|V_k(H')| = 1$. By the sparsity of (G, ψ), we know that, for all $c_0 + 1 \leq i \leq c$, $|E(H'_i)| \leq 2|V_1(H'_i)|$. If the fixed vertex is isolated, then $c_0 \geq 1$ and

$$\begin{aligned} |E(H')| &= \sum_{i=1}^{c} |E(H'_i)| \leq \sum_{i=1}^{c_0} [2|V(H'_i)| - 2] + \sum_{i=c_0+1}^{c} 2|V(H'_i)| \\ &= 2|V(H')| - 2c_0 \leq 2|V(H')| - 2. \end{aligned}$$

If the fixed vertex is not isolated, assume without loss of generality, that it lies in H'_{c_0+1}. Then,

$$\begin{aligned} |E(H')| &= \sum_{i=1}^{c} |E(H'_i)| \leq \sum_{i=1}^{c_0} [2|V(H'_i)| - 2] + [2|V(H'_{1+c_0})| - 2] + \sum_{i=c_0+2}^{c} 2|V(H'_i)| \\ &= 2|V(H')| - 2c_0 - 2. \end{aligned}$$

Since $c_0 \geq 0$, $|E(H')| \leq 2|V(H')| - 2$. Hence, in both cases we have

$$\begin{aligned}|E(H)| &\geq (2|V(H_1)| - 2) + (2|V(H_2)| - 2) - (2|V(H')| - 2)\\ &= 2|V(H)| - 2 = 2|V_1(H)|.\end{aligned}$$

By the sparsity of (G, ψ) and Proposition 6.1.2, this is a contradiction. So, H' has no fixed vertex, j is odd and $\langle H_i + f_i\rangle \simeq \mathbb{Z}_2$ for some $1 \leq i \leq 2$. Assume, without loss of generality, that $\langle H_1 + f_1\rangle \simeq \mathbb{Z}_2$. Then, since H' is a subgraph of $H_1 + f_1$ and j is odd, $|E(H')| \leq 2|V(H')| - 2c$, and so

$$\begin{aligned}|E(H)| &\geq (2|V(H_1)| - 2) + (2|V(H_2)| - 2) - (2|V(H')| - 2c)\\ &= 2|V(H)| + 2(c - 2). \qquad (7.4)\end{aligned}$$

By the sparsity of (G, ψ) and Proposition 6.1.2, this implies that $c = 1$ and that $|V_k(H)| = 0$. Hence, $|V_k(H_2)| = 0$, and we have $\langle H_2 + f_2\rangle \simeq \mathbb{Z}_2$. Since H' is connected, every closed walk W in $H + f_1 + f_2$ can be decomposed as a concatenation of closed walks in $H_1 + f_1$ and $H_2 + f_2$. Hence, $\langle H + f_1 + f_2\rangle \simeq \mathbb{Z}_2$. By the sparsity of (G, ψ), and by Equation (7.4), $|E(H)| = 2|V(H)| - 3 + \alpha^j_k(H + f_1 + f_2)$.

Case 2: $\alpha_1 = 2, \alpha_2 = 1$.

By the definition of α_1, $|V_k(H_1)| = 0$ and $H_1 + f_1$ is $S(k, j)$ or proper near-balanced. Notice that for each $1 \leq i \leq c_0$, $|E(H'_i)| = 2|V(H'_i)| - 2 < 2|V(H'_i)| - 1$. So, since $|V_k(H')| = 0$ and H' is a subgraph of $H_1 + f_1$, it must satisfy the inequality $|E(H')| \leq \sum_{i=1}^{c}[2|V(H'_i)| - 1] = 2|V(H')| - c$. Hence,

$$\begin{aligned}|E(H)| &\geq (2|V(H_1)| - 1) + (2|V(H_2)| - 2) - (2|V(H')| - c)\\ &= 2|V(H)| - 3 + c \geq 2|V(H)| - 2, \qquad (7.5)\end{aligned}$$

since $c \geq 1$. By Proposition 6.1.2, $|V_k(H)| = 0$. By the definition of α_2, H_2 is a $\mathbb{Z}_2$-blocker. Then, since H' is a subgraph of $H_2 + f_2$, each connected component of H' must be $(2, 2)$-sparse. It follows that H' satisfies $|E(H')| \leq 2|V(H')| - 2c$ and

$$|E(H)| \geq (2|V(H_1)| - 1) + (2|V(H_2)| - 2) - (2|V(H')| - 2c) = 2|V(H)| + 2c - 3.$$

This implies that $c = 1$, by the sparsity of (G, ψ). Since H' is unbalanced, $H_1 + f_1$ is not proper near-balanced: otherwise, $\langle H_1 + f_1 \rangle \simeq \mathbb{Z}_2$, by Lemma 3.2.22, which contradicts the definition of proper near-balancedness. It follows that $H_1 + f_1$ is $S(k, j)$. Then, by Lemma 3.2.15, $H + f_1 + f_2$ is $S(k, j)$ and so $\alpha_k^j(H + f_1 + f_2) = 2$. Hence,

$$|E(H)| \geq 2|V(H)| - 1 = 2|V(H)| - 3 + \alpha_k^j(H + f_1 + f_2). \tag{7.6}$$

By the sparsity of (G, ψ), Equation (7.6) holds with equality.

Case 3: $\alpha_1 = \alpha_2 = 2$.

In a similar way as we did in Case 2, we can see that $|E(H)| \geq 2|V(H)| - 2 + c$. If H' is not connected or if $V_k(H) \neq \emptyset$, then $|E(H)| \geq 2|V(H)|$, contradicting Proposition 6.1.2 or the sparsity of (G, ψ). So $c = 1$ and $V_k(H) = \emptyset$. Since H' is a subgraph of $H_1 + f_1$ and $V_k(H') = V_k(H_1 + f_1) = \emptyset$, it is $(2, 1)$-sparse. If $|E(H')| \leq 2|V(H')| - 2$, it is easy to see that $|E(H)| \geq 2|V(H)|$, contradicting Proposition 6.1.2 or the sparsity of (G, ψ). Hence, H' is $(2, 1)$-tight.

If exactly one of $H_1 + f_1, H_2 + f_2$ is near-balanced, then $H + f_1 + f_2$ is $S(k, j)$ by Lemma 3.2.27(ii). If both $H_1 + f_1, H_2 + f_2$ are $S(k, j)$, then they are both $S_i(k, j)$ for some $i \in \{0, -1, 1\}$, by Lemma 3.2.25. So, by Lemma 3.2.27(i), $H + f_1 + f_2$ is also $S_i(k, j)$. If neither $H_1 + f_1$ nor $H_2 + f_2$ is $S(k, j)$, then they are both proper near-balanced. Hence, H' is also proper near-balanced and so $H + f_1 + f_2$ is near-balanced by Lemma 3.2.19. By the sparsity of (G, ψ) and Proposition 6.1.2, $|E(H)| = 2|V(H)| - 1$ and $|V_k(H)| = 0$. Since $\alpha_k^j(H + f_1 + f_2) = 2$, we have $|E(H)| = 2|V(H)| - 3 + \alpha_k^j(H + f_1 + f_2)$, as required. □

Proposition 6.1.2, and Lemmas 7.1.2, 7.1.7 imply the following result.

Corollary 7.1.8. Let Γ be a cyclic group of order $k \geq 4$. For $2 \leq j \leq k - 2$, let (G, ψ) be a $\mathbb{Z}_k^j$-gain tight Γ-gain graph with a free vertex v of degree 3 which has no loop. Let $(G_1, \psi_1), (G_2, \psi_2)$ be obtained from (G, ψ) by applying two different 1-reductions at v, which add the edges f_1 and f_2, respectively. For $i = 1, 2$, assume that (G_i, ψ_i) has a blocker H_i, and suppose that $E(H_1 \cap H_2) \neq \emptyset$. If $|N_G(v)| \neq 3$, or if f_1 and f_2 do not share a fixed vertex, then $H_1 \cup H_2 + f_1 + f_2$ is proper near-balanced.

Proof. Let $H = H_1 \cup H_2$. Assume that $|N_G(v)| \neq 3$, or that f_1 and f_2 do not share a fixed vertex. Assume, for a contradiction, that $H + f_1 + f_2$ is not proper near-balanced. By Lemma 7.1.2, $\langle H + f_1 + f_2 \rangle = \langle H + v \rangle$. Therefore, we know that $\alpha_j(H + v) = \alpha_j(H + f_1 + f_2)$. By Lemma 7.1.7, $|E(H)| = 2|V(H)| - 3 + \alpha_k^j(H + v)$, which contradicts Proposition 6.1.2. Hence, $H + f_1 + f_2$ is proper near-balanced. $\square$

7.2 A gain tight graph admits a reduction

The following result is crucial for the combinatorial results of the paper. We show that, given a vertex v of degree 3, we may always apply an admissible 1-reduction at v except in one special case.

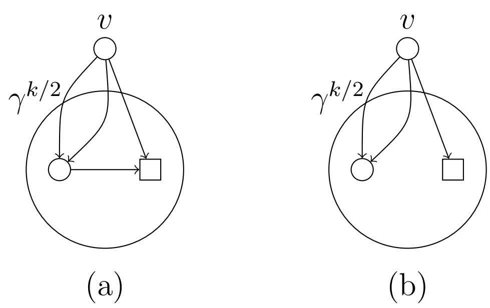


Figure 7.1: Two instances of a vertex v of degree 3 of a Γ-gain graph. In both cases v has two neighbours, one of which is fixed. In (a) there is an edge between the neighbours of v, in (b) there is not. In both cases, $\Gamma = \langle \gamma \rangle \simeq \mathbb{Z}_k$ through the isomorphism which maps γ to 1, and all unlabelled edges have identity gain.

Theorem 7.2.1. For $k \geq 4$, let $\Gamma = \langle \gamma \rangle \simeq \mathbb{Z}_k$ through the isomorphism defined by letting $\gamma \mapsto 1$. Let (G, ψ) be a Γ-gain graph with a free vertex v of degree 3 and no loop. Suppose that (G, ψ) is $\mathbb{Z}_k^j$-tight for some $2 \leq j \leq k-2$. If there is not an admissible 1-reduction at v, then k is even and j is odd, v has exactly two neighbours, only one of which is free, call it v_1. Moreover, the 2-cycle v, v_1, v has gain $\gamma^{k/2}$ (see Figure 7.1).

In a similar manner as we did for cyclic groups of order 2 and 3 (recall Section 6.2 in Chapter 6), we split the proof of Theorem 7.2.1 in the three separate cases where $|N_G(v)| = 1, 2$ and 3, in Subsections 7.2.1, 7.2.2 and 7.2.3, respectively.

7.2.1 v has exactly one neighbour

Proposition 7.2.2. For $k \geq 4$, let $\Gamma = \langle \gamma \rangle \simeq \mathbb{Z}_k$ through the isomorphism defined by $\gamma \mapsto 1$. For $2 \leq j \leq k-2$, let (G, ψ) be a $\mathbb{Z}_k^j$-gain tight Γ-gain graph with a vertex v of degree 3. Suppose that v has no loop, and exactly one neighbour u. Then, there is an admissible 1-reduction at v.

Proof. Let e_1, e_2, e_3 be the edges incident to u and v, with $g_i := \psi(e_i)$ for $1 \leq i \leq 3$. By Propositions 3.2.6 and 3.2.9, we may assume that $g_1 = \text{id}$. Moreover, $g_2, g_3, g_2 g_3^{-1} \neq \text{id}$ by the definition of gain graph. Let $(G_1, \psi_1), (G_2, \psi_2)$ and (G_3, ψ_3) be obtained from $G - v$ by adding the loops f_1, f_2, f_3 at u with gains $g_2, g_3, g_2 g_3^{-1}$, respectively. Assume, for a contradiction, that for each $1 \leq i \leq 3$, (G_i, ψ_i) has a blocker H_i, and for all such i let α_i denote $\alpha_k^j(H_i + f_i)$. Notice that, for each $1 \leq i \leq 3$, H_i is neither a balanced blocker (since $H_i + f_i$ contains a loop), nor a general-count blocker (by Proposition 6.1.2). Since $g_2, g_3, g_2 g_3^{-1} \neq \text{id}$, at most one of $g_2, g_3, g_2 g_3^{-1}$ is $\gamma^{k/2}$, and so at most one of H_1, H_2, H_3 is a $\mathbb{Z}_2$-blocker.

Notice that, for all $1 \leq s \neq t \leq 3$, $H_s \cup H_t + f_s + f_t$ contains a vertex with two different loops, and so it is not proper near-balanced. It follows, from Corollary 7.1.8 that $E(H_s \cap H_t) = \emptyset$ for all $1 \leq s \neq t \leq 3$. We now show that at most one of $H_1 + f_1, H_2 + f_2, H_3 + f_3$ is $S(k, j)$. To do so, fix some $1 \leq s \neq t \leq 3$ and assume, for a contradiction, that $H_s + f_s, H_t + f_t$ are both $S(k, j)$. Then, H_s is $(2, m_s, 1)$-tight and H_t is $(2, m_t, 1)$-tight, for some $0 \leq m_s, m_t \leq 1$. Since $u \in V(H_s \cap H_t)$ is free,

$$\begin{aligned}|E(H_s \cup H_t)| &= (2|V_1(H_s)| + m_s|V_k(H_s)| - 1) + (2|V_1(H_t)| + m_t|V_k(H_t)| - 1)\\ &= 2|V_1(H_s \cup H_t)| + 2|V_1(H_s \cap H_t)| - 2 + m_s|V_k(H_s)| + m_t|V_k(H_t)|\\ &\geq 2|V_1(H_s \cup H_t)|,\end{aligned}$$

contradicting Proposition 6.1.2 or the sparsity of (G, ψ). Therefore, we may assume that at most one of $H_1 + f_1, H_2 + f_2, H_3 + f_3$ is $S(k, j)$. This implies that, for some $1 \leq i \leq 3$, $H_i + f_i$ is proper near-balanced. (Since none of the H_i is a balanced blocker or a general-count blocker, at most one of the H_i is a $\mathbb{Z}_2$-blocker, and at most one of the $H_i + f_i$ is $S(k, j)$.) Hence, $\alpha_i = 2$. Without loss of generality, assume that $\alpha_3 = 2$. Let $H := H_1 \cup H_2 \cup H_3$, and $H' := H_1 \cap H_2 \cap H_3$. Since $u \in V(H_s \cap H_t)$ for all $1 \leq s \neq t \leq 3$, we have

$$\begin{aligned}|E(H)| &= \sum_{i=1}^{3} |E(H_i)| = 2\sum_{i=1}^{3} |V(H_i)| - 9 + \sum_{i=1}^{3} \alpha_i\\ &= 2|V(H)| + 2\sum_{1 \leq s \neq t \leq 3} [|V(H_s \cap H_t)| - |V(H')|] - 7 + \alpha_1 + \alpha_2\\ &\geq 2|V(H)| - 3 + \alpha_1 + \alpha_2.\end{aligned}$$

So, $\alpha_1 + \alpha_2 \leq 2$, by Proposition 6.1.2 and the sparsity of (G, ψ). If H has a fixed vertex, then we have $|E(H)| \geq 2|V(H)| - 3 + \alpha_1 + \alpha_2 = 2|V_1(H)| - 1 + \alpha_1 + \alpha_2$, and so $\alpha_1 + \alpha_2 = 0$, by Proposition 6.1.2 and the sparsity of (G, ψ). Since H_1, H_2 are not balanced blockers, the only case in which $\alpha_1 = \alpha_2 = 0$ is when $H_1 + f_1, H_2 + f_2$ are both $S_0(k, j)$ and $|V_k(H_1)| = |V_k(H_2)| = 1$. But this contradicts the fact that at most one of $H_1 + f_1, H_2 + f_2, H_3 + f_3$ is $S(k, j)$. So, we may assume that $V_k(H) = \emptyset$. This implies that, for $i = 1, 2$, $\alpha_i \geq 1$ with equality if and only if H_i is a $\mathbb{Z}_2$-blocker. So, the only way of having $\alpha_1 + \alpha_2 \leq 2$ is if H_1, H_2 are $\mathbb{Z}_2$-blockers. This contradicts the fact that at most one of $H_1, H_, H_3$ is a $\mathbb{Z}_2$-blocker. By contradiction, the result holds. □

7.2.2 v has exactly two neighbours

Proposition 7.2.3. For $k \geq 4$, let $\Gamma = \langle \gamma \rangle \simeq \mathbb{Z}_k$ through the isomorphism defined by letting $\gamma \mapsto 1$. For $2 \leq j \leq k-2$, let (G, ψ) be a $\mathbb{Z}_k^j$-gain tight Γ-gain graph with a free vertex v of degree 3. Suppose that v has no loop, and exactly two distinct neighbours v_1, v_2. Either there is an admissible 1-reduction at v or v has exactly one free neighbour, j is odd and the subgraph of G spanned by v, v_1, v_2 has gain $\gamma^{k/2}$.

Proof. Let $e_1, e_1' := (v, v_1)$ and $e_2 := (v, v_2)$, and let $g = \psi(e_1')$. By Propositions 3.2.6 and 3.2.9, we may assume that $\psi(e_1) = \psi(e_2) = \mathrm{id}$ and $g \neq \mathrm{id}$. We look at the cases where v_2 is free and fixed separately.

Case 1: *v_2 is free.*

Let $(G_1, \psi_1), (G_2, \psi_2), (G_3, \psi_3)$ be obtained from $G - v$ by adding, respectively, the edges $f_1 = (v_1, v_2)$ with gain id, the edge $f_2 = (v_2, v_1)$ with gain g, and a loop f_3 at v_1 with gain g. Assume, for a contradiction, that H_1, H_2 and H_3 are blockers for $(G_1, \psi_1), (G_2, \psi_2)$ and (G_3, ψ_3), respectively. Let $H = H_1 \cup H_2 \cup H_3$ and $H' = H_1 \cap H_2 \cap H_3$. By Proposition 6.1.2, H_1, H_2 are not general-count blockers. Moreover, H_3 is not a balanced blocker, since $H_3 + f_3$ contains a loop.

We start by showing that $E(H_s \cap H_t) = \emptyset$ for all $1 \leq s \neq t \leq 3$. So assume, for a contradiction, that $E(H_s \cap H_t) \neq \emptyset$ for some $1 \leq s \neq t \leq 3$. By Corollary 7.1.8, $H_s \cup H_t + f_s + f_t$ is proper near-balanced. Moreover, by Lemma 7.1.7, $H_s \cup H_t$ is $(2, 1)$-tight.

In particular, if $s = 1, t = 2$, then the base-vertices of near-balancedness must be v_1, v_2: otherwise, there is a gain ψ' equivalent to ψ such that $\psi'(f_1) = \psi'(f_2) = \mathrm{id}$, contradicting the definition of gain graph. This implies that every path W from v_1 to v_2 in $H_1 \cup H_2$ has gain id, or g^{-1}: W must have gain in $\{\mathrm{id}, g, g^{-1}\}$ because $f_1 \in E(H_1 \cup H_2 + f_1 + f_2)$, and it cannot have gain g, because $f_2 \in E(H_1 \cup H_2 + f_1 + f_2)$. Then $H_1 \cup H_2 + v$ is also near-balanced. Since H is $(2, 1)$-tight, this contradicts Proposition 6.1.2. Hence, $E(H_1 \cap H_2) = \emptyset$, and one of s, t is 3.

Assume, without loss of generality, that $E(H_1 \cap H_3) \neq \emptyset$, and recall that this implies that $H_1 \cup H_3 + f_1 + f_3$ is proper near-balanced, and that $H_1 \cup H_3$ is $(2,1)$-tight. By the sparsity of (G,ψ), $H_1 \cup H_3 + f_1$ is also proper near-balanced. It follows that $H_1' := H_1 \cup H_3$ is a blocker for (G_1,ψ_1). If $E(H_2 \cap H_3) \neq \emptyset$, then the same argument shows that $H_2' := H_2 \cup H_3$ is a blocker for (G_2,ψ_2). Since $E(H_1' \cap H_2') = E(H_3) \neq \emptyset$, $H_1' \cup H_2' + f_1 + f_2$ is proper-near balanced, by Corollary 7.1.8, and $H_1' \cup H_2'$ is $(2,1)$-tight by Lemma 7.1.7. Using a similar argument as in the previous paragraph, we can see that $H_1' \cup H_2' + v$ is proper near-balanced, contradicting Proposition 6.1.2. Hence, $E(H_2 \cap H_3) = \emptyset$. It follows that

$$\begin{aligned}|E(H)| &= |E((H_1 \cup H_3) \cup H_2)| \\ &= |E(H_1 \cup H_3)| + |E(H_2)| = (2|V(H_1 \cup H_3)| - 1) + (2|V(H_2)| - 3 + \alpha_2) \\ &= 2|V(H)| + 2|V((H_1 \cup H_3) \cap H_2)| - 4 + \alpha_2 \geq 2|V(H)| + \alpha_2 \\ &\geq 2|V(H)| \geq 2|V_1(H)|,\end{aligned}$$

since $v_1, v_2 \in V(H_1), V(H_2)$ and $\alpha_2 \geq 0$. This contradicts Proposition 6.1.2 or the sparsity of (G,ψ). Hence, $E(H_s \cap H_t) = \emptyset$ for all $1 \leq s \neq t \leq 3$. Since $E(H_1 \cap H_2) = \emptyset$,

$$\begin{aligned}|E(H_1 \cup H_2)| &= |E(H_1)| + |E(H_2)| = (2|V(H_1)| - 3 + \alpha_1) + (2|V(H_2)| - 3 + \alpha_2) \\ &= 2|V(H_1 \cup H_2)| + 2|V(H_1 \cap H_2)| - 6 + \alpha_1 + \alpha_2.\end{aligned}$$

If $|V(H_1 \cap H_2)| \geq 3$, or if $|V(H_1 \cap H_2)| = 2$ and $V_k(H_1 \cup H_2) \neq \emptyset$, this is at least $2|V_1(H_1 \cup H_2)|$, contradicting Proposition 6.1.2 or the sparsity of (G,ψ). Hence, $H_1 \cap H_2$ is composed of the two isolated vertices v_1, v_2, and $V_k(H_1) = V_k(H_2) = \emptyset$. Therefore, we have $|E(H_1 \cup H_2)| = 2|V(H_1 \cup H_2)| - 2 + \alpha_1 + \alpha_2$. Hence,

$$\begin{aligned}|E(H)| &= (2|V(H_1 \cup H_2)| - 2 + \alpha_1 + \alpha_2) + (2|V(H_3)| - 3 + \alpha_3) \\ &= 2|V(H)| + 2|V(H_1 \cup H_2) \cap H_3| - 5 + \sum_{i=1}^{3} \alpha_i. \end{aligned} \tag{7.7}$$

In particular, the intersection of $H_1 \cup H_2$ and H_3 must indeed be the isolated vertex v_3. To see this, assume, for a contradiction, that this is not the case. Then, we have $|E(H)| \geq 2|V(H)| - 1 + \sum_{i=1}^{3} \alpha_i$. If $V_k(H) \neq \emptyset$, this is at least

$2|V_1(H)|+1$, contradicting the sparsity of (G,ψ). If $V_k(H)=\emptyset$, then $\alpha_3 \geq 1$ (since H_3+f_3 is unbalanced), and so $|E(H)| \geq 2|V(H)| = 2|V_1(H)|$, which contradicts Proposition 6.1.2. So, $|V(H_1 \cup H_2) \cap H_3| = 1$ and

$$|E(H)| = 2|V(H)| - 3 + \sum_{i=1}^{3} \alpha_i. \tag{7.8}$$

Assume that $\alpha_1 = \alpha_2 = 0$, so that $|E(H)| = 2|V(H)| - 3 + \alpha_3$. Then, since all vertices of H_1, H_2 are free, H_1, H_2 are balanced blockers and, by Lemma 3.2.21, $H_1 \cup H_2 + f_1 + f_2$ is near-balanced with base vertex v_1 (and with base vertex v_2). Since $H_1 \cup H_2 + f_1 + f_2$ contains the 2-cycle f_1, f_2, it is near-balanced with gain g. So there is a gain ψ' equivalent to ψ such that $\psi'(e) \in \{\mathrm{id}, g, g^{-1}\}$ for all edges e in $E(H_1 \cup H_2)$ incident to v_1, and $\psi'(f) = \mathrm{id}$ for all other edges $f \in E(H_1 \cup H_2)$. In particular, $\langle H_1 \cup H_2 + f_1 + f_2 \rangle = \langle g \rangle$. Since $H_3 + f_3$ contains the loop f_3 with gain g, it follows that $\langle H_1 \cup H_2 + f_1 + f_2 \rangle \leq \langle H_3 + f_3 \rangle$, and so $\langle H + f_1 + f_2 + f_3 \rangle \simeq \langle H_3 + f_3 \rangle$. By Proposition 6.1.2 and Lemma 7.1.2, $H_3 + f_3$ must be proper near-balanced. Since it contains the loop f_3, it is near-balanced with base vertex v_1 and gain g. Recall that $H_1 \cup H_2 + f_1 + f_2$ is also near-balanced with base vertex v_1 and gain g, so $H + f_1 + f_2 + f_3$ and $H + v$ are proper near-balanced with base vertex v_1 and gain g. But then $|E(H)| = 2|V(H)| - 3 + \alpha_3 = 2|V(H)| - 3 + \alpha_k^j(H + f_1 + f_2 + f_3)$, which is a contradiction by Proposition 6.1.2.

Hence, $\alpha_1 + \alpha_2 \geq 1$. In particular, $V_k(H) = \emptyset$, for otherwise, by Equation (7.8), $|E(H)| \geq 2|V_1(H)|$, which contradicts Proposition 6.1.2 or the sparsity of (G,ψ). Since $H_3 + f_3$ is unbalanced, this implies that $\alpha_3 \geq 1$. Moreover, by Equation (7.8) and Proposition 6.1.2, $\sum_{i=1}^{3} \alpha_i \leq 2$. So, $(\alpha_1, \alpha_2, \alpha_3)$ is one of $(0,1,1)$ and $(1,0,1)$. Without loss of generality, assume that $\alpha_1 = 0$, $\alpha_2 = 1$ and $\alpha_3 = 1$. By the definition of α_2, α_3, H_2, H_3 are $\mathbb{Z}_2$-blockers. Hence, $g = \gamma^{k/2}$ and each path from v_1 to v_2 in H_2 has gain id or g. It follows that $\langle H_2 \cup H_3 + f_2 + f_3 \rangle \simeq \mathbb{Z}_2$. However,

$$\begin{aligned}|E(H_2 \cup H_3)| &= (2|V(H_2)| - 2) + (2|V(H_3)| - 2) \\ &= 2|V(H_2 \cup H_3)| + 2|V(H_2 \cap H_3)| - 4 = 2|V(H_2 \cup H_3)| - 2,\end{aligned}$$

contradicting Proposition 6.1.2 and Lemma 7.1.2. Hence, the result holds if v_2 is free.

Case 2: *v_2 is fixed.*
Let $(G_1, \psi_1), (G_2, \psi_2)$ be the graphs obtained from $G - v$ by adding, respectively, an edge $f_1 = (v_1, v_2)$, and a loop f_2 at v_1 with gain g. Notice that, if there is already an edge $(v_1, v_2) \in E(G)$, (G_1, ψ_1) is not a well-defined gain graph. Assume that $g \neq \gamma^{k/2}$ or j is even. We show that one of $(G_1, \psi_1), (G_2, \psi_2)$ is $\mathbb{Z}_k^j$-gain tight. So assume, for a contradiction, that H_2 is a blocker for (G_2, ψ_2) and, whenever $(v_1, v_2) \notin E(G)$, assume that H_1 is a blocker for (G_1, ψ_1). Since $H_2 + f_2$ contains the loop f_2, H_2 is not a balanced blocker. Moreover, since $g \neq \gamma^{k/2}$ or j is even, H_2 is not a $\mathbb{Z}_2$-blocker. So, if we show that $|V_k(H_2)| = 0$, then $\alpha_k^j(H_2 + f_2) \geq 2$ by definition.

Assume, for a contradiction, that $v_2 \in V(H_2)$. In particular, $H_2 + f_2$ is not near-balanced, since $V_k(H_2) \neq \emptyset$. Moreover, $\langle H_2 + v \rangle \simeq \langle H_2 + f_2 \rangle$, since v_2 is fixed. Since $|V_k(H_2 + v)| = |V_k(H_2 + f_2)|$, it follows that $\alpha_k^j(H_2 + v) = \alpha_k^j(H_2 + f_2)$. But this contradicts Proposition 6.1.2. Hence, $v_2 \notin V(H_2)$, and so $|V_k(H_2)| = \emptyset$. So, $\alpha_k^j(H_2 + f_2) \geq 2$ and $|E(H_2)| \geq 2|V(H_2)| - 1$. If $(v_1, v_2) \in E(G)$, then

$$|E(H_2 + v_2)| = |E(H_2)| + 1 \geq 2|V(H_2)| = 2|V_1(H_2 + v_2)|,$$

which contradicts Proposition 6.1.2 or the sparsity of (G, ψ). Hence, $(v_1, v_2) \notin E(G)$, and $(G_1, \psi_1), H_1$ are well-defined. Let $H = H_1 \cup H_2$ and $H' = H_1 \cap H_2$. Notice that $H + f_1 + f_2$ is neither balanced nor near-balanced, since it contains the loop f_2 and the fixed vertex v_0. Hence, by Corollary 7.1.8, $E(H') = \emptyset$. Then,

$$\begin{aligned}
|E(H)| &= (2|V(H_1)| - 3 + \alpha_k^j(H_1 + f_1)) + (2|V(H_2)| - 3 + \alpha_k^j(H_2 + f_2)) \\
&= 2|V(H)| + 2|V(H')| - 6 + \alpha_k^j(H_1 + f_1) + \alpha_k^j(H_2 + f_2) \\
&\geq 2|V(H)| - 4 + \alpha_k^j(H_1 + f_1) + \alpha_k^j(H_2 + f_2) \\
&\geq 2|V(H)| - 2 = 2|V_1(H)|.
\end{aligned}$$

This contradicts Proposition 6.1.2. Hence, there is an admissible 1-reduction at v, and the result holds. □

7.2.3 v has exactly three neighbours

Proposition 7.2.4. Let Γ be a cyclic group of order $k \geq 4$. For $2 \leq j \leq k-2$, let (G, ψ) be a $\mathbb{Z}_k^j$-gain tight Γ-gain graph with a free vertex v of degree 3. Suppose that v has no loop, and exactly three distinct neighbours v_1, v_2, v_3. Then there is an admissible 1-reduction at v.

Proof. For $i = 1, 2, 3$, let $e_i = (v, v_i)$ be the edges incident with v. Let f_1, f_2 and f_3 denote the edges $(v_1, v_2), (v_2, v_3)$ and (v_3, v_1), respectively. By Propositions 3.2.6 and 3.2.9, we may assume $\psi(e_1) = \psi(e_2) = \psi(e_3) = \text{id}$. For $1 \leq i \leq 3$, let (G_i, ψ_i) be obtained by applying a 1-reduction at v, during which we add the edge f_i with gain id, and assume that (G_i, ψ_i) has a blocker H_i. Let $H := H_1 \cup H_2 \cup H_3$ and $H' := H_1 \cap H_2 \cap H_3$. We will consider the following cases separately: $E(H_s \cap H_t) = \emptyset$ for at most two pairs of s, t; and $E(H_s \cap H_t) = \emptyset$ for all pairs s, t. In both cases, we show that there is a contradiction.

Case 1: *$E(H_s \cap H_t) = \emptyset$ for at most two pairs s, t.*
Without loss of generality, we may assume $E(H_1 \cap H_2) \neq \emptyset$. By Corollary 7.1.8, either $H_1 \cup H_2 + f_1 + f_2$ is proper near-balanced or v_2 is fixed. If $H_1 \cup H_2 + f_1 + f_2$ is near-balanced, say with base vertex u, then so is $H_1 \cup H_2 + v$, since every walk which contains u, from v_1 to v_2, from v_2 to v_3, and from v_3 to v_1 must have gain id, g or g^{-1}, for some $g \in \Gamma$. However, by Lemma 7.1.7, $H_1 \cup H_2$ is $(2,1)$-tight, which contradicts Proposition 6.1.2.

Hence, we may assume that v_2 is fixed, and so v_1, v_3 are free. By the same argument as in the previous paragraph, it is easy to see that $E(H_1 \cap H_3)$ and $E(H_2 \cap H_3)$ are both empty. Therefore, by Lemma 7.1.7, and by the fact that $v_1, v_3 \in V((H_1 \cup H_2) \cap H_3)$, we have

$$\begin{aligned}
|E(H)| &= |E((H_1 \cup H_2) \cup H_3)| = |E(H_1 \cup H_2)| + |E(H_3)| \\
&= (2|V(H_1 \cup H_2)| - 3 + \alpha_{12}) + (2|V(H_3)| - 3 + \alpha_k^j(H_3 + f_3)) \\
&= 2|V(H)| + 2|V((H_1 \cup H_2) \cap H_3)| - 6 + \alpha_{12} + \alpha_k^j(H_3 + f_3) \\
&\geq 2|V(H)| - 2 + \alpha_{12} + \alpha_k^j(H_3 + f_3) \geq 2|V_1(H)|,
\end{aligned}$$

where α_{12} denotes $\alpha_k^j(H_1 \cup H_2 + f_1 + f_2)$. This contradicts Proposition 6.1.2 or the sparsity of (G, ψ).

Case 2: *$E(H_s \cap H_t) = \emptyset$ for all pairs s, t.*
For simplicity, let $\alpha_i := \alpha_k^j(H_i + f_i)$ for $1 \leq i \leq 3$. We have

$$\begin{aligned} |E(H)| &= \sum_{i=1}^{3} |E(H_i)| = 2\sum_{i=1}^{3} |V(H_i)| - 9 + \sum_{i=1}^{3} \alpha_i \\ &= 2[|V(H)| + \sum_{1 \leq s \neq t \leq 3} |V(H_s \cap H_t)| - |V(H')|] - 9 + \sum_{i=1}^{3} \alpha_i \\ &\geq 2|V(H)| - 3 + \sum_{i=1}^{3} \alpha_i. \end{aligned} \tag{7.9}$$

By the sparsity of (G, ψ) and Proposition 6.1.2, $0 \leq \sum_{i=1}^{3} \alpha_i \leq 2$. Moreover, $|V(H_s \cap H_t)| \geq 2$ for at most one pair $1 \leq s \neq t \leq 3$. Otherwise, it is easy to see that $\sum_{1 \leq s \neq t \leq 3} |V(H_s \cap H_t)| - |V(H')| \geq 5$, and so $|E(H)| \geq 2|V(H)| + 1$, contradicting the sparsity of (G, ψ).

First, let $\sum_{i=1}^{3} \alpha_i = 0$ and $|E(H)| \geq 2|V(H)| - 3$. Then, for each $1 \leq i \leq 3$, $H_i + f_i$ is either balanced or it is $S_0(k, j)$ with $|V_k(H_i)| = 1$. First, assume that each H_i is a balanced blocker. If $|V(H_s \cap H_t)| = 1$ for all pairs $1 \leq s \neq t \leq 3$, then $H + f_1 + f_2 + f_3$ is balanced: each path in H_1 (respectively H_2 and H_3) between v_1 and v_2 (respectively v_2 and v_3, and v_1 and v_3) has gain id. So, $H + v$ is also balanced. Since $|E(H)| \geq 2|V(H)| - 3$, this contradicts Proposition 6.1.2 or the sparsity of (G, ψ). So, without loss of generality, assume that $|V(H_1 \cap H_2)| = 2$, and $|V(H_1 \cap H_3)| = |V(H_2 \cap H_3)| = 1$, so that $|E(H)| \geq 2|V(H)| - 1$. If $V_k(H) \neq \emptyset$, then $|E(H)| \geq 2|V_1(H)| + 1$, contradicting the sparsity of (G, ψ). So $V_k(H) = \emptyset$. By Lemma 3.2.21, $H_1 \cup H_2 + f_1 + f_2$ is near-balanced with base vertex v_2. Since each path in H_3 from v_1 to v_3 has gain id, it follows that $H + f_1 + f_2 + f_3$ is near-balanced with base vertex v_2. So $H + v$ is also near-balanced with base vertex v_2. Since $|E(H)| \geq 2|V(H)| - 1$, this contradicts Proposition 6.1.2 or the sparsity of (G, ψ).

Now, assume that $H_i + f_i$ is $S_0(k, j)$ with $|V_k(H_i)| = 1$ for some $1 \leq i \leq 3$. Without loss of generality, let $H_1 + f_1$ be $S_0(k, j)$. If $|V(H_s \cap H_t)| \geq 2$ for some pair $1 \leq s \neq t \leq 3$, then $|E(H)| \geq 2|V(H)| - 1 = 2|V_1(H)| + 1$, contradicting the sparsity of (G, ψ). So $|V(H_s \cap H_t)| = 1$ for all pairs $1 \leq s \neq t \leq 3$. In particular, $H_1 + f_1, H_2 + f_2, H_3 + f_3$ cannot all be $S_0(k, j)$: otherwise, they all share a fixed vertex and, since $v_1, v_2, v_3 \notin V(H')$, $|V(H_s \cap H_t)| \geq 2$ for all $1 \leq s \neq t \leq 3$. So, without loss of generality, consider the following cases separately: $H_1 + f_1, H_2 + f_2$ are $S_0(k, j)$ and H_3 is a balanced blocker; $H_1 + f_1$ is $S_0(k, j)$, and H_2, H_3 are balanced blockers.

First, assume that $H_1 + f_1, H_2 + f_2$ are $S_0(k, j)$ and H_3 is a balanced blocker. Let $n_1, n_2 \in S_0(k, j)$ be such that $\langle H_1 + f_1 \rangle \simeq \mathbb{Z}_{n_1}, \langle H_2 + f_2 \rangle \simeq \mathbb{Z}_{n_2}$. Since $|V(H_1 \cap H_2)| = 1$ and H_1, H_2 share the fixed vertex, v_2 is the fixed vertex. So, $\langle H + f_1 + f_2 + f_3 \rangle$ is the group generated by $\psi(W)$, for all closed walks W in $H + f_1 + f_2 + f_3$ not containing v_2, which in turn is the group

$$\langle \psi(W) : W \text{ is a closed walk in } H_1 \text{ or } H_2 \text{ not containing } v_2, \text{ or in } H_3 + f_3 \rangle \simeq \mathbb{Z}_l,$$

where $l = \operatorname{lcm}(n_1, n_2) \in S_0(k, j)$. So $H + f_1 + f_2 + f_3$ is $S_0(k, j)$, which Proposition 6.1.2 or the sparsity of (G, ψ), since $|E(H)| \geq 2|V_1(H)| - 1$.

Now, let $\langle H_1 + f_1 \rangle \simeq \mathbb{Z}_n$ for some $n \in S_0(k, j)$, and H_2, H_3 be balanced blockers. Then the gain of $H + f_1 + f_2 + f_3$ is composed of the gain of every closed walk in H_i not containing the fixed vertex, for $1 \leq i \leq 3$, and the gain of every walk obtained by concatenating a walk from v_1 to v_2 (in H_1), a walk from v_2 to v_3 (in H_2), and a walk from v_3 to v_1 (in H_3). Since every walk from v_1 to v_2 has gain in $\mathbb{Z}_n$ (since f_1 has identity gain), and every closed walk in H_1 has gain in $\mathbb{Z}_n$ (since $H_1 \subset H_1 + f_1$), and every closed walk in H_2, H_3, as well as every walk from v_2 to v_3 and from v_3 to v_1 has gain id, $\langle H + f_1 + f_2 + f_3 \rangle \simeq \mathbb{Z}_n$. By Lemma 7.1.2, $H + v$ is $S_0(k, j)$. Since $|E(H)| \geq 2|V_1(H)| - 1$, this is a contradiction, by the sparsity of (G, ψ) and Proposition 6.1.2.

So, let the triple $(\alpha_1, \alpha_2, \alpha_3)$ be one of $(1, 0, 0), (2, 0, 0), (1, 1, 0)$. In particular, since $\sum_{i=1}^{3} \alpha_i \geq 1$, $|V(H_s \cap H_t)| = 1$ for all $1 \leq s \neq t \leq 3$. Otherwise, $\sum_{1 \leq s \neq t \leq 3} |V(H_s \cap H_t)| - |V(H')| \geq 4$, and so, by Equation (7.9), $|E(H)| \geq 2|V(H)|$, contradicting Proposition 6.1.2 or the sparsity of (G, ψ). Moreover, if $|V_k(H)| = 1$, then $|E(H)| \geq |V_1(H)|$ by Equation (7.9). This contradicts Proposition 6.1.2 or the sparsity of (G, ψ), so $|V_k(H)| = 0$.

If the $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 0)$, then H_1, H_2 are $\mathbb{Z}_2$-blockers, and H_3 is a balanced blocker. Since $|V(H_s \cap H_t)| = 1$ for all $1 \leq s \neq t \leq 3$, the gain of $H + f_1 + f_3 + f_2$ is given by the gain of each closed walk in $H_1 + f_1$, $H_2 + f_2$ and $H_3 + f_3$, and the gain of every walk obtained by concatenating a walk from v_1 to v_2 (in H_1), a walk from v_2 to v_3 (in H_2), and a walk from v_3 to v_1 (in H_3). So, $\langle H + v\rangle = \langle H + f_1 + f_2 + f_3\rangle \simeq \mathbb{Z}_2$. Since $|E(H)| \geq 2|V(H)| - 2$ and j is odd, this contradicts Proposition 6.1.2 or the sparsity of (G, ψ).

So assume that $(\alpha_2, \alpha_3) = (0, 0)$. Then $H_2 \cup H_3 + f_2 + f_3$ is balanced, since $H_2 \cap H_3$ is the isolated vertex v_3. Hence, $\langle H + v\rangle = \langle H + f_1 + f_2 + f_3\rangle = \langle H_1 + f_1\rangle$. Moreover, it is easy to see that $H + f_1 + f_2 + f_3$ (and hence also $H + v$) is near-balanced whenever $H_1 + f_1$ is near-balanced. Since $|V_k(G)| = 0$, this implies that $\alpha_j^k(H+v) = \alpha_1$. Since $|E(H)| = 2|V(H)| - 3 + \alpha_1$, this contradicts Proposition 6.1.2 or the sparsity of (G, ψ). □

7.3 Main results

For some integer $k \geq 4$, let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_k$-generic framework with $\mathcal{C}_k$-gain framework (G, ψ, p). Recall that (G, ψ) is $(2, 0, 3, 1)$-gain tight whenever $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic, that (G, ψ) is $(2, 1, 3, 1)'$-gain tight whenever $(\tilde{G}, \tilde{p})$ is ρ_1-isostatic or ρ_{k-1}-isostatic, and that, for $2 \leq j \leq k - 2$, (G, ψ) is $\mathbb{Z}_k^j$-gain tight whenever $(\tilde{G}, \tilde{p})$ is ρ_j-isostatic (see Proposition 5.1.8 in Subsection 5.1.3). In this section, we show that the converse statements are also true whenever $5 \leq k \leq 1000$ is odd or $k = 4, 6$.

This was already done in Chapter 6 for the fully-symmetric case and, with the restriction that k is odd, for the ρ_1-symmetric and ρ_{k-1}-symmetric cases (recall Theorem 6.5.1). In this section, we conclude the study by considering the ρ_1-symmetric and ρ_{k-1}-symmetric cases for $k = 4, 6$ and, the ρ_j-symmetric case for all odd $5 \leq k \leq 1000$ and for $k = 4, 6$, where $2 \leq j \leq k-2$. The case where $V_k(G) = \emptyset$ was already shown in [27] for odd $k \leq 1000$, and in [8] for $k = 4, 6$. Here, we unite the results and state them as Theorem 7.3.1.

Theorem 7.3.1 ([8], [27])**.** For odd $5 \leq k \leq 1000$ or for $k = 4, 6$, let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_k$-generic framework with $\mathcal{C}_k$-gain framework (G, ψ, p). Suppose that $V(G) = V_1(G)$. $(\tilde{G}, \tilde{p})$ is ρ_1-isostatic and ρ_{k-1}-isostatic if and only if (G, ψ) is $(2,1,3,1)'$-gain tight. For $2 \leq j \leq k-2$, $(\tilde{G}, \tilde{p})$ is ρ_j-isostatic if and only if (G, ψ) is $\mathbb{Z}_k^j$-gain tight.

The proofs of Theorem 7.3.1 apply inductive arguments. The base case for the ρ_1-symmetric case and the ρ_{k-1}-symmetric case is a single vertex with a loop, similarly as we had for the 3-fold rotation symmetry case (recall Figure 6.5 in Subsection 6.5). However, in this case the loop is not allowed to have gain g such that $\langle g \rangle \simeq \mathbb{Z}_2$. For the ρ_j-symmetric case, where $2 \leq j \leq k-2$, the base cases are a combination of disjoint unions of certain base graphs, which may be grouped into three classes. The first class is composed of the graphs in Figure 7.2. The second class consists of all $\mathbb{Z}_k^j$-gain tight 4-regular graphs which may be obtained from an $S(k,j)$ $\mathbb{Z}_k$-gain graph by adding an edge. The third class consists of all $\mathbb{Z}_k^j$-gain tight 4-regular graphs (with j odd) which can be obtained from a $\mathbb{Z}_k$-gain graph G with $\langle G \rangle \simeq \mathbb{Z}_2$ by adding two edges.

Similarly, we use inductive arguments which employ reduction moves on the gain graph (G, ψ). When a fixed vertex is present, we will see that we obtain exactly one additional base graph for the ρ_1-symmetric and ρ_{k-1}-symmetric case, and exactly one additional connected component of a base graph for the ρ_j-symmetric case where $2 \leq j \leq k-2$: in both cases, this is the isolated fixed vertex. Since we employ a proof by induction, we will need the following result.

Lemma 7.3.2. For $k = 4, 6$, let $\Gamma = \langle \gamma \rangle \simeq \mathbb{Z}_k$ through the isomorphism defined by letting $\gamma \mapsto 1$. For $2 \leq j \leq k-2$, let (G, ψ) be a $\mathbb{Z}_k^j$-gain tight (respectively, $(2,1,3,1)'$-gain tight) Γ-gain graph with $V_k(G) = \{v_0\}$ and $|V(G)| \geq 2$. Suppose that $\deg(v_0) \geq 1$. Then there is a reduction of (G, ψ) which yields a $\mathbb{Z}_k^j$-gain

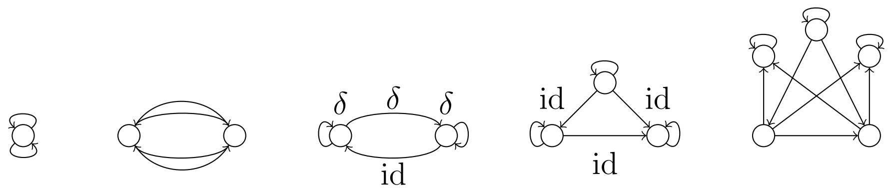


Figure 7.2: Base graphs for k-fold rotation for $2 \leq j \leq k-2$, where $V(G) = V_1(G)$. All (unlabelled) edges of such graphs may be labelled freely, with the restrictions that loops must not have non-identity gains, the non-looped edges of the last graph are labelled id, and each graph must be $\mathbb{Z}_k^j$-gain tight.

tight (respectively, $(2,1,3,1')$-gain tight) graph (G', ψ'). The reduction which yields (G', ψ') is one of the following: a 0-reduction, a loop-1-reduction, a 1-reduction or a 2-vertex reduction.

Proof. By Lemma 6.1.1, there is a free vertex in $V(G)$ of degree 2 or 3. We may assume that G has no free vertex of degree 2 and no free vertex of degree 3 with a loop. Otherwise, we may apply a 0-reduction or loop-1-reduction to (G, ψ). Further, we may assume that j is odd if (G, ψ) is $\mathbb{Z}_k^j$-gain tight, and that for all free vertices v of degree 3, v has exactly 2 distinct neighbours, one of which is v_0, and the 2-cycle v forms with its free neighbour has gain $\gamma^{k/2}$ (see Figure 7.1). Otherwise, we may apply a 1-reduction to (G, ψ), by Theorems 6.2.1 and 7.2.1. If $\mathbb{Z}_k^j$-gain tight, this forces k and j to be 6 and 3, respectively.

Let $v_1, \dots, v_t$ be the free vertices of degree 3 in G. For $1 \leq i \leq t$ let u_i be the free neighbour of v_i, and $e_i := (u_i, v_0)$. By Lemma 6.1.1, $\deg(v_0) \leq t$. So, if the edge e_i is present for some $1 \leq i \leq t$, then u_i must be a vertex of degree 3. Hence, we can apply a 2-vertex reduction at u_i, v_i. So, we may assume that $e_i \notin E(G)$ for all $1 \leq i \leq t$.

For $1 \leq i \leq t$, let (G_i, ψ_i) be obtained from (G, ψ) by removing v_i and adding e_i with gain id. We will show that, for some $1 \leq i \leq t$, (G_i, ψ_i) is an admissible 1-reduction. Assume, for a contradiction, that for all $1 \leq i \leq t$ there is a blocker H_i for (G_i, ψ_i). Suppose that (G, ψ) is $\mathbb{Z}_k^j$-gain tight. If there is some $1 \leq i \leq t$ such that $\alpha_k^j(H_i + e_i) \geq 1$, then

$$|E(H_i)| = 2|V(H_i)| - 3 + \alpha_k^j(H_i + e_i) = 2|V_1(H_i)| - 1 + \alpha_k^j(H_i + e_i) \geq 2|V_1(H_i)|,$$

since H_i contains the fixed vertex v_0. This contradicts Proposition 6.1.2 or the sparsity of (G, ψ). So for all $1 \leq i \leq t$, $\alpha_k^j(H_i + e_i) = 0$ and H_i is $(2,3)$-tight. Similarly, if (G, ψ) is $(2,1,3,1)'$-gain tight, then each H_i is $(2,3)$-tight: H_i cannot be a general count blocker by Proposition 6.1.2, and it cannot be a $\mathbb{Z}_2$-blocker, as otherwise $|E(H_i)| = 2|V_1(H_i)| + |V_k(H_i)| - 1$, contradicting Proposition 6.1.2. It follows that, for all $1 \leq i \leq t$, $H_i + e_i$ is either balanced or, just for the case where (G, ψ) is $\mathbb{Z}_k^j$-gain tight, $S_0(6,3)$. Since $S_0(6,3) = \{3\}$, it follows that $\langle H_i + e_i \rangle$ is either $\{\text{id}\}$ or, just for the case where (G, ψ) is $\mathbb{Z}_k^j$-gain tight, $\{\text{id}, \gamma^2, \gamma^4\}$.

Moreover, for each $1 \leq i \neq s \leq t$, $v_s \notin V(H_i)$. To see this, suppose, for a contradiction, that $v_s \in V(H_i)$. Since $\langle H_i + e_i \rangle$ is either $\{\text{id}\}$ or $\{\text{id}, \gamma^2, \gamma^4\}$, it cannot contain the 2-cycle $(v_s, u_s)(u_s, v_s)$ of gain γ^3. Hence, there is an edge e incident to v_s, u_s such that $e \notin E(H_i)$. By Lemma 6.1.1(i), since H_i i $(2,3)$-tight, all of its vertices have degree 2 in H_i. In particular, u_s has degree 2 in H_i, so two edges incident to u_s lie in H_i. Then,

$$|E(H_i + e)| = 2|V(H_i)| - 2 = 2|V_1(H_i + e)| = 2|V_1(H_i + e)| + |V_k(H_i + e)| - 1,$$

since $v_0 \in V(H_i)$. This contradicts Proposition 6.1.2. Hence, for all $1 \leq i \neq s \leq t$, $v_s \notin V(H_i)$.

Claim: *$E(H_i \cap H_s) = \emptyset$ and $V(H_i \cap H_s) = \{v_0\}$ for all $1 \leq i \neq s \leq t$.*

Proof. Choose some $1 \leq i \neq s \leq t$. Assume for a contradiction that $E(H_i \cap H_s) \neq \emptyset$. By the proof of Lemmas 7.1.5 and 7.1.6, we can see that

$$\begin{aligned} |E(H_i \cup H_s)| &= 2|V(H_i \cup H_s)| - 3 \\ &= 2|V_1(H_i \cup H_s)| - 1 = 2|V_1(H_i \cup H_s)| + |V_k(H_i \cup H_s)| - 2. \end{aligned}$$

But then,

$$\begin{aligned}|E(H_i \cup H_s + v_i + v_s)| &= |E(H_i \cup H_s)| + 6 = 2|V_1(H_i \cup H_s)| + 5\\ &= 2|V_1(H_i \cup H_s + v_i + v_s)| + 1\\ &= 2|V_1(H_i \cup H_s + v_i + v_s)| + |V_k(H_i \cup H_s + v_i + v_s)|,\end{aligned}$$

contradicting the sparsity of (G, ψ). So $E(H_i \cap H_s) = \emptyset$ for all $1 \leq i \neq s \leq t$. Now, if $V(H_i \cap H_s) \neq \{v_0\}$, then $H_i \cap H_s$ contains a free vertex, and so

$$\begin{aligned}|E(H_i \cup H_s)| &= |E(H_i)| + |E(H_s)| = 2|V(H_i \cup H_s)| + 2|V(H_i \cap H_s)| - 6\\ &\geq 2|V(H_i \cup H_s)| - 2 = 2|V_1(H_i \cup H_s)|\\ &= 2|V_1(H_i \cup H_s)| + |V_k(H_i \cup H_s)| - 1,\end{aligned}$$

since $H_i \cup H_s$ contains v_0. This contradicts Proposition 6.1.2 or the sparsity of (G, ψ), so $V(H_i \cap H_s) = \{v_0\}$. Since i, s were arbitrary, the claim holds. $\square$

Let $H := \bigcup_{i=1}^{t} H_i$. By the Claim,

$$|E(H)| = \sum_{i=1}^{t} |E(H_i)| = 2\sum_{i=1}^{t} |V(H_i)| - 3t = 2(|V(H)| + (t-1)) - 3t = 2|V(H)| - t - 2.$$

So, $H' := H + v_1 + \cdots + v_t$ satisfies $|E(H')| = 2|V(H_1')| - 2$. Since $v_0 \in V(H')$, it also satisfies $|E(H')| = 2|V_1(H_1')| = 2|V_1(H_1')| + |V_1(H_1')| - 1$. This implies that there is no edge $e \in E(G) \setminus E(H')$ that joins two vertices in $V(H')$ (both when (G, ψ) is $\mathbb{Z}_k^j$-gain tight and when it is $(2,1,3,1)'$-gain tight).

Next, we show that H' is a connected component of G. Clearly, H' is connected. Suppose G has a non-empty subgraph G' such that $V(G)$ is the disjoint union of $V(H')$ and $V(G')$. Let $d(H', G')$ be the number of edges joining a vertex in H' with one in G'. We aim to show that $d(H', G') = 0$. Let $\alpha \geq 0$ be such that $|E(G')| = 2|V(G')| - \alpha = 2|V_1(G')| - \alpha = 2|V_1(G')| + |V_k(G')| - \alpha$. (Note that $\alpha \geq 1$ if (G, ψ) is $(2,1,3,1)'$-gain tight.) Then,

$$\begin{aligned}|E(G)| &= |E(H')| + |E(G')| + d(H', G') = 2|V_1(H')| + 2|V_1(G')| - \alpha + d(H', G')\\ &= 2|V_1(G)| - \alpha + d(H', G') = |E(G)| - \alpha + d(H', G'),\end{aligned}$$

since $|E(G)| = 2|V_1(G)| = 2|V_1(G)| + |V_k(G)| - 1$. So $\alpha = d(H', G')$. Since every vertex in G' has degree at least 4 in G,

$$4|V(G')| \leq \sum_{v \in V(G')} \deg_G(v) = 2|E(G')| + d(H', G')$$
$$= 4|V(G')| - 2\alpha + \alpha = 4|V(G')| - \alpha,$$

and so $d(H', G') = \alpha = 0$, as required. If (G, ψ) is $(2, 1, 3, 1)'$-tight, we already have a contradiction. Hence, we may assume that (G, ψ) is $\mathbb{Z}_k^j$-gain tight.

Consider H_1 and let n, m be the vertices of degree 2 and 3 in H_1, respectively. Let $\hat{\rho}, \rho_{\min}$ be the average degree and minimum attainable degree of H_1, respectively. Since H_1 is $(2, 3)$-tight, $|V(H_1)|\hat{\rho} = 4|V(H_1)| - 6$. Moreover, $\rho_{\min}$ is attained when all vertices of H_1 have degree 2,3 or 4, and hence we have $|V(H_1)|\rho_{\min} = 4|V(H_1)| - 2n - m$. Since $\rho_{\min} \leq \hat{\rho}$, $2n + m \geq 6$. Hence, there are at least three vertices of degree 2 or 3 in H_1. If two of the vertices are v_0, v_1, there is still a free vertex w in H_1 of degree 2 or 3. Since H_1 is a connected component of G, it follows that w has degree 2 or 3 in G. But this contradicts our assumption that the only free vertices of degree 2 or 3 in G are $v_1, \ldots, v_t$. Hence, our result holds by contradiction. □

We now prove the main result of this section.

Theorem 7.3.3. For odd $5 \leq k \leq 1000$ or for $k = 4, 6$, let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_k$-generic framework with $\mathcal{C}_k$-gain framework (G, ψ, p). The following hold:

(1) If (G, ψ) is $(2, 0, 3, 1)$-gain tight, then $(\tilde{G}, \tilde{p})$ is fully-symmetrically isostatic.

(2) If (G, ψ) is $(2, 1, 3, 1)'$-gain tight, then $(\tilde{G}, \tilde{p})$ is ρ_1-isostatic and ρ_{k-1}-isostatic.

(3) If (G, ψ) is $\mathbb{Z}_k^j$-gain tight for some $2 \leq j \leq k - 2$, then $(\tilde{G}, \tilde{p})$ is ρ_j-isostatic.

Proof. Since (1) is an example of Theorem 6.5.1(1), we need only prove (2) and (3). We use induction on $|V(G)|$. The base cases for (2) are the same that were given in Figure 6.5. It is easy to check, in a similar way as we did in the proof of Theorem 6.5.2, that in both cases $O_1(G, \psi, p)$ and $O_{k-1}(G, \psi, p)$ have full rank and nullity 1. For the base cases of (3), let $V(G) = V_k(G) = \{v_0\}$. Then (G, ψ) is an isolated fixed vertex, and so it is easy to see that $(\tilde{G}, \tilde{p})$ is ρ_j-symmetrically isostatic for $1 \leq j \leq k-1$. The Γ-liftings of the graphs in Figure 7.2 were shown to have ρ_j-isostatic realisations for $2 \leq j \leq k-2$ in [27]. The base cases of our induction argument are exactly the disjoint combinations of the base graphs given in [27], and of the isolated fixed vertex.

We may assume that $V_1(G) \neq \emptyset$ (since otherwise we obtain a base graph). Assume further that the statement is true for all graphs on at most t vertices, for some integer $t \geq 1$, that $|V(G)| = t+1$, and that G is not a base graph.

For (2), suppose that (G, ψ) is $(2,1,3,1)'$-gain tight. By Lemma 7.3.2, (G, ψ) admits a reduction using one of the moves listed in the statement of the lemma. Let (G', ψ') be a $(2,1,3,1)'$-gain tight graph obtained by applying such a reduction to (G, ψ). By induction, all $\mathcal{C}_k$-generic realisations of $\tilde{G}'$ are ρ_1-isostatic and ρ_{k-1}-isostatic. Let $\tilde{q}'$ be a $\mathcal{C}_k$-generic configuration of $\tilde{G}'$ which also satisfies the conditions of Lemma 5.3.11 (respectively, Lemma 5.3.8) if $\tilde{G}'$ is obtained from $\tilde{G}$ by applying a 1-reduction (respectively, a loop-1-reduction). By Lemmas 5.3.4, 5.3.8, 5.3.11 and 5.3.13, there is a realisation $(\tilde{G}, \tilde{p})$ of $\tilde{G}$ which is ρ_1-isostatic (or ρ_{k-1}-isostatic). (2) follows from the fact that $\tilde{p}$ is $\mathcal{C}_k$-generic.

Now we prove (3). If $V_k(G) = \emptyset$, or if $V(G)$ has an isolated fixed vertex, then the graph (G', ψ') obtained from (G, ψ) by removing its fixed vertex (if it has one), is $\mathbb{Z}_k^j$-gain tight. By Theorem 7.3.1, $(\tilde{G}', \tilde{p}|_{V(G')})$ is ρ_j-symmetrically isostatic. Since $O_j(G, \psi, p) = O_j(G', \psi', p|_{V(G')})$, $(\tilde{G}, \tilde{p})$ is also ρ_j-symmetrically isostatic. So, we may assume that G has a connected component H which contains a fixed vertex, and which is not a base graph. Hence, the fixed vertex has degree at least 1.

If $|V_1(G)| = 1$, then $V(G) = \{v_0, v\}$, where v_0 is a fixed vertex and v is free, and $E(G)$ is composed of a loop e at v, and an edge between v and v_0. Since (G, ψ) is $\mathbb{Z}_k^j$-gain tight, if $k = 6$ and $j = 3$, then e does not have gain $\gamma^{k/2}$. Moreover, G is not $S_0(k, j)$. We may apply a loop-1-reduction at v to obtain a $\mathbb{Z}_k^j$-gain tight graph (G', ψ') on t vertices. By the inductive hypothesis, every $\mathcal{C}_k$-generic realisation of $\tilde{G}'$ is ρ_j-symmetrically isostatic. Let $(\tilde{G}', \tilde{q}')$ be a $\mathcal{C}_k$-generic realisation of $\tilde{G}'$. By Lemma 5.3.8, there is a $\mathcal{C}_k$-symmetric realisation $(\tilde{G}, \tilde{q})$ of $\tilde{G}$ which is ρ_j-symmetrically isostatic. Then, since $(\tilde{G}, \tilde{p})$ is $\mathcal{C}_k$-generic, it is also ρ_j-symmetrically isostatic.

So, we may assume that $|V_1(G)| \geq 2$. If $k = 4, 6$, by Lemma 7.3.2, there is a $\mathbb{Z}_k^j$-gain tight graph (G', ψ') on at most t vertices obtained from (G, ψ) by applying a reduction given in the statement of the lemma (exactly t if we apply a 0-reduction, loop-1-reduction or 1-reduction, and exactly $t-1$ if we apply a 2-vertex reduction). By induction, every $\mathcal{C}_k$-generic realisation of $\tilde{G}'$ is ρ_j-symmetrically isostatic. Moreover, if we apply a loop-1-reduction at a vertex v which removes a loop e, by the sparsity of (G, ψ), the following hold: if $k = 6, j = 3$, then e does not have gain $\gamma^{k/2}$; if the vertex incident to v is fixed, call it v_0, then the graph spanned by v, v_0 is not $S_0(k, j)$. So the conditions in Lemma 5.3.8 hold.

Let $\tilde{q}'$ be a $\mathcal{C}_k$-generic configuration of $\tilde{G}'$, which also satisfies the condition in Lemma 5.3.11 if the move applied is a 1-reduction. Notice that such a configuration does exist, since small symmetry-preserving perturbations of the points of a $\mathcal{C}_k$-generic framework maintain $\mathcal{C}_k$-genericity. By Lemmas 5.3.4, 5.3.8, 5.3.11 and 5.3.13 there is a $\mathcal{C}_k$-symmetric realisation $(\tilde{G}, \tilde{q})$ of $\tilde{G}$ which is ρ_j-symmetrically isostatic. By $\mathcal{C}_k$-genericity, $(\tilde{G}, \tilde{p})$ is also ρ_j-symmetrically isostatic.

So, assume that k is odd. By Lemma 6.1.1, H has a free vertex v of degree 2 or 3. If v has degree 2, or if it has degree 3 with a loop, then we may apply a 0-reduction or loop-1-reduction at v to obtain a $\mathbb{Z}_k^j$-gain tight graph (G', ψ') on t vertices. Moreover, if v has a loop, and the vertex incident to v is fixed, call it v_0, then the graph spanned by v, v_0 is not $S_0(k, j)$. By the inductive hypothesis, all $\mathcal{C}_k$-generic realisations of $\tilde{G}'$ are ρ_j-symmetrically isostatic. Then, our result holds by Lemmas 5.3.4 and 5.3.8. So, assume that v has degree 3 and no loop. Then, by Theorem 7.2.1, there is a $\mathbb{Z}_k^j$-tight graph (G', ψ') on t vertices obtained by applying a 1-reduction at v. By the inductive hypothesis, all $\mathcal{C}_k$-generic realisations of $\tilde{G}'$ are ρ_j-symmetrically isostatic. Let $\tilde{q}'$ be a $\mathcal{C}_k$-generic realisation of $\tilde{G}'$ which satisfies the condition of Lemma 5.3.11. Then, our result holds by Lemma 5.3.11. □

We finally have our main combinatorial characterisation for $\mathcal{C}_k$ for all odd integers $5 \leq k \leq 1000$ and for $k = 4, 6$, which is a direct result of Proposition 5.1.8 and Theorem 7.3.3.

Theorem 7.3.4. Let $k = 4, 6$ or $5 \leq k \leq 1000$ be odd. Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_k$-generic framework, and let (G, ψ) be the $\mathbb{Z}_k$-gain graph of $\tilde{G}$. $(\tilde{G}, \tilde{p})$ is infinitesimally rigid if and only if (G, ψ) has a $(2, 0, 3, 1)$-gain tight spanning subgraph, a $(2, 1, 3, 1)'$-gain tight spanning subgraph and a $\mathbb{Z}_k^j$-gain tight spanning subgraph for all $2 \leq j \leq k-2$.

The restriction $k \leq 1000$ in Theorem 7.3.1 arises from the difficulty of computationally checking the rank of the corresponding orbit matrices for a growing list of base graphs. Both in [8] and in [27], it is conjectured that this restriction may be dropped if $V(G) = V_1(G)$. For even $k \geq 8$, there are counterexamples to Theorem 7.3.1, as we will see in Section 7.4. Our final result relies on Theorem 7.3.1. Hence, we must maintain all restrictions on k.

7.4 Rotation groups of even order at least 8

In this section, we provide, for all even $|\Gamma| \geq 8$, examples of Γ-gain graphs that satisfy all conditions of Theorem 7.3.4, but whose $\mathcal{C}_{|\Gamma|}$-generic lifting frameworks are still not infinitesimally rigid.

Let $k := |\Gamma| \geq 4$ be even, and let G be the multigraph with exactly one free vertex v, which is free, and two loops f_1, f_2 at v (see Figure 7.3(a)). Let γ be the generator of Γ which corresponds to 1 in $\mathbb{Z}_k$. Let $\psi : E(G) \to \Gamma$ be defined by letting $\psi(f_1) = \gamma$ and $\psi(f_2) = \gamma^3$. If $k \geq 6$, (G, ψ) is a well-defined Γ-gain graph. Moreover, if $k \geq 8$, then (G, ψ) is $\mathbb{Z}_k^j$-gain tight for all $2 \leq j \leq k-2$. Since $G - f_1$ is both $(2, 0, 3, 1)$-gain tight and $(2, 1, 3, 1)'$-gain tight, (G, ψ) satisfies all three conditions of Theorem 7.3.4. For all odd $1 \leq k-1$, we have

$$\rho_{\frac{k}{2}}(\gamma^j) = \exp(\pi\sqrt{-1}j) = \cos(\pi j) + \sqrt{-1}\sin(\pi j) = -1.$$

Therefore, for any injective $p : (V) \to \mathbb{R}^2$, the $\rho_{k/2}$-orbit rigidity of (G, ψ, p) is

$$\begin{pmatrix} [(p(v) - C_k p(v) + p(v) - C_k^{-1} p(v)]^T \\ [(p(v) - C_k^3 p(v) + p(v) - C_k^{-3} p(v)]^T \end{pmatrix} = \begin{pmatrix} -2\sin(\frac{2\pi}{k})y & 2\sin(\frac{2\pi}{k})x \\ -2\sin(\frac{6\pi}{k})y & 2\sin(\frac{6\pi}{k})x \end{pmatrix},$$

where x and y denote the x-coordinate and the y-coordinate of $p(v)$, respectively. Suppose $x \neq 0$. Then, for any $m_1 \in \mathbb{R}$, it is easy to see that the column vector in $\mathbb{R}^2$ whose first entry is m_1 and whose second entry is $m_1 y/x$ lies in the kernel of $O_{k/2}(G, \psi, p)$. Therefore, (G, ψ, p) is not $\rho_{k/2}$-isostatic.

This result is not unexpected. Let $\tilde{G}$ be the Γ-lifting of (G, ψ), and take an arbitrary $\mathcal{C}_k$-symmetric realisation $(\tilde{G}, \tilde{p})$ of $\tilde{G}$. By definition, the realisation of the vertices in $V(G)$ form a regular k-gon. Moreover, it is easy to see that the vertices of the k-gon alternate between vertices of the two partite sets of a bipartite graph (see e.g. Figure 7.3(b) for the case when $k = 8$), as no odd cycles are created. Clearly, the framework is $\mathcal{C}_k$-generic. It is well known that such a framework has an 'in-and-out' infinitesimal motion m which, for $\tau(\delta) = C_k$, satisfies the system of equations

$$m(\delta^t v) = \begin{cases} C_k^t m(v) & \text{if } t \text{ is even} \\ -C_k^t m(v) & \text{if } t \text{ is odd,} \end{cases}$$

where v is an arbitrary vertex of $\tilde{G}$ (here, $m(v)$ is a vector on the line from the origin to p_v), and $0 \leq t \leq k-1$ (see e.g. [72]). Equivalently, for all $v \in V(\tilde{G})$ and $0 \leq t \leq k-1$,

$$\begin{aligned} m(\gamma^t v) &= \cos(\pi t) C_k^t m(v) = \cos(-\pi t) C_k^t m(v) \\ &= \exp(-\pi i t) C_k^t m(v) = \overline{\rho_{k/2}(\gamma^t)} C_k^t m(v). \end{aligned}$$

So, m is a $\rho_{k/2}$-symmetric infinitesimal motion.

This example may be extended to the case in which the Γ-gain graph has a fixed vertex. Let G be a multigraph with exactly two free vertices u, v, and one fixed vertex v_0. Let the edge set of G be composed of two loops f_1, f_2 at u, one loop f_3 at v, and the edges $e_1 = (u, v)$ and $e_2 = (v, v_0)$ (see Figure 7.3(c)). Let $\psi : E(G) \to \Gamma$

be defined by letting $\psi(f_1) = \gamma, \psi(f_2) = \gamma^3, \psi(f_3) = \gamma^2$, and $\psi(e_1) = \psi(e_2) = \mathrm{id}$. Similarly as in the previous examples, (G, ψ) is well-defined for all $k \geq 6$. Moreover, it has the following spanning subgraphs: $G - f_1 - f_3$, which is $(2, 0, 3, 1)$-gain tight; $G - f_1$, which is $(2, 1, 3, 1)'$-gain tight; and $G - f_3$, which is $\mathbb{Z}_k^j$-gain tight for all $2 \leq j \leq k - 2$, provided $k \geq 8$. Hence, for $k \geq 8$, (G, ψ) satisfies all conditions in Theorem 7.3.4. Since $\rho_{k/2}(\gamma^2) = \exp(2\pi\sqrt{-1}) = 1$, for some $p : V(G) \to \mathbb{R}^2$, the $\rho_{k/2}$-orbit rigidity matrix of (G, ψ, p) is

$$\begin{pmatrix} -2\sin(\frac{2\pi}{k})y_u & 2\sin(\frac{2\pi}{k})x_u & 0 & 0 \\ -2\sin(\frac{6\pi}{k})y_u & 2\sin(\frac{6\pi}{k})x_u & 0 & 0 \\ x_u - x_v & y_u - y_v & x_v - x_u & y_v - y_u \\ 0 & 0 & (2 - 2\cos(\frac{4\pi}{k}))x_v & (2 - 2\cos(\frac{4\pi}{k}))y_v \\ 0 & 0 & x_v & y_v \end{pmatrix},$$

where $p(u) = (x_u \ \ y_u)^T$ and $p(v) = (x_v \ \ y_v)^T$. Clearly, the first two rows of the matrix are linearly dependent, as are the bottom two. Hence, $\operatorname{rank} O_{k/2}(G, \psi, p) \leq 3$. By the Rank-nullity Theorem, $\operatorname{null} O_{k/2}(G, \psi, p) \geq 1$. Therefore, (G, ψ, p) is not $\rho_{k/2}$-isostatic.

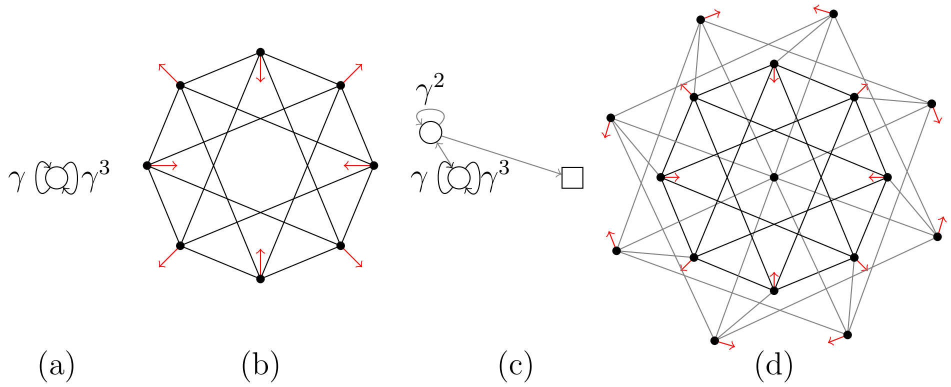


Figure 7.3: (a,c) show Γ-gain graphs with $\mathcal{C}_8$-symmetric frameworks (b,d), respectively. Though (a,c) satisfy the conditions in Theorem 7.3.4, (b,d) are ρ_4-symmetrically flexible. Here, γ denotes the generator of Γ which corresponds to rotation by $\pi/4$.

Similarly as with the free action case, the result is not unexpected. Let $\tilde{G}$ be the lifting of (G, ψ) and consider a $\mathcal{C}_k$-generic realisation $(\tilde{G}, \tilde{p})$ of $\tilde{G}$. Since this is an extension of the previous example, $(\tilde{G}, \tilde{p})$ still contains a regular k-gon P, and the graph induced by the vertices of P is bipartite. In addition, (G, ψ) contains two regular $k/2$-gons, P_1 and P_2, such that all vertices of P_1, P_2 are adjacent to the origin, and they are adjacent with the vertices of P as shown in Figure 7.3(d). Then, the infinitesimal motion from the previous example extends to an infinitesimal motion m of $(\tilde{G}, \tilde{p})$ which rotates P_1 and P_2 clockwise and anti-clockwise, respectively.

Chapter 8

Further work

When I started my PhD, I was presented with the idea of extending the combinatorial characterisations of infinitesimally rigid plane $\mathcal{C}_s, \mathcal{C}_2$ and $\mathcal{C}_3$-generic frameworks given in [56] to the case where the symmetry group need not act freely on the joints. The mathematical motivation was to start closing a gap in the knowledge of infinitesimally rigid plane symmetric frameworks. The gap has partly been closed now: we have a combinatorial characterisation for infinitesimally rigid $\mathcal{C}_s$-generic and $\mathcal{C}_k$-generic frameworks, where $k = 2, 4, 6$ or $3 \leq k < 1000$ is odd. However, there are still many question to be answered. In fact, for certain symmetry groups, the infinitesimal rigidity of symmetry-generic plane frameworks has not yet been characterised, even with the assumption that the symmetry group acts freely on the joints of the framework.

8.1 Cyclic groups

For all odd $k > 1000$ and for all even $k \geq 8$, we do not have a combinatorial characterisation of infinitesimally rigid $\mathcal{C}_k$-generic frameworks, even with the assumption that $\mathcal{C}_k$ acts freely on the joints of the framework. In Chapter 5, we presented necessary conditions for the infinitesimal rigidity of $\mathcal{C}_k$-symmetric framework, where $k \geq 3$ is arbitrary. (Analogous necessary conditions were given for the case where $\mathcal{C}_k$ acts freely on the joints of a given framework in [27] and in [56].) In Chapters 6 and 7, we showed that the conditions given in Chapter 5 were also sufficient for the ρ_0-infinitesimal rigidity, the ρ_1-infinitesimal rigidity and ρ_{k-1}-infinitesimal rigidity of $\mathcal{C}_k$-generic frameworks. Therefore, the problem lies in proving that, for $2 \leq j \leq k-2$, a $\mathbb{Z}_k^j$-gain tight graph has a ρ_j-isostatic 'realisation'.

The proof of our final combinatorial characterisation (see Theorem 7.3.4) strongly relies on the free action equivalent of the result given in [8] and [27] (see Theorem 7.3.1). Even assuming that a $\mathbb{Z}_k^j$-gain tight graph (G, ψ) has a fixed vertex v_0, we still rely on Theorem 7.3.1: since $\mathbb{Z}_k^j$-gain graphs need not be connected, it is possible that (G, ψ) has a connected component which does not contain v_0. Further, assuming that (G, ψ) has a fixed vertex v_0 *and* it is connected, we still need Theorem 7.3.1: when applying a reduction at a vertex of (G, ψ), it is possible we obtain a disconnected graph, and again we may have a connected component which does not contain v_0. Therefore, our inductive argument implicitly uses Theorem 7.3.1 throughout.

Since Theorem 7.3.1 assumes that $5 \leq k \leq 1000$ is odd or $k = 4, 6$, we must keep these bounds on the order of the group. As mentioned in Chapter 7, the upper bound of 1000 on k is due to the computational technique adopted in [27] to study the base cases of an inductive argument. In [27] it was (strongly) conjectured that this upper bound may be dropped. Since our only additional base case is a simple fixed vertex, we conjecture the following.

Conjecture 8.1.1. Let Γ be a cyclic group of odd order $k \geq 5$. Let $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_k$-generic framework with underlying Γ-symmetric graph $\tilde{G}$. Let (G, ψ) be the Γ-gain graph of $\tilde{G}$. Then, $(\tilde{G}, \tilde{p})$ is infinitesimally rigid if and only if (G, ψ) has a $(2, 0, 3, 1)$-gain tight spanning subgraph, a $(2, 1, 3, 1)$-gain tight spanning subgraph and a $\mathbb{Z}_k^j$-gain tight spanning subgraph for all $2 \leq j \leq k-2$.

Proving Conjecture 8.1.1 requires an analysis of the phase-symmetric orbit rigidity matrices of the base cases through algebraic tools.

Dropping the bound on even k would require more work. At the end of Chapter 7 we have seen through counterexamples that for all even $k \geq 8$, the lifting of a $\mathbb{Z}_k^{k/2}$-gain tight graphs need not have a $\rho_{k/2}$-infinitesimally rigid $\mathcal{C}_k$-symmetric realisation. Therefore, more refined conditions are needed for a full characterisation of infinitesimally rigid $\mathcal{C}_k$-generic frameworks for even $k \geq 8$.

Let $\Gamma = \langle \gamma \rangle$ be a cyclic group of even order $k \geq 8$, and let it be isomorphic to $\mathbb{Z}_k$ through the isomorphism which maps γ to 1. By definition, a $\mathbb{Z}_k^{k/2}$-gain tight Γ-gain graph (G, ψ) is allowed to have two loops f_1, f_2 at a vertex, with some restrictions on the gains $\psi(f_1), \psi(f_2)$. If, say $\psi(f_1) = \gamma$, then f_2 is allowed to have any gain except $\mathrm{id}, \gamma^{-1}, \gamma$ and $\gamma^{k/2}$. In particular, $\psi(f_2)$ can be γ^n for any odd n which is not equivalent (modulo k) to $-1, 1$ or $\frac{k}{2}$. As we have seen in Section 7.4, for all odd n, $\rho_{k/2}(\gamma^n) = -1$. Hence, for any injective $p : V(G) \to \mathbb{R}^2$, the rows corresponding to two loops at a vertex v with gains γ^n, γ^m, where n, m are odd, in $O_{k/2}(G, \psi, p)$ are

$$\begin{pmatrix} -2\sin(\frac{2n\pi}{k})y & 2\sin(\frac{2n\pi}{k})x \\ -2\sin(\frac{2m\pi}{k})y & 2\sin(\frac{2m\pi}{k})x \end{pmatrix},$$

where $p(v) = (x \quad y)^T$. This clearly presents a row dependence. Thus, two loops at a vertex are not allowed to have gains γ^n, γ^m where n, m are both odd (see, e.g., Figure 8.1).

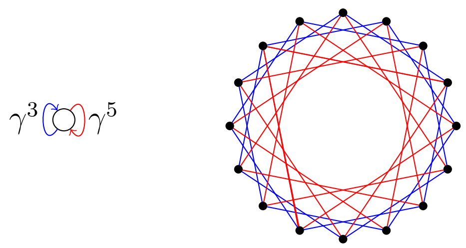


Figure 8.1: A $\mathbb{Z}_{16}$-gain graph with a $\mathcal{C}_{16}$-symmetric realisation of its $\mathbb{Z}_{16}$-lifting. Here, γ denotes the generator of $\Gamma \simeq \mathbb{Z}_{16}$ which corresponds to rotation by $\pi/8$, and the loop in blue (respectively, red) lifts to the edge orbit in blue (respectively, red). Note that the $\mathcal{C}_{16}$-symmetric framework is a bipartite graph realised on a conic.

On the other hand, for all even n, $\rho_{k/2}(\gamma^n) = 1$. Hence, for any injective $p : V(G) \to \mathbb{R}^2$, the rows in $O_{k/2}(G, \psi, p)$ corresponding to two loops at a vertex v with gains γ^n, γ^m, where n, m are even, are

$$\begin{pmatrix} (2-2\cos(\frac{2n\pi}{k}))x & (2-2\cos(\frac{2n\pi}{k}))y \\ (2-2\cos(\frac{2m\pi}{k}))x & (2-2\cos(\frac{2m\pi}{k}))y \end{pmatrix},$$

where $p(v) = (x \;\; y)^T$. Again, this clearly presents a row dependence. Therefore, two loops at a vertex are not allowed to have gains γ^n, γ^m where n, m have the same parity. Note, similarly as we did in the definition of $\mathbb{Z}_k^j$-gain tight given in Chapter 3, we must also consider subgraphs of (G, ψ) which are symmetric with respect to a subgroup of Γ. Therefore, for all even $k \geq 8$ and all $2 \leq j \leq k-2$, we define the following set in a similar way as we defined $S_0(k, j)$ and $S_{\pm 1}(k, j)$:

$$S_{k/2}(k, j) = \{n \in \mathbb{N}; n \geq 8 \text{ is even}, n|k, j \equiv n/2 (\operatorname{mod} n)\}.$$

Then, we say a connected subgraph H of G is $S_{k/2}(k, j)$ if $\langle H \rangle_\psi \simeq \mathbb{Z}_n$ for some $n \in S_{k/2}(k, j)$. For all $S_{k/2}(k, j)$ subgraphs H of (G, ψ), we expect that H does not have two loops at a vertex with gains γ^n, γ^m, where n, m have the same parity. However, more refined conditions may be needed.

8.2 Dihedral groups

More challenging, though still tractable for some special classes, is the study of plane frameworks which are symmetric with respect to a dihedral group. In [29], Jordán, Kaszanitzky and Tanigawa combinatorially characterise fully-symmetrically infinitesimally rigid $\mathcal{C}_{kv}$-generic frameworks on the plane, where $k \geq 3$ is odd and the symmetry group acts freely on the joints of the framework. We present their result in our terminology.

Theorem 8.2.1 ([29], Theorem 8.2)**.** Let $k \geq 3$ be an odd integer, and $(\tilde{G}, \tilde{p})$ be a $\mathcal{C}_{kv}$-generic framework with $\mathcal{C}_{kv}$-gain framework (G, φ, ψ, p). Suppose that all vertices of G are free. Then, $(\tilde{G}, \tilde{p})$ is fully-symmetrically infinitesimally rigid if and only if (G, φ, ψ) is $\mathbb{D}_{2k}$-gain tight.

Using a similar proof strategy as we did for the case of cyclic groups of order $2, 4, 6$ or of odd order $3 \leq k \leq 1000$, we expect that this result can be generalised to the setting where $V(G) \neq V_1(G)$. In Chapter 5 we proved the necessity of the these sparsity conditions (see Proposition 5.2.6). It remains to show that the conditions are also sufficient. Take a connected $\mathbb{D}_{2k}$-gain tight graph (G, φ, ψ) for some odd $k \geq 3$. If we let G have semi-free vertices, $\langle H \rangle_{v,\psi}$ and $\langle H \rangle_{v,\psi'}$ can be very different for two equivalent edge-gain maps ψ, ψ' and for some non-fixed $v \in V(G)$. Therefore, the problem becomes significantly more complex. However, the problem should still be tractable, through a proof by induction on $|V(G)|$.

It is easy to see, through a combinatorial argument similar to that used for the proof of 6.1.1, that the following result holds.

Lemma 8.2.2. Let (G, φ, ψ) be a connected $\mathbb{D}_{2k}$-gain tight graph with $|V(G)| \geq 1$ and $V(G) \neq V_1(G)$. Suppose that G is not an isolated fixed vertex. Then, G has either a semi-free vertex of degree 1 or 2 or a free vertex of degree 2 or 3.

The definitions of fix-0-reduction, 0-reductions, loop-1-reductions, 1-reductions (as well as fix-0-extensions, 0-extensions, loop-1-extensions, 1-extensions) can easily be adapted to our new setting, as can the proofs of Lemmas 5.3.2, 5.3.4, 5.3.8 and 5.3.11. Moreover, arguments similar to those used in Chapters 6 and 7 (though slightly more laborious) show that there is always an ‘admissible’ reduction at a free vertex of degree 3 of a $\mathbb{D}_{2k}$-gain tight graph, where $k \geq 3$ is odd.

Theorem 8.2.3. For odd $k \geq 3$, let (G, φ, ψ) be a $\mathbb{D}_{2k}$-gain tight graph with a free vertex v of degree 3. There is a reduction at v which yields a $\mathbb{D}_{2k}$-gain tight graph.

Therefore, it remains to:

1. Generalise the notions of 0-extension and 0-reduction to our new setting (note, we now add a semi-free vertex), and show that a 0-extension can always be applied to a fully-symmetrically isostatic $\mathcal{C}_{kv}$-gain graph in a way that does not add a row dependence in the ρ_0-orbit rigidity matrix.

2. Show that there is always an admissible semi-1-reduction at a semi-free vertex $v \in V_2(G)$ of degree 2, provided such a vertex exists.

3. Check that a $\mathbb{D}_{2k}$-gain tight graph with $k \geq 3$ odd has a fully-symmetrically isostatic '$\mathcal{D}_{kv}$-realisation' when $|V(\tilde{G})|$ is small.

If $k \geq 2$ is even, Theorem 8.2.1 does not hold. The smallest known counterexample is the Bottema mechanism, a $\mathcal{C}_{2v}$-generic framework $(\tilde{G}, \tilde{p})$ (see Figure 8.2). It is easy to see that the $\mathbb{D}_4$-gain graph of the underlying $\mathbb{D}_4$-symmetric graph of $(\tilde{G}, \tilde{p})$ is $\mathbb{D}_4$-gain tight. However, it is well-known that $(\tilde{G}, \tilde{p})$ is fully-symmetrically flexible: it is a well-known fact in kinematics [74]; it is also easy to see that $(\tilde{G}, \tilde{p})$ is a bipartite graph realised on a conic, and hence it has an 'in-and-out' infinitesimal motion, i.e. an infinitesimal motion which shifts the vertices of one partite set of the graph in the direction of the origin, and the other partite set in the direction opposite to the origin [72]. Further counterexamples are also given in [[29], Section 8.2]. Moreover, if $V(G) \neq V_1(G)$, there is the added difficulty that for two vertices $u, v \in V_2(G)$, $\varphi(u)$ and $\varphi(v)$ need not be conjugate. It is important to note that the sparsity conditions given in Theorem 8.2.1 are necessary for all even $k \geq 2$. This was shown in [29] for the case where $V(G) = V_1(G)$ and in this thesis for the case where $V(G) \neq V_1(G)$.

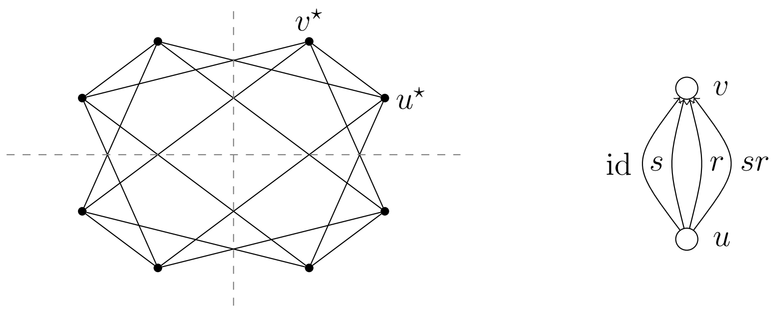


Figure 8.2: The Bottema mechanism and the $\mathbb{D}_4$-gain graph of its underlying graph.

It can also be interesting to consider the infinitesimal motions of a $\mathcal{C}_{kv}$-symmetric framework which break its symmetry. Given a 2-dimensional irreducible representation μ_j of $\mathbb{D}_{2k}$ (for $k \geq 3, 1 \leq j \leq \lfloor k/2 \rfloor$), the velocity vector corresponding to a μ_j-symmetric infinitesimal motion $\tilde{m}$ at a joint $\tilde{p}(v)$ does not uniquely determine the velocity vectors corresponding to $\tilde{m}$ at $\tilde{p}(\gamma v)$ for all $\gamma \in \mathbb{D}_{2k}$. This ambiguity creates two obstacles:

(A) It is not clear how to write the definition of μ_j-symmetric infinitesimal motion in terms of given vertex representatives in a way that uniquely determines the motion on the whole vertex orbits.

(B) None of the notions of gain graph given in Chapter 3 enclose sufficient information to be considered a good mathematical model to describe a combinatorial characterisation of infinitesimally rigid $\mathcal{C}_{kv}$-generic frameworks.

Therefore, for all $k \geq 3$, even just providing necessary combinatorial conditions for the infinitesimal rigidity of $\mathcal{C}_{kv}$-symmetric frameworks is a challenge. (If $k = 2$, the necessary conditions are given in Chapter 5.) In [31], engineers Kangwai and Guest present a possible approach to the problem: given a $\mathcal{C}_{kv}$-symmetric framework $(\tilde{G}, \tilde{p})$, they use the column space of a the projection operator matrix

$$O^{\mu_j}_{mn} = \sum_{\gamma \in \mathbb{D}_{2k}} [\mu_j(\gamma)]_{m,n} (\tau \otimes P_{V(\tilde{G})}(\gamma))$$

as a basis for the $\tau \otimes P_{V(\tilde{G})}(\gamma)$-invariant subspace of $\mathbb{R}^{|V(\tilde{G})|}$ corresponding to μ_j. (See also the papers [19, 30, 32].) This projection operator may be useful to obtain a general description of the phase-symmetric orbit rigidity matrices.

8.3 Different types of symmetric frameworks

In this thesis we considered the infinitesimal rigidity of symmetric *bar-joint* frameworks. However, for a variety of reasons, mathematicians (and non-mathematicians) are also interested in other types of symmetric structures.

8.3.1 Body-bar-hinge frameworks and molecular graphs

A *body* in $\mathbb{R}^d$ is a set of points which affinely span $\mathbb{R}^d$, and an *infinitesimal motion* of a body B is an isometric linear transformation of B. In other words, B can be seen as an infinitesimally rigid structure, and so the space of infinitesimal motions of B has dimension $\binom{d+1}{2}$. A *body-and-bar framework* is a structure composed of bodies, connected by rigid bars.

Similarly as for bar-joint frameworks, body-and-bar frameworks can be modelled mathematically by a graph G and a configuration $p : V(G) \to \mathbb{R}^d$ which determines the end-points of the bars. However, since joints are now substituted by rigid structures, we allow up to $\binom{d+1}{2}$ parallel edges between any pair of vertices u, v of $V(G)$. The notions of genericity, infinitesimal rigidity and rigidity matrix naturally extend to this setting, as do the ideas of symmetric framework, symmetric genericity and gain graph [58, 51, 68, 71].

Unlike infinitesimally rigid generic bar-joint frameworks, infinitesimally rigid generic body-and-bar frameworks have been combinatorially characterised in all dimensions.

Theorem 8.3.1 ([68], Theorem 5.3)**.** A generic body-and-bar framework (G, p) in $\mathbb{R}^d$ is isostatic if and only if G is $\left(\binom{d+1}{2}, \binom{d+1}{2}\right)$-tight; or equivalently if and only if it is the union of $\binom{d+1}{2}$ edge-disjoint spanning trees.

This powerful result spiked the interest of mathematicians. In particular, it was noticed that analogous techniques which were used for the study of infinitesimally rigid 'symmetry-generic' bar-joint frameworks could also be adapted to the setting of symmetric body-and-hinge frameworks [25, 58, 51]. In [58], it was noted that a similar proof as that given in [49] for Theorem 4.3.1 shows that the rigidity matrix of a symmetric body-and-bar framework also diagonalises in a way that each block correspond to an irreducible representation of the symmetry group.

This allowed a characterisation of infinitesimally rigid body-and-bar frameworks $(\tilde{G}, \tilde{p})$ in $\mathbb{R}^d$ which are symmetric (and symmetry-generic) with respect to a symmetry group $\tau(\Gamma) \simeq \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, provided the group acts freely on the bodies of the framework [58]. The conditions are given in terms of sparsity counts on the Γ-gain graph of the underlying Γ-symmetric graph $\tilde{G}$. The proofs are significantly more complex than that used for the bar-joint framework setting, as they employ Plücker coordinates. However, the general approach is similar, as are some of the specific techniques. Therefore, I expect that the content of this thesis can be translated to the body-and-bar setting, in order to extend the result in [58] to the case where the symmetry group does not act freely on the bodies.

One of the reasons body-and-bar frameworks have sparked such interest is their strong connection to body-and-hinge frameworks, key objects in the study of protein-folding, engineering, robotics and other applied sciences. A *body-and-hinge framework* in $\mathbb{R}^d$ is a structure composed of bodies in $\mathbb{R}^d$, which are connected in pairs along *hinges*, i.e. $(d-1)$-affine subspaces of $\mathbb{R}^d$ (see Figure 8.3).

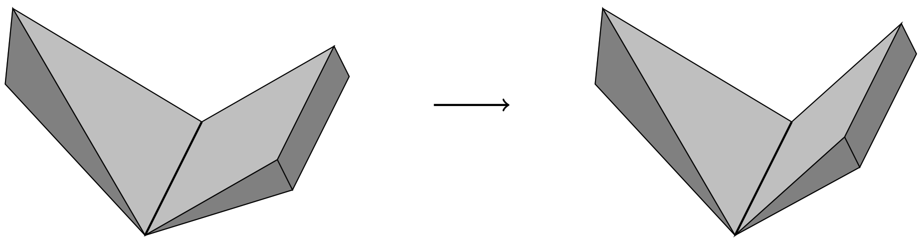


Figure 8.3: Continuous motion of a body-and-hinge framework in $\mathbb{R}^3$.

Replacing a hinge by 5 independent bars which intersect the hinge line, we can see that body-and-hinge frameworks are special cases of body-and-bar-frameworks [70, 71]. Further, the same conditions which characterise generically rigid body-and-bar frameworks characterise generically rigid body-and-hinge frameworks [65, 66, 73]. Similarly as for body-and-bar frameworks, [58] also characterises infinitesimally rigid body-and-hinge frameworks $(\tilde{G}, \tilde{p})$ in $\mathbb{R}^d$ which are symmetric (and symmetry-generic) with respect to a symmetry group $\tau(\Gamma) \simeq \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, provided the group acts freely on the bodies of the framework. Again, I expect that the content of this thesis can be used to extend this result to the case where the symmetry group need not act freely on the bodies of the body-and-hinge framework.

An even more special class of body-and-bar frameworks is the class of panel-and-hinge frameworks. *Panel-and-hinge frameworks* are body-and-hinge frameworks for which all the hinges incident to each body are forced to lie in a common hyperplane (see Figure 8.4). Panel-and-hinge frameworks are often referred to as *molecular graphs*, as they are a good model to describe molecules. In 1984, Tay and Whiteley proposed the Molecular Conjecture, which states that a multigraph can be realised as an infinitesimally rigid body-and-hinge framework in $\mathbb{R}^d$ if and only if it can be realised as an infinitesimally rigid panel-and-hinge framework in $\mathbb{R}^d$ [70]. The statement was famously proved by Tanigawa and Katoh in 2011 [35]. It would be interesting to prove the symmetric version of the Molecular Conjecture, stated in [51], which asserts that body-and-hinge frameworks and panel-and-hinge frameworks share the same infinitesimal rigidity properties, both for the free and non-free action case, provided they are 'symmetry-generic'.

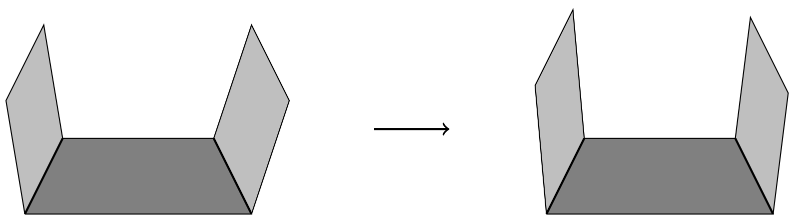


Figure 8.4: Continuous motion of a panel-and-hinge framework in $\mathbb{R}^3$.

8.3.2 Linearly constrained frameworks

A multigraph $G = (V, E)$ can sometimes be written as a triple $G = (V, E, L)$, where V, E and L denote, respectively, the vertices, non-loop edges, and loops of G. A *linearly constrained framework* in $\mathbb{R}^d$ is a triple (G, p, q), where $G = (V, E, L)$ is a graph with no parallel edges and $p : V \to \mathbb{R}^d, q : L \to \mathbb{R}^d$ are functions. Note that $(G - L, p)$ is a bar-joint framework. An *infinitesimal motion* of (G, p, q) is an infinitesimal motion $m : V \to \mathbb{R}^2$ of $(G - L, p)$ such that $\langle m(v), q(l)\rangle = 0$ for all $v \in V, l \in L$ with $l = uu$.

This additional condition on m forces the velocity vector associated to a vertex $v \in V$ with a loop $l \in L$ to lie on the hyperplane through the joint $p(v)$ with normal $q(l)$. Therefore, linearly constrained frameworks can be seen as bar-joint frameworks for which some joints are constrained to lie on certain hyperplanes (see Figure 8.5), and therefore are a good model for slide joints in engineering. The notions of genericity, infinitesimal rigidity, rigidity matrix and isostaticity extend naturally to this setting, as do the notions of symmetric framework and symmetry-genericity.

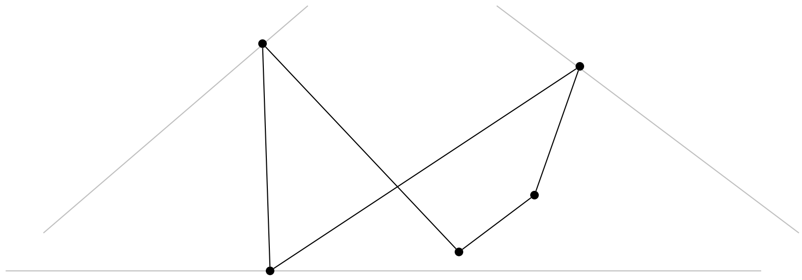


Figure 8.5: Example of a linearly constrained framework.

It was shown in [63] that a generic linearly constrained framework (G, p) with $G = (V, E, L)$ is isostatic in $\mathbb{R}^2$ if and only if (V, E) is $(2, 3)$-sparse, G is $(2, 0)$-sparse and $|E| + |L| = 2|V|$. This result was extended in [14] to the case where $d \geq 3$, provided each vertex has sufficiently many linear constraints with respect to d.

Theorem 8.3.2 ([14], Theorem 1.2)**.** Let $G = (V, E, L)$ be a graph and $d, t \geq 1$ be integers such that $d \geq \max\{2t, t(t-1)\}$. Suppose that for all $v \in V$, there are at least $d - t$ loops at v. Then, G can be realised as an infinitesimally rigid linearly constrained framework in $\mathbb{R}^d$ if and only if G has a $(t, 0)$-tight spanning subgraph.

In [44] it was shown, in a similar way as in [49], that the rigidity matrix of a symmetric linearly constrained framework block-diagonalises. The same paper uses this result, together with the characterisation in [63], to characterise isostatic linearly constrained plane frameworks which are symmetry-generic with respect to $\mathcal{C}_k$, where $k = 2$ or $k \geq 3$ is odd. Notice that the restriction on the order k of the symmetry group is very similar to the restriction we had in Chapter 7. In [44], the restriction is due to a problem encountered while proving that a vertex of degree 3 admits a reduction, and there are no known counterexamples to show that the result does not extend to all $k \geq 2$. It could be interesting to either find counterexamples for even $k \geq 4$, or to find an alternative argument to show that the result extends to all $k \geq 4$.

The other symmetry group for which we do not yet have a combinatorial characterisation of infinitesimally rigid symmetry-generic linearly constrained plane frameworks is the group $\mathcal{C}_s$. In this case, the main obstacles encountered by the authors of [44] are due to the fact that the number of fixed joints, bars and linear constrains of a $\mathcal{C}_s$-symmetric linearly constrained framework can be arbitrarily large (unlike for the rotation symmetry case in which, for instance, at most one fixed joint is allowed). Perhaps some of the techniques in the thesis can be adapted to the study of linearly constrained frameworks in order to overcome this difficulty.

Moreover, higher dimensions may be explored. Since the infinitesimal rigidity of generic bar-joint frameworks has not been characterised in dimensions $d \geq 3$, a lot of results concerning symmetric bar-joint frameworks are restricted to plane frameworks or, perhaps, frameworks which are known to be generically rigid in $\mathbb{R}^d$ for $d \geq 3$, such as body-and-bar frameworks, or frameworks obtained from generically rigid frameworks by applying a series of Henneberg moves, or triangulated polytopes. Theorem 8.3.2 gives us the possibility to tackle the problem of studying symmetry-generic linearly constrained frameworks in $\mathbb{R}^d$.

Appendix A

Additional proofs

In Sections 3.2.3 and 5.1.3 we stated some results related to near-balanced graphs which were proved in [27]. This documents is hard to access. We therefore present the proofs in our own language.

Lemma A.0.1 (Lemma 4.1 in [27])**.** For a group Γ, let (G, ψ) be a proper near-balanced Γ-gain graph. Then, G is unbalanced and there is some $\gamma \in \Gamma$ and a gain map ψ' equivalent to ψ such that $\psi'(e) \in \{\mathrm{id}, \gamma\}$ for all edges $e \in E(G)$ directed to v, and $\psi'(e) = \mathrm{id}$ for all edges $e \in E(G)$ not incident to v.

Proof. The statement clearly holds if $|V(G)| = 1$. So assume that $|V(G)| \geq 2$.

By definition, there is a non-identity element $\gamma \in \Gamma$ such that $\psi(W) \in \{\mathrm{id}, \gamma, \gamma^{-1}\}$ for every closed walk W containing v as its initial vertex but not as its internal vertex. Moreover, $G - v$ is balanced, as $\langle G \rangle_\psi \not\simeq \mathbb{Z}_2, \mathbb{Z}_3$. Take a spanning tree T of G such that $T \cap E(G - v)$ is a spanning forest of $G - v$. By Proposition 3.2.6, there is a gain function ψ'' equivalent to ψ such that $\psi''(e) = \mathrm{id}$ for all $e \in T$. By Lemma 3.2.11, $\psi''(e) = \mathrm{id}$ for all $e \in E(G - v)$.

Let $E_1, \ldots, E_t$ be the connected edge subsets of $G - v$, and let E_v be the set of edges in G directed to v. For each $1 \leq i \leq t$ define the set

$$E'_i := \{e \in E_v : e \text{ is incident to a vertex in } V(E_i)\}.$$

Note that $E(G) = (E_1 \cup E'_1) \cup \cdots \cup (E_t \cup E'_t)$. For each $1 \leq i \leq t$ consider the connected edge set $E_1 \cup E_1'$. Notice that $\psi''(e) = \text{id}$ for all $e \in E_i$ and there is at least one edge in E_i' with identity gain. By the definition of the proper near-balancedness, either $\psi''(e) \in \{\text{id}, \gamma\}$ for all $e \in E_i'$ or $\psi''(e) \in \{\text{id}, \gamma^{-1}\}$ for all $e \in E_i'$. If the latter holds, we may apply a switching at each vertex in $V(E_i)$ with gain γ, in order to obtain a gain map ψ' equivalent to ψ'' (and ψ) such that $\psi'(e) = \psi''(e)$ for all $e \in E(H) \setminus (E_i \cup E_i')$, $\psi'(e) = \text{id}$ for all $e \in E_i$ and $\psi'(e) \in \{\text{id}, \gamma\}$ for all $e \in E'$. The result follows. $\square$

Lemma A.0.2 (Lemma 4.2 in [27])**.** Let Γ be a group and (G, ψ) be a proper near-balanced Γ-gain graph which satisfies $|E(G)| \geq 2|V(G)| - 1$. Suppose that all balanced subgraph of (G, ψ) are $(2,3)$-sparse. Then, the base vertex of G is unique. Moreover, for all near-balanced subgraphs H of G with $|E(H)| = 2|V(H)| - 1$, the unique base vertex of H coincides with the base vertex of G.

Proof. Let v be a base vertex of G, and let E_v denote the set of edges in $E(G)$ directed to v. We start by showing that v is the unique base vertex of G. If v has a loop, this is clearly true. So, we may assume that v has no loop. By Lemma A.0.1, there is a gain function ψ' equivalent to ψ and some $\gamma \in \Gamma$ such that $\psi'(e) \in \{\text{id}, \gamma\}$ for all $e \in E_v$, and $\psi'(e) = \text{id}$ for all $e \in E(G) \setminus E_v$.

Since $G - v$ is balanced and H is unbalanced, v lies on every unbalanced cycle. It suffices to show that there are two vertex-disjoint unbalanced cycles passing through v. Notice that

$$\begin{aligned}|E(G)| &= \deg_G(v) + |E(G - v)| \leq \deg_G(v) + 2|V(G - v)| - 3 \\ &= \deg_G(v) + 2|V(G)| - 5 \leq \deg_G(v) + |E(G)| - 4,\end{aligned}$$

and so $\deg_G(v) \geq 4$. Define the two set $S := \{u \in N_G(v) : \psi'(uv) = \text{id}\}$ and $T := \{u \in N_G(v) : \psi'(uv) = \gamma\}$. (Note that S and T are not necessarily disjoint, as parallel edges are allowed.) Since $\deg_G(v) \geq 4$ and every balanced subgraph of G is $(2,3)$-sparse, $|S|, |T| \geq 2$. Let r be the be the maximum number of vertex-disjoint paths in $G - v$ from a vertex in S to a vertex in T. We look at the following cases separately: $r = 0$; $r = 1$; and $r \geq 2$.

Case 1: $r = 0$**.**

There are two sets W_1 and W_2 such that $S \subseteq W_1, T \subseteq W_2, W_1 \cup W_2 = V(G)$ and $W_1 \cap W_2 = \{v\}$. This implies

$$2|V(G)|-1 \leq |E(G)| = |E(W_1)|+|E(W_2)| \leq (2|W_1|-3)+(2|W_2|-3) = 2|V(G)|-4,$$

a contradiction.

Case 2: $r = 1$**.**

For some $u \in V(G)$, there are two sets W_1 and W_2 such that $S \subseteq W_1, T \subseteq W_2$, $W_1 \cup W_2 = V(G)$ and $W_1 \cap W_2 = \{u, v\}$. Define the sets of edges F_1, F_2 by letting $F_1 = E(W_1) \setminus \{e \in E_v : \psi'(e) = \gamma\}$ and $F_2 = E(W_2) \setminus \{e \in E_v : \psi'(e) = \mathrm{id}\}$. Then,

$$2|V(G)| - 1 \leq |E(G)| = |F_1| + |F_2| \leq (2|W_1| - 3) + (2|W_2| - 3) = 2|V(G)| - 2,$$

a contradiction.

Case 3: $r \geq 2$

Then there are two vertex-disjoint paths from a vertex in S to a vertex in T. Combining the two paths and E_v, we obtain two unbalanced cycles both of which contain v. Hence, v is the unique base vertex of G.

Now, let u be the unique base vertex of H. Then, by a similar argument as above, there are two vertex-disjoint unbalanced cycles which contain u in H. Hence, $u = v$. This completes the proof. □

In [27], R. Ikeshita adopts a matroidal approach to his proofs, and hence the statements of Lemmas A.0.4, A.0.5 and A.0.6 are slightly stronger in our setting. However, the proofs in [27] can be applied almost directly. First, we need the following preliminary result.

Lemma A.0.3 (Lemma 4.3(a,d) in [27])**.** Let Γ be a group, and let (G, ψ) be a $(2, 1)$-tight Γ-gain graph. Suppose that every balanced subgraph of (G, ψ) is $(2, 3)$-tight. Then, every 2-connected component of G is unbalanced. Moreover, every $(2, 3)$-tight graph is 2-connected.

Proof. Assume, for a contradiction, that G has a balanced 2-connected component H. There is a vertex $v \in V(H)$ and a subgraph H' of G such that $V(H \cap H') = \{u\}$ and $E(G) = E(H) \dot{\cup} E(H')$. But then

$$\begin{aligned}2|V(G)| - 1 = |E(G)| &= |E(H)| + |E(H')| \\ &\leq (2|V(H)| - 3) + (2|V(H)| - 1) = 2|V(G)| - 2\end{aligned}$$

This is a contradiction. Hence, all 2-connected components of G are unbalanced. Similarly, assume for a contradiction that there is a $(2,3)$-tight graph H which is not connected. Then, there is a vertex $v \in V(H)$ and two graphs H_1, H_2 such that $H_1 \cup H_2 = H, E(H_1 \cap H_2) = \emptyset$ and $V(H_1 \cap H_2) = \{v\}$. But then

$$\begin{aligned}2|V(G)| - 3 = |E(G)| &= |E(H)| + |E(H')| \\ &\leq (2|V(H)| - 3) + (2|V(H)| - 3) = 2|V(G)| - 4.\end{aligned}$$

Since this is a contradiction, the result holds. □

Lemma A.0.4 (Lemma 4.4 in [27])**.** Let Γ be a group, (G, ψ) be a Γ-gain graph and H_1, H_2 be proper near-balanced subgraphs of G such that $H_1 \cap H_2$ is $(2,1)$-tight and proper near-balanced. Assume that for $1 \leq i \leq 2$ there is an edge $f_i \in E(H_i)$ such that $H_i - f_i$ is $(2,1)$-tight. Assume further that every balanced subgraph of $H_1 - f_1, H_2 - f_2$ is $(2,3)$-sparse. Then $H_1 \cup H_2$ is proper near-balanced.

Proof. By Lemma 3.2.30 $H_1 - f_1, H_2 - f_2, H_1 \cap H_2$ are connected, as are H_1, H_2. By Lemma A.0.2, $H_1, H_2, H_1 \cap H_2$ have a common unique base vertex v. By Lemma A.0.3 the 2-connected components of $H_1, H_2, H_1 \cap H_2$ are unbalanced. Hence, v is not a separating vertex for $H_1, H_2, H_1 \cap H_2$. It follows that v is not a separating vertex for $H_1 \cup H_2$.

Let E_v be the set of edges in G directed to v. By Lemma A.0.1, we may assume that there is some $\gamma \in \Gamma$ such that $\psi(e) \in \{\mathrm{id}, \gamma\}$ for all $e \in H_1 \cap H_2 \cap E_v$ and $\psi(e) = \mathrm{id}$ for all $e \in (H_1 \cap H_2) \setminus E_v$. Choose a 2-connected component H of $H_1 \cup H_2$. We need only show that we can obtain a gain map ψ' equivalent to ψ such that

$\psi'(e) \in \{\mathrm{id}, \gamma\}$ for all $e \in E(H) \cap E_v$ and $\psi'(e) = \mathrm{id}$ for all $e \in E(H) \setminus E_v$ by performing type I switchings at the vertices of $H - v$.

For some $c \geq 1$, let $H'_1, \dots, H'_c$ be the 2-connected components of $H_1 \cap H_2$ in H, and let $H' = \bigcup_{i=1}^{c} H'_i$. We look at the cases where $c = 0$ and $c \geq 1$ separately.

Case 1: $c = 0$.
Here, H is a 2-connected component of H_1 or H_2. Take a spanning tree T in H such that $T \setminus E_v$ is a spanning tree of $H - v$. By Proposition 3.2.6, we may assume that $\psi(e) = \mathrm{id}$ for all $e \in T$. Since $H - v$ is balanced, $\psi(e) = \mathrm{id}$ for all $e \in E(H) \setminus E_v$. Hence, either $\psi(e) \in \{\mathrm{id}, \gamma\}$ for all $e \in E_v$ or $\psi(e) \in \{\mathrm{id}, \gamma^{-1}\}$ for all $e \in E_v$. In the former case, we have the desired gain map. In the latter, we may apply a type I switching at each vertex in $H - v$ with gain γ, in order to obtain the desired gain map.

Case 2: $c \geq 1$.
Take a spanning tree T of H' such that $T \setminus E_v$ is a maximal spanning forest of $H' - v$ and that $\psi(e) = \mathrm{id}$ for all $e \in T \cap E_v$, and take a maximal forest F of $H - V(H')$. For some $s \geq 1$, let $G_1, \dots, G_s$ be the connected components of the graph induced by F. For each $1 \leq i \leq s$, choose an edge $e_i = u_i v_j$, where $u_i \in V(G_i)$ and $v_j \in V(H_j - v)$ for some $1 \leq j \leq s$, and add e_i to G_i. By Proposition 3.2.6, we may assume that $\psi(e) = \mathrm{id}$ for all $1 \leq i \leq s$.

Let $T' := T \cup E(G_1) \cup \cdots \cup E(G_s)$, and notice that T' is a spanning tree of H, $T' \cap E(H_i)$ is a spanning tree of $H \cap H_i$ for $1 \leq i \leq 2$, and $\psi(e) = \mathrm{id}$ for all $e \in T'$. Let Z be the set $\{uw \in E(G) : w \in V(H'), u \in V(G_i) \text{ for some } 1 \leq i \leq s\}$, and notice that $E(H) = Z \,\dot{\cup}\, E(H') \,\dot{\cup}\, E(H - V(H'))$. Since $H \cap H_1 - v, H \cap H_2 - v$ are balanced, $\psi(e) = \mathrm{id}$ for all $e \in E(H - V(H'))$.

If $c = 1$, then given an edge $e = uw \in Z \setminus E_v$, $u \in V(G_i), w \in V(H'_1)$ for some $1 \leq i \leq s$. Then, there is a cycle C composed of e, edges of $E(H'_1 - v)$, the edge e_i, and edges of G_i. So, by balancedness we have $\psi(e) = \mathrm{id}$. Given an edge $e = uw \in Z \cap E_v$, $u \in V(G_i)$ for some $1 \leq i \leq c$. Then, there are two cycles passing C_1, C_2 composed of e, edges in $E(H_1)$, the edge e_i, and edges of G_i, such that

$\psi(C_1) = \psi(e)$ and $\psi(C_2) = \psi(e)\gamma^{-1}$. By proper near-balancedness, $\psi(e) \in \{\text{id}, \gamma\}$, as required. So, we may assume that $c \geq 2$.

Let $e = uw \in Z$. If $e \notin E_v$, then $u \in V(G_i), w \in V(H'_j)$ for some $1 \leq i \leq s, 1 \leq j \leq c$. If $e_i = u_i v_j$, then there is a cycle C composed of e_i, edges of $E(H'_j - v)$, the edge e_i, and edges of G_i. By balancedness, we then $\psi(C) = \psi(e) = \text{id}$. Otherwise, $e_i = u_i v_k$ for some $k \neq j$, and there are three cycles C_1, C_2, C_3 composed of e, edges in $E(H_j)$, edges in $E(H_k)$, the edge e_i, and edges of G_i, such that $\psi(C_1) = \psi(e), \psi(C_2) = \psi(e)\gamma^{-1}$ and $\psi(C_3) = \psi(e)\gamma$. By proper near-balancedness, $\psi(e) = \text{id}$. If $e \in E_v$, then $u \in V(G_i)$ for some $1 \leq i \leq s$. Similarly as above, there are two cycles C_1, C_2 passing through v with $\psi(C_1) = \psi(e)$ and $\psi(C_2) = \psi(e)\gamma^{-1}$, and so $\psi(e) \in \{\text{id}, \gamma\}$. This proves the result. □

Lemma A.0.5 (Lemma 4.5 in [27])**.** Let Γ be a group, (G, ψ) be a Γ-gain graph and H_1, H_2 be subgraphs of G such that $H_1 \cap H_2$ is connected, balanced and $(2,3)$-tight. Assume that there is an edge $f_1 \in E(H_1)$ such that $H_1 - f_1$ is $(2,1)$-tight and that H_1 is proper near-balanced. Assume further that H_2 is connected and balanced, and that $V_{|\Gamma|}(H_2) = \emptyset$. Then $H_1 \cup H_2$ is proper near-balanced.

Proof. First, notice that $V_{|\Gamma|}(H_1 \cup H_2) = \emptyset$, since $V_{|\Gamma|}(H_1) = \emptyset = V_{|\Gamma|}(H_2) = \emptyset$. Let H_1 be proper near-balanced with base vertex $v \in V(H_1)$ and gain $\gamma \in \Gamma$. Let E_v denote the set of edges in $H_1 \cup H_2$ directed to v. By Lemma A.0.1, we may assume that $\psi(e) = \text{id}$ for all $e \in E(H_1) \setminus E_v$ and that $\psi(e) \in \{\text{id}, \gamma\}$ for all $e \in E(H_1) \cap E_v$. Let E'_v be the set $\{e \in E(H_1) \cap E_v : \psi(e) = \gamma\}$. We look at the cases where $E'_v \cap E(H_1 \cap H_2) = \emptyset$ and $E'_v \cap E(H_1 \cap H_2) \neq \emptyset$ separately.

First, assume that $E'_v \cap E(H_1 \cap H_2) = \emptyset$. Take a spanning tree T of H_2 such that $T \cap E(H_2)$ is a spanning tree of $H_1 \cap H_2$. Since $E'_v \cap E(H_1 \cap H_2) = \emptyset$, we know that $\psi(e) = \text{id}$ for all $e \in E(H_1) \cap T$. Then, we may apply a series of switchings at the vertices in $V(T) \setminus V(H_1)$ in order to obtain a gain map ψ' equivalent to ψ such that $\psi'(e) = \psi(e)$ for all $e \in E(H_1)$ and $\psi'(e) = \text{id}$ for all $e \in T$. By Lemma 3.2.11, $\psi'(e) = \text{id}$ for all $e \in E(H_2)$. Hence, $H_1 \cup H_2$ is proper near-balanced by Lemma A.0.1.

Now suppose that $E'_v \cap E(H_1 \cap H_2) \neq \emptyset$, and let $e' \in E(H_1 \cap H_2) \cap E_v$ be an edge satisfying $\psi(e) = \gamma$. By Lemma A.0.3, $H_1 \cap H_2$ is 2-connected. Hence, given an edge $e \in (E(H_1 \cap H_2) \cap E_v) \setminus \{e'\}$, there is a cycle C in $H_1 \cap H_2$ which passes through e, e'. By balancedness of H_1, we have that $\psi(C) = \mathrm{id}$, and so $\psi(e) = \gamma$. It follows that $\psi(e) = \gamma$ for all $e \in E(H_1 \cap H_2) \cap E_v$. Apply a switching operation at v with gain γ^{-1} in order to obtain a gain map ψ' equivalent to ψ such that $\psi'(e) = \mathrm{id}$ for all $e \in E(H_1 \cap H_2) \cap E_v$ and for all $e \in H_1 \setminus E_v$, and such that $\psi'(e) \in \{\mathrm{id}, \gamma^{-1}\}$ for all $e \in E(H_1) \cap E_v$. Hence, we may apply the same proof as in the case where $E'_v \cap E(H_1 \cap H_2) = \emptyset$ to show that $H_1 \cup H_2$ is proper near-balanced. □

Lemma A.0.6 (Lemma 4.6 in [27])**.** Let Γ be a group, (G, ψ) be a Γ-gain graph and H_1, H_2 be balanced subgraphs of G such that $H_1 \cap H_2$ consists of two connected components, one of which is an isolated vertex v. Suppose that there is an edge $f_1 \in E(H_1)$ such that $H_1 - f_1$ is $(2,3)$-tight, and that H_2 is connected. Suppose further that $V_{|\Gamma|}(H_1 \cup H_2) = \emptyset$. Then $H_1 \cup H_2$ is near-balanced with base vertex v.

Proof. Let E_v be the edges in H_1 directed to v. By Lemma A.0.3, $H_1 - f_1$ is 2-connected, as is H_1. In particular, $H_1 - v$ is connected. By assumption, $(H_1 \cap H_2) - v$ is connected. Since $H_1 - v$ and H_2 are balanced, $(H_1 - v) \cup H_2$ is balanced by Lemma 3.2.14. Then, by Proposition 3.2.6 and Lemma 3.2.11, we may assume that $\psi(e) = \mathrm{id}$ for all $e \in E((H_1 - v) \cup H_2)$. Let $\gamma := \psi(e)$ for some $e' \in E_v$. Given an edge $e \in E_v$, there is a cycle C in H_1 which passes through e, e'. Since H_1 is balanced, $\psi(C) = \mathrm{id}$, and so $\psi(e) = \psi(e') = \gamma$. It follows that $\psi(e) = \gamma$ for all $e \in E_v$. By Lemma A.0.1, this proves the result. □

Lemma A.0.7. Let Γ be a group, (G, ψ) be a Γ-gain graph and H_1, H_2 be connected subgraphs of G such that $H_1 \cap H_2$ is connected and unbalanced. Assume that H_1 is proper near-balanced and that $\langle H_2 \rangle \simeq \mathbb{Z}_p$ for some prime $p \geq 2$. Then, we have $\langle H_1 \rangle = \langle H_2 \rangle = \langle H_1 \cap H_2 \rangle = \langle H_1 \cup H_2 \rangle$.

Proof. Since H_1 is near-balanced, $V_{|\Gamma|}(H_1 \cap H_2) = V_{|\Gamma|}(H_1) = \emptyset$. Then, since $H_1 \cap H_2$ is connected and unbalanced, we have $\langle H_1 \cap H_2 \rangle \simeq \langle H_2 \rangle \simeq \mathbb{Z}_p$.

Let v be a base vertex of H_1 and let E_v be the set of edges in $H_1 \cup H_2$ directed to v. Then, there is a gain map ψ' equivalent to ψ, and some $\gamma \in \Gamma$ such that $\psi'(e) = \text{id}$ for all $e \in E(H_1 - E_v)$ and $\psi'(e) \in \{\text{id}, \gamma\}$ for all $e \in E(H_1) \cap E_v$. Since $H_1 \cap H_2$ is unbalanced, we have $\gamma \in \langle H_1 \cap H_2 \rangle_{\psi'}$. Moreover, there is an edge $e \in (E(H_1 \cap H_2) \cap E_v)$ such that $\psi'(e) = \text{id}$, as otherwise $H_1 \cap H_2$ would be balanced. It follows, by Lemma 3.2.11, that $\langle H_1 \rangle \simeq \langle H_1 \cap H_2 \rangle \simeq \mathbb{Z}_2$. The result then follows by Lemma 3.2.15. □

Lemma A.0.8. (Lemma 5.5 in [27]) Let $k := |\Gamma| \geq 4, 0 \leq j \leq k-1$, $\tau : \Gamma \to \mathcal{C}_k$ be an injective homomorphism, (G, ψ) be a Γ-gain graph, and $p : V(G) \to \mathbb{R}^2$. If $O_j(G, \psi, p)$ is row independent, $|E(H)| \leq 2|V(H)| - 1$ for any near-balanced subgraph H of G.

Proof. Recall that the cyclic group $\Gamma = \langle \gamma \rangle$ is isomorphic to $\mathbb{Z}_k$, through the isomorphism which maps γ to 1. Let H be a near-balanced subgraph of G. If $\langle H \rangle \simeq \mathbb{Z}_2, \mathbb{Z}_3$, the result holds by Lemma 5.1.7. Hence, assume that H is proper near-balanced. Let v be a base vertex of H. By Propositions 3.2.6, 4.1.4 and 4.1.5, and by Lemma 3.2.17, we can assume that there is an integer $0 \leq t \leq k-1$ such that $\psi(e) = \text{id}$ for all $e \in E(H)$ not incident to v and $\psi(e) \in \{\text{id}, \gamma^t\}$ for all $e \in E(H)$ directed to v. Let M be the matrix obtained from $O_j(G, \psi, p)$ by removing the columns which correspond to vertices in $V(G) \setminus V(H)$, together with the rows corresponding to their incident edges.

Let $v_1, \ldots, v_n$ be the vertices incident to v in H and, for all $1 \leq i \leq n$, let $e_i := (v_i, v)$. Let $0 \leq m \leq n$ be such that $e_1, \ldots, e_m$ have gain γ^t and $e_{m+1}, \ldots, e_n$ have gain id. Note that, by the definition of near-balancedness, there is at most one loop at v. If there is such a loop, let it be e_1. For each $1 \leq i \leq m$, the row in M corresponding to e_i has the form

$$e_i = \begin{pmatrix} \ldots & \rho_j(\gamma^t)(p(v) - \tau(\gamma^{-t})p(v_i))^T & \ldots & (p(v_i) - \tau(\gamma^t)p(v))^T & \ldots \end{pmatrix}$$

if $v \neq v_i$, and it has the form

$$e_i = \begin{pmatrix} \ldots & (p(v) - \tau(\gamma^t)p(v) + \rho_j(\gamma^t)p(v) - \rho_j(\gamma^t)\tau(\gamma^{-t})p(v))^T & \ldots \end{pmatrix}$$

otherwise. Order the rows of M such that, for all $1 \leq i \leq t$, the i^{th} row in M is the row corresponding to e_i. Then, add the following column to M:

$$\begin{pmatrix} \rho_j(\gamma^t)(p(v) - \tau(\gamma^{-t})p(v_1))^T \\ \vdots \\ \rho_j(\gamma^t)(p(v) - \tau(\gamma^{-t})p(v_m))^T \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then, subtracting the new columns from the columns corresponding to v and multiplying the new columns by $\rho_j(\gamma^t)^{-1}\tau(\gamma^t)$, we can see that M is

$$\left(\begin{array}{c|c|c} (\tau(\gamma^t)p(v) - p(v))^T & (p(v) - \tau(\gamma^t)p(v))^T & 0 \\ (\tau(\gamma^t)p(v) - p(v_2))^T & 0 & \star \\ \vdots & \vdots & \vdots \\ (\tau(\gamma^t)p(v) - p(v_m))^T & 0 & \vdots \\ 0 & (p(v) - p(v_{m+1}))^T & \vdots \\ \vdots & \vdots & \vdots \\ 0 & (p(v) - p(v_n))^T & \star \\ \hline 0 & 0 & O_j(H - v, \psi|_{E(H-v)}, p|_{V(H-v)}) \end{array}\right),$$

where the first row is removed if there is no loop at v. It is easy to see that the matrix obtained is the standard rigidity matrix for a graph F with $|V(H)| + 1$ vertices. Therefore, $|E(H)| \geq (|V(H)|+1) - 3 = 2|V(H)| - 1$, proving the result. $\square$

Appendix B

Character tables of cyclic and Dihedral groups

Maschke's Theorem implies that all the information of the representations of a group is held in its irreducible representations. In fact, a lot of this information is held in the characters of its irreducible representations. Hence, we sometimes use character tables to study the representations of a group. The *character table* of a group Γ is a table which presents the characters of its irreducible representations. The rows of the character table of a group correspond to the characters of the irreducible representations of the group. The columns correspond to the group elements. The entry at row χ and column $\gamma \in \Gamma$ represents $\chi(\gamma)$.

Given a group Γ and an irreducible matrix representation ρ of Γ with trace χ, $\chi(\gamma) = \chi(\delta)$ for all conjugate $\gamma, \delta \in \Gamma$. Hence, the number of columns in the character tables of a group Γ is exactly the number of conjugacy classes of Γ. Moreover, $\chi(\mathrm{id}) = \dim \rho$. Therefore, the entry at row χ and column id gives exactly $\dim \rho$. We consider the irreducible representations over the complex numbers.

For the scope of this thesis, we only present the character tables of arbitrarily large (but finite) cyclic groups (see Section B.1) and arbitrarily large (but finite) dihedral groups (see Section B.2).

B.1 Character tables of cyclic groups

$\mathcal{C}_s$	id	s
χ_0	1	1
χ_1	1	-1

$\mathcal{C}_2$	id	C_2
χ_0	1	1
χ_1	1	-1

$\mathcal{C}_3$	id	C_3	C_3^2
χ_0	1	1	1
χ_1	1	$\frac{i\sqrt{3}-1}{2}$	$\frac{i\sqrt{3}+1}{2}$
χ_2	1	$\frac{i\sqrt{3}+1}{2}$	$\frac{i\sqrt{3}-1}{2}$

For an arbitrary integer $k \geq 2$, the character table of $\mathcal{C}_k$ is:

$\mathcal{C}_k$	id	C_k	C_k^2	$\ldots$	C_k^{k-1}
χ_0	1	1	1	$\ldots$	1
χ_1	1	$\exp \frac{2\pi i}{k}$	$\exp \frac{4\pi i}{k}$	$\ldots$	$\exp \frac{(k-1)\pi i}{k}$
χ_2	1	$\exp \frac{4\pi i}{k}$	$\exp \frac{8\pi i}{k}$	$\ldots$	$\exp \frac{2(k-1)\pi i}{k}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_{k-1}	1	$\exp \frac{2(k-1)\pi i}{k}$	$\exp \frac{4(k-1)\pi i}{3}$	$\ldots$	$\exp \frac{2(k-1)^2\pi i}{k}$

B.2 Character tables of Dihedral groups

The only dihedral group whose irreducible representations are all 1-dimensional is the dihedral group of order 4:

$\mathcal{C}_{2v}$	id	s	C_2	sC_2
χ_0	1	1	1	1
χ_1	1	1	-1	-1
χ_2	1	-1	-1	1
χ_3	1	-1	1	-1

For $k \geq 3$, all dihedral groups of order $2k$ have at least one irreducible representation of order 2. The irreducible representations of $\mathcal{C}_{kv}$ depend on the parity of k. If $k = 2n$ for some integer $n \geq 2$, then the character table of $\mathcal{C}_{kv}$ is:

$\mathcal{C}_{kv}$	id	s	C_k	sC_k	$\dots$	C_k^{k-1}	sC_k^{k-1}
χ_0	1	1	1	1	$\dots$	1	1
χ_1	1	1	-1	-1	$\dots$	-1	-1
χ_2	1	-1	-1	1	$\dots$	-1	1
χ_3	1	-1	1	-1	$\dots$	1	-1
E_1	2	0	$2\cos\frac{2\pi}{k}$	0	$\dots$	$2\cos\frac{(k-1)\pi}{k}$	0
E_2	2	0	$2\cos\frac{4\pi}{k}$	0	$\dots$	$2\cos\frac{2(k-1)\pi}{k}$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	0
E_{n-1}	2	0	$2\cos\frac{2(n-1)\pi}{k}$	0	$\dots$	$2\cos\frac{2(k-1)(n-1)\pi}{k}$	0

If $k = 2n + 1$ for some integer $n \geq 1$, then the character table of $\mathcal{C}_{kv}$ is:

$\mathcal{C}_{kv}$	id	s	C_k	C_k^2	$\dots$	C_k^{k-1}
χ_0	1	1	1	1	$\dots$	1
χ_1	1	-1	1	1	$\dots$	1
E_1	2	0	$2\cos\frac{2\pi}{k}$	$2\cos\frac{4\pi}{k}$	$\dots$	$2\cos\frac{(k-1)\pi}{k}$
E_2	2	0	$2\cos\frac{4\pi}{k}$	$2\cos\frac{8\pi}{k}$	$\dots$	$2\cos\frac{2(k-1)\pi}{k}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
E_n	2	0	$2\cos\frac{2n\pi}{k}$	$2\cos\frac{4n\pi}{3}$	$\dots$	$2\cos\frac{2(k-1)n\pi}{k}$

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