Universality of zero-dispersion peaks in the fluctuation spectra of underdamped nonlinear oscillators

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Zero-dispersion peaks (ZDP's), which can arise in the fluctuation spectra of noise-driven underdamped oscillators for which the dependence of the eigenfrequency $\omega(E)$ on energy E possesses a maximum or minimum, have been investigated by means of analog electronic experiments. Two different model systems were studied: a tilted Duffing oscillator and a model of a bistable superconducting quantum-interference device (SQUID). It is demonstrated experimentally that, for strong enough intensity T of Gaussian pseudowhite noise (equivalent to temperature in a thermal system), the shape of a ZDP becomes universal, independent of the system under investigation. Its evolution with T is also shown to exhibit universal features, being governed by a single parameter provided that T exceeds a critical value T_c, below which the ZDP disappears abruptly. The hierarchy of universalities connected to particular types of extrema in $\omega(E)$ is discussed. The results are of relevance to underdamped SQUID's and, in particular, to the recently discovered phenomenon of zero-dispersion stochastic resonance.

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I. INTRODUCTION

The fluctuation spectra [1] of noise-driven oscillators are of interest because they characterize the susceptibility and consequently determine the frequency dependence of experimentally accessible quantities such as absorption, conductivity and polarizability [2]. The model of a classical nonlinear oscillator subject to noise (or interacting with a thermal bath) is applicable to a diverse range of physical systems, including electrical oscillating circuits [3], Josephson junctions [4], local vibrations in certain doped crystals [5], polymer molecules [6], and many others [2]. The fluctuation spectra of classical oscillators subject to noise have been studied intensively both theoretically [2,7–19] and experimentally [7,20–22].

The oscillatory character of the fluctuation-induced motion is manifested in such spectra only when the relevant damping constant is small, i.e., in the case of underdamped systems. Because the energy of the oscillator fluctuates in time, the shape of the fluctuation spectrum is largely determined by the dependence of its eigenfrequency $\omega(E)$ on energy E [23]. It was originally pointed out in [24] that, if $\omega(E)$ possesses an extremum, i.e., a point where the dispersion of the frequency is zero,

$$\left.\frac{d\omega(E)}{dE}\right|_{E_m} = 0, \qquad (1)$$

then the fluctuation spectrum will contain sharp peaks at the extremal frequency $\omega(E_m)$ and its overtones. In a sense, these *zero-dispersion peaks* (ZDP's) are analogous to Van Hove singularities [25] in solid-state physics.

A formalism was developed in [26] which demonstrated that, for oscillators performing one-dimensional potential motion with sufficiently weak damping, the shape of ZDP's should be described by a universal function. It was also shown [27] that the evolution of the shape of a ZDP with temperature (noise intensity) should be universal. We will show in Sec. II below that there is actually a hierarchy of universalities corresponding to the hierarchy of types of extrema of $\omega(E)$. The theoretical prediction of the ZDP's was recently confirmed in an analog electronic experiment [28].

It should be noted that the class of systems in which ZDP's can in principle arise is not restricted to oscillators performing potential motion, although it was the latter type of oscillator that was considered in [24,26–28] (and that will also be considered below). Quite generally, ZDP's are to be anticipated in dynamical systems for which slow and fast variables exist and for which the generalized eigenfrequency has either a minimum or a maximum in the space of the slow variables $\mathbf{E}$

$$\frac{d\psi}{dt} = \omega(\mathbf{E}) + \epsilon F_\psi(\mathbf{E},\psi,f_\psi(t)),$$
$$\frac{d\mathbf{E}}{dt} = \epsilon F_{\mathbf{E}}(\mathbf{E},\psi,f_{\mathbf{E}}(t)), \qquad (2)$$
$$\left.\frac{d\omega(\mathbf{E})}{d\mathbf{E}}\right|_{\mathbf{E}=\mathbf{E}_m} = 0, \qquad \epsilon \ll 1,$$

where $\mathbf{F}$ is some function of the dynamical variables and of the random force $\mathbf{f}(t)$. The spectrum of fluctuations for any quantity periodic in ψ will have a sharp peak at the extremal frequency $\omega(\mathbf{E}_m)$.

The occurrence of ZDP's is important, not only as an interesting physical phenomenon in its own right, but also because of their significance in relation to stochastic resonance (SR), a stochastic amplification effect which is normally assumed to be confined to bistable systems [29]. The perception of SR as a linear response phenomenon [30], describable in terms of a linear susceptibility obtained from the fluctuation dissipation theorem [1], has led to the discovery [31] of a new form of SR in monostable systems, arising from the large susceptibility associated with the ZDP's; called *zero-dispersion stochastic resonance*, it has been shown [32] to yield large enhancements of the signal-to-noise ratio if the damping of the system is sufficiently small.

The aim of the present paper is to present the results of a detailed study of ZDP's for systems with linear friction performing potential motion driven by white noise,

$$\ddot{q} = -\frac{\partial U}{\partial q} - \Gamma\dot{q} + f(t), \tag{3a}$$

where $\Gamma \ll 1$ is a friction coefficient and $f(t)$ is a white noise

$$\langle f(t)\rangle = 0, \qquad \langle f(t_1)f(t_2)\rangle = 2T\Gamma\delta(t_1 - t_2), \tag{3b}$$

and the noise intensity T is equivalent to a temperature in cases where the dissipation and noise both arise from coupling to a thermal bath. The study was based on experimental measurements of the fluctuation spectra of electronic models, which were used to check the predictions of the universality of the shape of the ZDP's. With the latter purpose in mind, we chose for the investigation two quite different potentials, describing different physical systems. The first of these was a tilted Duffing oscillator

$$U(q) = \frac{q^2}{2} + \frac{q^4}{4} + Aq, \tag{4}$$

where A is constant. This model could be used to describe, e.g., an electrical oscillator with a battery, or local vibrations in doped crystals [5] subject to a constant field. Note that, although $U(q)$ in (4) is a single-well potential as shown in Fig. 1(a), the dependence of its eigenfrequency $\omega(E)$ on energy displays a minimum, provided $|A| > 8/(7)^{\frac{3}{2}}$, as shown in Fig. 1(b) [33].

The second potential

$$U(q) = \cos q + \frac{B}{2}(q - q_0)^2. \tag{5}$$

With appropriate choice of the constants B and q_0, the potential (5) is multistable, as shown in Fig. 2(a). This model describes a single-junction superconducting quantum-interference device (SQUID) [4] and thus is of potential importance in terms of applications. Its variation of eigenfrequency $\omega(E)$ with energy is more complicated than for the model (4), typically exhibiting a number of extrema as shown in Fig. 2(b).

A brief review of the theory is presented in Sec. II. The electronic experiments used to reveal the ZDP's are described in Sec. III. The results obtained are discussed and compared with theory in Sec. IV. The work is summarized, and conclusions are drawn, in Sec. V. Details of the calculation of numerical values of parameters for the

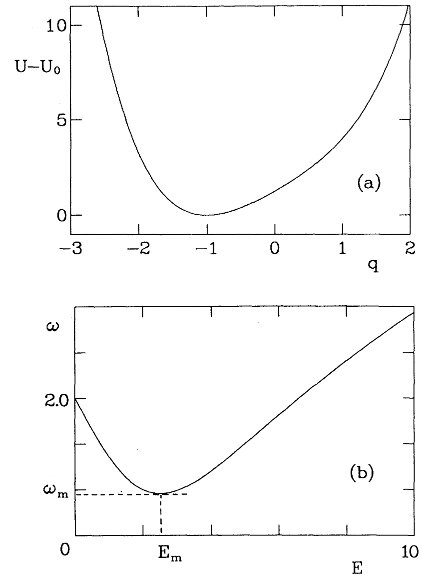

FIG. 1. (a) Plot of the potential $U(q)$ given by Eq. (4) with $A = 2$ for the tilted Duffing oscillator. The zero of potential energy has been shifted such that the energy of the minimum is zero. (b) Plot of the eigenfrequency $\omega(E)$ as a function of energy E for the same potential. The minimum of $\omega(E)$ is at $\omega_m = 1.791\,25$, $E_m = 2.51$.

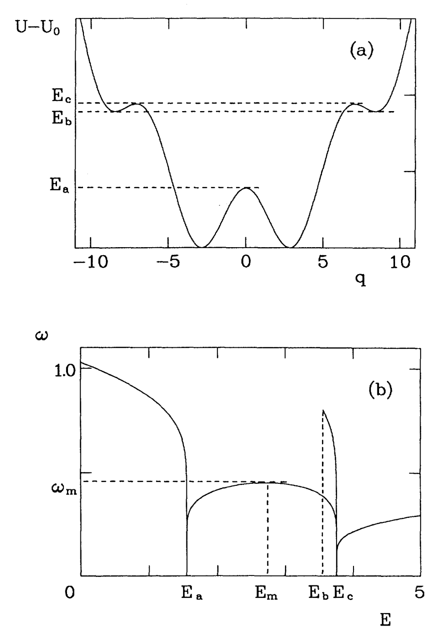

FIG. 2. (a) Plot of the potential $U(q)$ given by Eq. (5) with $B = 0.1$, $q_0 = 0$, for the SQUID model. The zero of potential energy has been shifted such that the energy of the lowest minima is zero. (b) Plot of the eigenfrequency $\omega(E)$ as a function of energy E for the same potential. The relevant numerical values are $\omega_0 = 1.029$, $\omega_m = 0.451$, $E_a = 1.55$; $E_m = 2.72$, $E_b = 3.56$, and $E_c = 3.76$.

two particular models selected for the investigation are given in the Appendix.

II. THEORY

We consider a system described by the stochastic equation of motion (3). The stationary distribution for the system is the Boltzmann distribution [2],

$$W_{st}(q,\dot{q}) = \frac{1}{Z}\exp(-E/T), \tag{6}$$

where $E = \frac{1}{2}\dot{q}^2 + U(q) - U_{\min}$ is an energy which, for convenience, we measure from the minimum of the potential, and $Z = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq\, d\dot{q}\exp(E/T)$ is the partition function.

The quantity in which we are interested is the spectral density of the fluctuations [1,2]

$$Q(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \exp(-i\Omega t)\langle(q(t)-\langle q\rangle)(q(0)-\langle q\rangle)\rangle, \tag{7}$$

where brackets $\langle\ \rangle$ denote averaging over both the equilibrium ensemble (6) [i.e., over initial conditions for the equation of motion (3a)] and realizations of the noise $f(t)$.

In the zero-friction limit the initial energy of the system does not vary in time, and the dynamics of the phase of the vibrations is determined completely by the eigenfrequency at this energy [34]

$$\frac{d\psi}{dt} = \omega(E), \qquad \Gamma = 0. \tag{8}$$

Allowing for the periodicity of the coordinate over phase [35], $q(E,\psi)$ can be expanded as a Fourier series

$$q \equiv q(E,\psi) = \sum_{n=-\infty}^{\infty} q^{(n)}(E)e^{-in\psi}, \qquad q^{(-n)} = (q^{(n)})^*. \tag{9}$$

Note that, in fact, (9) can be considered as the definition of the phase variable ψ.

In order to find the spectrum $Q(\Omega)$, one has only to average δ-shaped peaks at frequencies $\omega(E)$, and at corresponding harmonics, over a statistical (Boltzmann) distribution of energies. Thus it may readily be shown [24,26] that, in this zero-friction limit, the spectrum is given by

$$Q(\Omega) = Q^{(0)}(\Omega) = \frac{2\pi}{Z}\sum_{n=1}^{\infty}\sum_j \frac{1}{n}\left[\frac{\exp(-E/T)}{\omega(E)}|q^{(n)}|^2\frac{1}{|d\omega(E)/dE|}\right]\Bigg|_{E=E_j(\frac{\Omega}{n})}, \quad \Omega \neq 0, \tag{10a}$$

where $E_j(x)$ are determined by the equation

$$\omega(E_j(x)) = x \tag{10b}$$

and the summation over j in (10a) implies a summation over all the roots of Eq. (10b).

It is seen immediately from (10) that, if $\omega(E)$ has an extremum at some point E_m, then there are peaks of infinite magnitude at the extreme frequency $\omega_m = \omega(E_m)$ and at its harmonics, because the spectral density of oscillations [23] $q_n(\Omega) = \sum_j \left(|d\omega(E)/dE|\,|_{E_j(\Omega/n)}\right)^{-1}$ tends to infinity as Ω tends to $n\omega_m$ [cf. (1)]. These peaks are called *zero-dispersion peaks* (ZDP's).

In reality, of course, friction is always nonzero. Hence, the energy varies over time due to fluctuations and dissipation, and this results in a decay of the phase correlation. It is the dynamics of this decay that determines the shape of ZDP's. A special theoretical technique, based on an asymptotic solution of the Fokker-Planck equation, developed in [26], allows one to derive the shape of the ZDP explicitly. The main result is that its shape is universal, whereas the width of the peak and its magnitude are proportional respectively to $\sqrt{\Gamma}$ and $\Gamma^{-\frac{1}{4}}\exp(-E_m/T)$:

$$Q^{(\mathrm{ZDP})}(\Omega) = C_{\mathrm{scale}}S\left(\frac{\Omega - n\omega_m}{\Delta\Omega}\right), \qquad |\Omega - n\omega_m| \ll \omega_m, \qquad n = 1,2,3,\ldots \tag{11a}$$

where

$$\Delta\Omega = \mathrm{sgn}(\omega'')\sqrt{n\Gamma|\omega''|T\overline{p^2}},$$
$$C_{\mathrm{scale}} = \frac{4\sqrt{\pi}\exp(-E_m/T)|q^{(n)}(E_m)|^2}{Z\omega_m(n|\omega''|)^{3/4}(\Gamma T\overline{p^2})^{1/4}},$$
$$\omega'' \equiv \frac{d^2\omega(E_m)}{dE_m^2},$$
$$\overline{p^2} \equiv \frac{1}{2\pi}\int_0^{2\pi} d\psi\left(\omega_m\frac{\partial q(E,\psi)}{\partial\psi}\right)^2 = 2\omega_m^2\sum_{n=1}^{\infty} n^2|q^{(n)}(E_m)|^2, \tag{11b}$$

and

$$S(x) = \left|\mathrm{Re}\left[\int_0^{\infty} d\tau\frac{\exp(-ix\tau)}{\sqrt{(1-i)\sinh[(1-i)\tau]}}\right]\right|. \tag{11c}$$

The function $S(x)$, which determines the universal shape of the ZDP, is plotted in Fig. 3.

It is assumed here that $d^2\omega(E_m)/dE_m^2 \neq 0$, the most frequently occurring case in real systems. At the same time, it is possible in principle that the order of the lowest nonzero derivative in the extremum is higher than 2

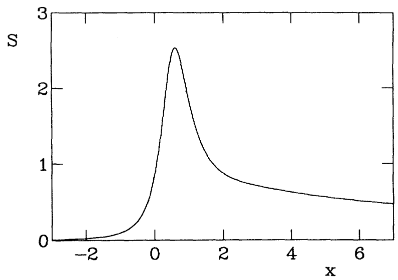

FIG. 3. Plot of the function $S(x)$ [see Eq. (11c)], which determines the universal shape of the ZDP.

(as for an inflection point, for instance). In this case, the shape of the ZDP will be different from (11c), although still universal, i.e., described by a function that is independent of any parameter. Thus the hierarchy of universal shapes of the ZDP depends on the order of the extremum of the functions $\omega(E)$. The quantitative analysis of this hierarchy, based on the reduction of the full Fokker-Planck equation to the asymptotic dimensionless one, and its further solution, will be presented elsewhere [36]. Here, we will present instead a qualitative explanation of the hierarchy in universality and corresponding scaling factors.

We suppose that, near the extremum, $\omega(E)$ is of the form

$$\omega(E) = \omega_m + \frac{1}{k!}\omega^{(k)}(E - E_m)^k, \quad k = 2, 3, 4, \ldots . \tag{12}$$

The characteristic time $t_c^{(k)}$ for the decay of phase correlation is determined by the fluctuations of energy, giving rise in turn to fluctuations of the phase derivative with respect to time $d\psi/dt \approx \omega(E)$ [cf. (8)] and correspondingly to a loss of phase correlation. In order to estimate $t_c^{(k)}$ we need to take into account the diffusionlike growth with time of the energy distribution (for an initially definite value of energy) [2]:

$$\Delta E(t) \sim \sqrt{\Gamma T \overline{p^2} t}. \tag{13}$$

Correspondingly, the distribution of the phase of vibration for energies close to E_m (in which region of energy $t_c^{(k)}$ is at its largest) is of the order of

$$\Delta\psi(t) \sim |\omega(E_m \pm \Delta E(t)) - \omega(E_m)| t \sim \frac{1}{k!}|\omega^{(k)}|(\Gamma T \overline{p^2})^{k/2} t^{\frac{k+2}{2}}. \tag{14}$$

The decay time $t_c^{(k)}$ of phase correlation corresponds to

$$\Delta\psi(t_c^{(k)}) \sim \pi. \tag{15}$$

Then it follows from (14) and (15) that

$$t_c^{(k)} \sim \frac{1}{|\omega^{(k)}/k!|^{\frac{2}{2+k}} (\Gamma T \overline{p^2})^{\frac{k}{k+2}}}. \tag{16a}$$

The corresponding scaling factor for the width of the ZDP is

$$|\Delta\Omega^{(k)}| = \frac{1}{t_c^{(k)}}. \tag{16b}$$

The characteristic band of energies near E_m, which are responsible for the ZDP, is of width $\sim \Delta E(t_c^{(k)})$, i.e., using (13) and (16),

$$\Delta E(t_c^{(k)}) \sim \left(\frac{\Gamma T \overline{p^2}}{|\omega^{(k)}|/k!}\right)^{\frac{1}{k+2}}. \tag{17}$$

The scaling factor for the magnitude of the ZDP at ω_m is [cf. the definition (11b)]

$$\begin{aligned} C_{\text{scale}}^{(k)} &\sim \frac{\exp(-E_m/T)}{Z\omega_m} \Delta E(t_c^{(k)}) |q^{(1)}(E_m)|^2 t_c^{(k)} \\ &\sim \frac{\exp(-E_m/T)|q^{(1)}(E_m)|^2}{Z\omega_m |\omega^{(k)}/k!|^{\frac{3}{k+2}} (\Gamma \overline{p^2} T)^{\frac{k-1}{k+2}}}. \end{aligned} \tag{18}$$

For the case $k = 2$, the expressions for the scaling factors (16), (17), and (18) reduce to (11b).

The dynamics of the decay of phase correlation is determined [cf. (8)] by a diffusionlike growth (13) of the energy distribution and by its influence on phase fluctuations, i.e., by the power-type function $\omega(E)$ (12). Correspondingly, a change of parameters leads only to a rescaling of time and energy, and thus the shape of the ZDP depends only on the value of k in the expression for $\omega(E)$ (12). This universality can be proven rigorously by means of a Fokker-Planck equation treatment [36].

Now we return again to consider the most widespread and important (for applications) case of an extremum for which $k = 2$, i.e., the situation for which the dependence of eigenfrequency upon energy $\omega(E)$ (12) has either a minimum or maximum of parabolic form. It has recently been shown theoretically [27] that, not only is the shape of the ZDP universal in the low-friction limit, but the evolution with temperature *towards* this shape has a scaling property as well, i.e., after the ZDP near $\Omega = n\omega_m$ arises at some critical temperature $T_c^{(n)}$, the evolution of its shape towards the universal one with further growth of temperature is governed by only one parameter $\gamma/n^{\frac{1}{4}}$, where

$$\gamma = \frac{\Delta E}{T} \equiv \left(\frac{\Gamma \overline{p^2}}{|\omega''| T^3}\right)^{\frac{1}{4}}, \tag{19a}$$

and

$$\Delta E \equiv \Delta E(t_c^{(2)}) \tag{19b}$$

is the characteristic scale of energy (17) for the case of parabolic maximum or minimum ($k = 2$):

$$Q(\Omega) \simeq C_{\text{scale}} \tilde{S}\left(\Delta\Omega,\ \frac{\gamma}{2n^{\frac{1}{4}}},\ \frac{x_m n^{\frac{1}{4}}}{2}\right),$$

$$|\Omega - n\omega_m| \ll \omega_m, \qquad n = 1, 2, 3, \ldots . \quad (20a)$$

Here, C_{scale} and $\Delta\Omega$ are defined in (11b),

$$x_m = \frac{E_m}{\Delta E} \equiv \left(\frac{E_m^4 |\omega''|}{\Gamma T p^2}\right)^{\frac{1}{4}}, \quad x_m \gg 1, \quad (20b)$$

$$\gamma x_m < 5 \ln(x_m n^{\frac{1}{4}}/2),$$

and

$$\tilde{S}(x,y,z) = \left| \text{Re} \int_0^\infty d\tau \frac{\exp\left[-ix\tau + y^2\left(\frac{2/(1-i)}{\tanh[(1-i)\tau/2]} - \tau\right)\right]}{2\{(1-i)\sinh[(1-i)\tau/2]\}^{\frac{1}{2}}} \times \text{erfc}\left(-\{-(1-i)\tanh[(1-i)\tau]\}^{\frac{1}{2}} z + y\frac{1+\text{sech}[(1-i)\tau]}{\{(1-i)\tanh[(1-i)\tau]\}^{\frac{1}{2}}}\right)\right|, \quad (20c)$$

where $\text{erfc}(x) \equiv 2/\sqrt{\pi}\int_x^\infty dy \exp(-y^2)$ is the complement of the error function with respect to 1.

It was shown in [27] that the function $\tilde{S}(x,y,z)$ has the following structure for $z \gg 1$:

$$\tilde{S}(x,y,z) = S^{(\text{ZDP})}(x,y) + \frac{\epsilon}{8\sqrt{\pi}} \frac{\exp(4yz)}{z^5},$$

$$\epsilon \sim 1, \quad S^{(\text{ZDP})} \sim 1 \quad \text{at} \quad |x| <\sim 1, \quad y <\sim 1, \quad z \gg 1. \quad (21)$$

Thus for $1 \ll z < (5/4y)\ln(5/4y)$ (under which conditions the ZDP is manifested), $\tilde{S}(x,y,z)$ does not depend on z. The evolution of $\tilde{S}$ with y is shown in Fig. 4. As $y \to 0$, $S^{(\text{ZDP})}(x,y) \to S(x)$ (11c).

The existence of a critical temperature $T_c^{(n)}$ for the manifestation of the ZDP is due to contributions to the spectrum at $\Omega = n\omega_m$ from lower energies in the distribution, in particular from the relaxation-induced tail caused by vibrations in the bottom of the potential well. The latter can be calculated at low temperatures, as for a harmonic oscillator of eigenfrequency equal to the eigenfrequency in the bottom of the potential [2,27]

$$Q^{(\text{harmonic})}(\Omega, \Omega_0) = \frac{\Gamma T}{\pi(\Omega^2 - \Omega_0^2)^2},$$

$$\Omega_0 \equiv \omega(0), \qquad |\Omega - \Omega_0| \gg \Gamma, \qquad T \ll E_m. \quad (22)$$

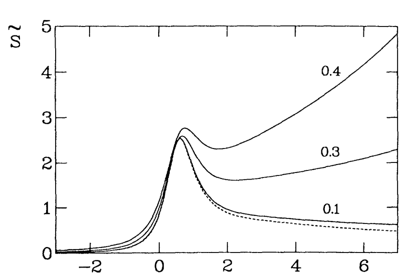

FIG. 4. Plots of the shape $\tilde{S}(x,y,z)$ of the ZDP (21) for $z = 3$ and various values of y (numbers adjacent to curves) showing how it evolves towards the universal shape $S(x)$ of Eq. (11c) (dashed curve) as $y \to 0$.

Since the magnitude of the ZDP is proportional to the Boltzmann factor $\exp(-E_m/T)$ [see (11b)], its contribution to the spectrum becomes negligible at small enough temperatures. In order to find the critical temperature, one should compare C_{scale} (11b) with the relaxation-induced tail (22) and with the "nondissipative" contribution [see (10a)] from the lowest root of (10b), i.e., the energy E_l at which $\omega(E_l) = (n/i)\omega(E_m)$, $i = 1, 2, 3, \ldots$, $E_l \neq E_m$ (if such an energy exists).

In the asymptotic limit $\Gamma \to 0$ (when Γ is smaller than any other relevant parameter)

$$T_c^{(n)} \equiv T_c = \frac{4\max\{E_m/5; E_m - E_l\}}{\ln(\omega_m/\Gamma)}, \quad \Gamma \to 0, \quad (23)$$

although, strictly T_c is described by the formula (23) only at extremely small Γ (cf. Sec. III) because it is the smallness of $\ln^{-1}(\Gamma^{-1})$ that is important, rather than that of Γ itself.

As can be seen from (19) and (23), the parameter γ, which determines the shape of the ZDP (20), is much less than 1 at $T = T_c$ in the asymptotic limit $\Gamma \to 0$. Thus, in this limit, the ZDP's rise *extremely* rapidly as T is increased beyond its critical value of T_c; their shape is described by the universal function S (11c).

III. THE ANALOG EXPERIMENT

In order to test the above ideas, we have investigated the properties of the predicted ZDP's through analog electronic experiments. To provide a check on the predictions of universality, two very different models were chosen: the tilted Duffing oscillator (TDO) described by Eqs. (3) and (4) with $A = 2$ [see also Fig. 1(a)]; and the SQUID model described by Eqs. (3) and (5) with $B = 0.01$ [see also Fig. 2(a)]. The calculations of $\omega(E)$ and of some relevant parameters for these models are presented in the Appendix. Their eigenfrequencies as functions of energy, $\omega(E)$, are plotted in Figs. 1(b) and 2(b) respectively, where they can be seen immediately to be very different from each other.

The basis of the analog technique has been described in detail elsewhere [22] but, in essence, is extremely simple.

An electronic model of the stochastic differential equation under study is built using standard analog components (operational amplifiers, multipliers, etc.). This is then driven by noise from an external noise generator, and the response of the model is analyzed with the aid of a digital data processor. Systematic uncertainties in measurements of this kind arise from departures of passive components (e.g., resistors, capacitors) from their nominal values and, usually of greater significance, from the internal noise and slightly nonideal responses of the active components (e.g., operational amplifiers, analog multipliers). Consequently, there is a net systematic error of typically a few percent [22], as is the case for the particular models to be considered below. The statistical uncertainties in the results show up as scatter, and can be estimated from the departure of the data points from a smooth curve drawn through them. The statistical errors can in principle always be reduced by extending the time of data acquisition, i.e., by increasing the number of realizations included in the average. Compromise is necessary, however, partly because of the need to complete the research on a reasonable schedule, and partly because extended acquisition periods can exacerbate the effect of thermal drift in the system, leading, in turn, to increased systematic errors.

The circuit model of the TDO was exactly the same as described previously when used for the investigation of noise-induced spectral narrowing [33], except that, in order to provide the required extremely small value of Γ, the corresponding feedback resistor on the first integrator was necessarily rather larger ($\sim$ 100 MΩ) than would normally [22] be used in circuit modeling experiments. This exacerbated the effect of leakage currents, stray capacitance, and other nonidealities of the components. Consequently, a precise value of Γ could not be calculated from the nominal values of the components. Instead, Γ was measured experimentally, using two entirely different methods. First, having checked that the oscillator modeled (3) and (4) correctly and that it resonated at the calculated frequency, it was driven precisely at resonance by adding an extremely weak periodic force of amplitude F from a frequency synthesizer in place of $f(t)$ in (3), enabling Γ to be determined directly from the amplitude F/Γ of the resultant velocity $\dot{q}(t)$. Second, Γ was measured by driving the circuit with quasiwhite Gaussian noise of mean-square voltage $\langle V_N^2\rangle$ from a noise generator [equivalent to $f(t)$ in (3)]. Measurement of the second moment of the velocity $\langle \dot{q}^2\rangle$ and use of the principle of equipartition of energy $\langle \dot{q}^2\rangle = T$ then enabled Γ to be calculated from the circuit's noise-scaling relation [22]

$$T = \frac{\tau_N}{\Gamma\tau_I}\langle V_N^2\rangle,$$

where τ_I is the integrators' time constant and $\tau_N \ll \tau_I$ is the correlation time of the noise. The values of Γ obtained by means of these two independent methods were larger by 13% than those calculated from the nominal component values, but agreed with each other to within 2%, which was well within the experimental error of the measurements, leading to a final result of $\Gamma = (2.4 \pm 0.1) \times 10^{-3}$.

The circuit of the SQUID model is shown in (slightly simplified) block form in Fig. 5. It was designed and scaled in the standard way [22] so as to optimize use of the dynamic range of the active components. Thus, the actual equation simulated was the integral form (see Fig. 5) of

$$f(t)\frac{R_4}{R_1} + 10\sin q - \frac{R_4}{R_5}q = R_4C_1R_3C_2\ddot{q} + \frac{R_4R_3C_2}{R_2}\dot{q},$$

with nominal values

$$R_1 = 22\ \mathrm{k}\Omega,$$

$$R_2 \simeq 10^8\ \Omega\ \ \text{(see text)},$$

$$R_3 = R_4 = R_5 = 10^5\ \Omega,$$

$$C_1 = C_2 = 10\ \mathrm{n}F,$$

$$\tau' = \tau_1 = R_4C_1 = \tau_2 = R_3C_2 = 10^{-3}\ \mathrm{s}.$$

With the exception of R_2, the resistances and capacitances all lay within $\pm 1\%$ of their nominal values. Thus, the circuit actually simulated

$$\tau'^2\ddot{q} + \Gamma'\tau'\dot{q} = 10\sin q - q + \frac{100}{22}f'(t),$$

where τ' and Γ' are readily related to the τ and Γ in the theory by means of the scaling relations

$$\tau = \frac{\tau'}{\sqrt{10}}, \qquad \Gamma = \frac{\Gamma'}{\sqrt{10}}, \qquad f(t) = \frac{10}{22}f'(t).$$

The sine function was obtained from an AD639 integrated circuit [37]. As in the case of the TDO (above), the value of Γ was made extremely small by use of a very large feedback resistor R_2 across the first integrator. Using the same two independent experimental methods described above, Γ was measured as $(2.8 \pm 0.2) \times 10^{-4}$.

The experimental results are presented in Figs. 6 and 7 for the TDO and SQUID models, respectively. The insets show the spectrum in the immediate vicinity of the ZDP in each case, plotting the spectral density as

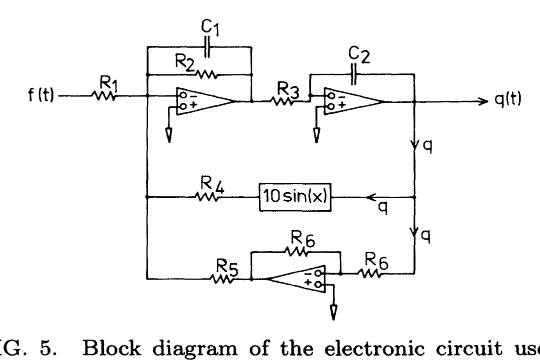

FIG. 5. Block diagram of the electronic circuit used to model the SQUID, Eqs. (3) and (5).

the discrete data points obtained from the fast-Fourier-transform (FFT) averaging technique used to measure it. In Fig. 6, the insets refer to the ZDP's at the fundamental frequency; in Fig. 7, the ZDP's at the fundamental are plotted in the left-hand insets and those at the third harmonic in the right-hand insets. In the latter case, the statistics are relatively poor except [Fig. 7(c)] at relatively strong noise intensity.

IV. DISCUSSION

Before discussing the results in detail, we note that it was impossible in practice to obtain reliable measurements of spectra in the ZDP region for small values of T. The reason is that the *statistical* average of the theory is replaced, in the experiments, with an average *over time*, which is correct if the latter extends over an infinite period of time. However, the period of data acquisition is of course always finite in real experiments. For temperatures $T \ll E_m$, the probability of the system having an energy E_m is small and, correspondingly, the time interval that elapses between such events for the random process $q(t)$ given in (3) will be large [be-

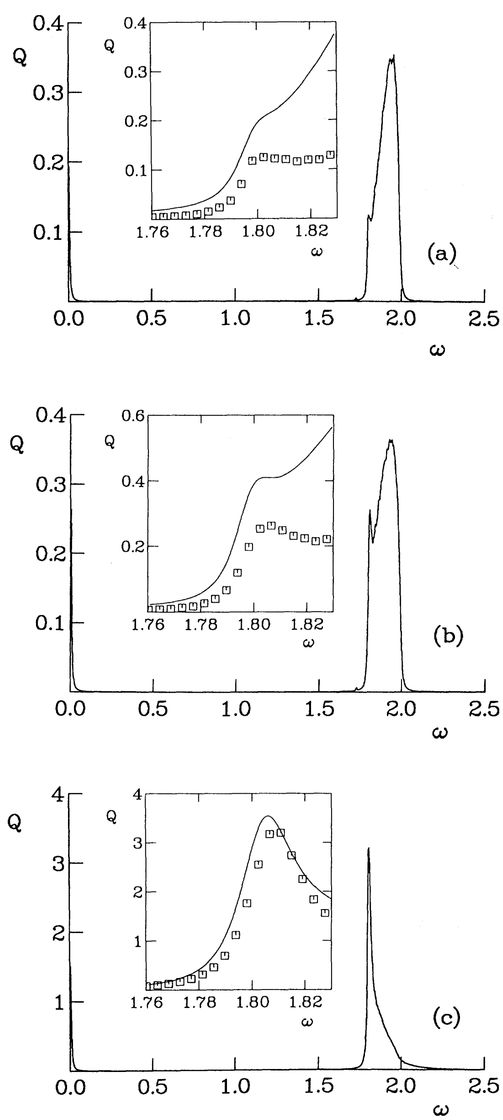

FIG. 6. Spectral densities of fluctuations measured experimentally for the electronic model of the TDO, Eqs. (3) and (4), for various temperatures (noise intensities) T: (a) $T = 0.370$; (b) $T = 0.445$; (c) $T = 1.5$. The insets in each case show a comparison, on expanded scales, of the experimental ZDP with the theoretical prediction for the fundamental peak.

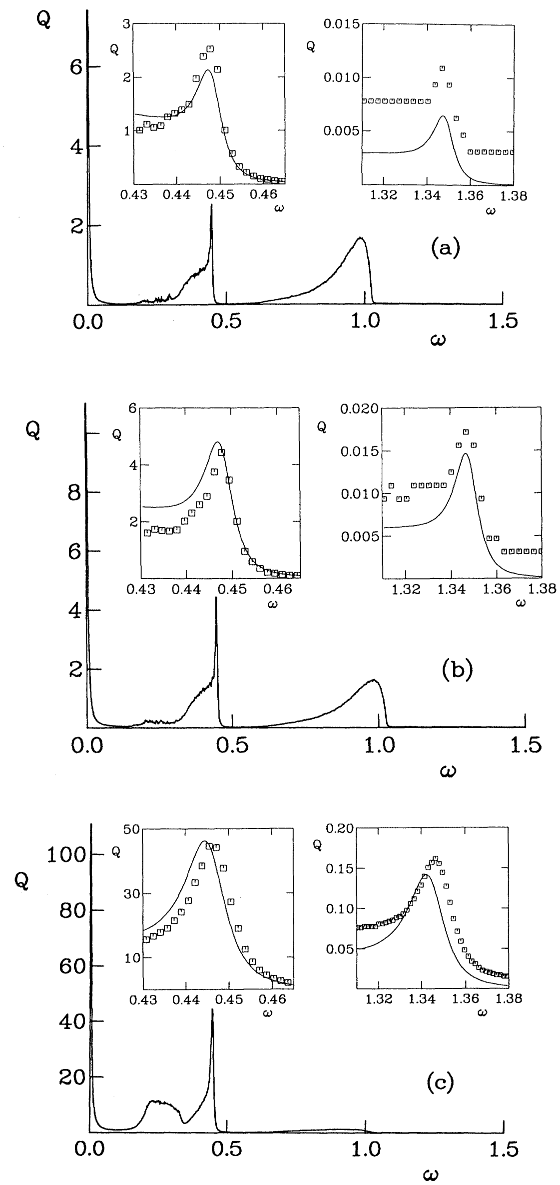

FIG. 7. Spectral densities of fluctuations measured experimentally for the electronic model of the SQUID, Eqs. (3) and (5), for various temperatures (noise intensities) T: (a) $T = 0.394$; (b) $T = 0.463$; (c) $T = 1.3$. The insets in each case show a comparison, on expanded scales, of the experimental ZDP with the theoretical prediction for the fundamental (left-hand insets) and third-harmonic (right-hand insets) peaks.

ing $\sim \Gamma^{-1}\exp(-E_m/T)$]. Given that the practical upper limit on the time of data acquisition $\tau_{\rm expt} \sim 1$ day, and that $\Gamma^{-1} \sim 3$ s in real time, the lower limit of T was determined as

$$T_l \simeq \frac{E_m}{\ln(\Gamma\tau_{\rm expt})} \sim \frac{E_m}{9}. \tag{24}$$

Thus, for the parameters of the experiments, $T_l \sim 0.3$ for both the TDO and SQUID models. If we calculate critical temperatures for the various harmonics of the ZDP's as described in Sec. II, we obtain: for the TDO, $T_c^{(1)} \simeq$ 0.18–0.2; and for the SQUID, $T_c^{(1)} \simeq$ 0.15–0.16, $T_c^{(3)} \simeq$ 0.30–0.32. Consequently, we were able to observe the *full* evolution of a ZDP (i.e., right from first onset until the peak has attained its universal shape) only for the third harmonic for the SQUID model. The theoretical predictions for the shape of the ZDP (full curves) are compared with the experimental measurements (data points) in the insets of Figs. 6 and 7.

The evolution of the shape of the experimental ZDP in its third harmonic for the SQUID model (right-hand insets of Fig. 7) can be seen to be in remarkably good agreement with theory: note that the theoretical curves plot only the ZDP itself, and do not include the other contributions to the spectrum. In Fig. 7(a), for $T = 0.394$, which is close to the threshold temperature, the ZDP is clearly manifested but is superimposed on the large relatively flat "pedestal" due to other spectral contributions; the statistics are relatively poor, but the *shape* of the ZDP is nonetheless seen to be very close to the theoretical prediction. In Fig. 7(b), with $T = 0.463$, the ZDP has already become the dominant contribution to the spectrum; with further increase of T, its shape evolves towards the universal one [Fig. 7 (c)].

The results for the fundamental frequency also allow us to draw important conclusions about the evolution of the ZDP's shape. First, it may be noted from the inset to Fig. 6(c), and the left-hand inset to Fig. 7(c), that the ZDP's approach the universal shape for large enough T, when $\gamma \ll 1$ (cf. Fig. 4). The small discrepancies in frequency are attributable to nonidealities in the circuit model, and lie well within the experimental error of $\sim 2\%$; the discrepancies in magnitude arise from the experimental uncertainty of $\pm 2\%$ in T, which in turn gives rise to a much larger uncertainty in the scaling factor $C_{\rm scale} \propto \exp(-E_m/T)$.

The evolution with T of the ZDP's on the fundamental coincides qualitatively with the universal evolution described by Eq. (20) and plotted in Fig. 4. In particular [cf. Fig. 6(a)], at small T where the parameter γ is not small, the ZDP is more like a step than a peak. The quantitative agreement between theory and experiment becomes poorer as T decreases. One reason (see above) is the sensitivity of $C_{\rm scale}$ to any inaccuracy in T; but the main reason is presumably that, when the parameter γ becomes comparable with unity, the shape of the ZDP will differ from that of the universal function $\tilde{S}$ given by (20) and will be defined, instead, by the individual characteristics of the particular system being investigated. The results are thus very much in accord with theory [27], which predicts that the departure from universal shape in the vicinity of the ZDP, i.e., for $|\Omega - n\omega_m| <\sim |\Delta\Omega| \equiv (n\Gamma|\omega''|T\overline{p^2})^{\frac{1}{2}}$,

$$\delta Q(\Omega) \sim \frac{\max\{1, \gamma x_m\}}{x_m^2}. \tag{25}$$

Thus, the deviation is expected to increase with distance from $n\omega_m$, consistent with the experimental results shown in Figs. 6(a)–6(c) insets, and 7(a)–7(c) left-hand insets.

V. CONCLUSIONS

The experimental data presented above, and their comparison with theory, allow a number of conclusions to be drawn. If the dependence of eigenfrequency $\omega(E)$ on energy E for the underdamped oscillator (3) possesses a maximum or minimum at some energy E_m, then the spectral density of its fluctuations will exhibit sharp peaks (ZDP's) at the extremal frequency $\omega_m = \omega(E_m)$ and at its harmonics $n\omega_m$.

For large enough temperature (noise intensity) T or, equivalently, for sufficiently small values of the dissipation parameter Γ, the shape of the ZDP's can be described in terms of the universal function $S((\Omega - n\omega_m)/\Delta\Omega)$ of Eq. (11), plotted in Fig. 3; the parameters of the particular system and noise determine only the scaling factors.

The magnitude of the ZDP decreases rapidly with decreasing T [being $\propto \exp(-E_m/T)$] and, at some critical temperature $T_c^{(n)}$, depending on the particular system considered, the ZDP becomes smaller than a pedestal determined by the relaxation-induced tail of the ordinary intrawell vibrations and other contributions from energies far from E_m.

For $T > T_c^{(n)}$, the evolution of the shape of the ZDP with increasing T is governed primarily by a single parameter (18),

$$\gamma_n \equiv \left[\frac{\Gamma\overline{p^2}}{n|\omega''|T^3}\right]^{\frac{1}{4}},$$

and the shape is described by the function $S^{\rm ZDP}((\Omega - n\omega_m)/\Delta\Omega, \gamma_n)$ given by (20) and (21) and plotted in Fig. 4. The universal shape $S(x)$ given by (11) corresponds to the limit where $\gamma_n \ll 1$. The accuracy of this universal description is good in the close vicinity of the ZDP [i.e., $(|\Omega - n\omega_m| <\sim \Delta\Omega)$], but it decreases with increasing distance from $n\omega_m$.

For sufficiently small Γ, the parameter $\gamma_n \ll 1$, even at the critical temperature $T_c^{(n)}$. In such cases, the shape of the ZDP transforms to the universal shape almost discontinuously as T exceeds $T_c^{(n)}$. For larger Γ, the ZDP acquires a steplike shape for T close to $T_c^{(n)}$, which then evolves relatively slowly towards the universal shape S given by Eq. (11) with a further increase of T.

The hierarchy of universalities should depend only on the type of extremum in $\omega(E)$. The universal function S given by Eq. (11), and described above, corresponds to the most widespread case in which the extremum is quadratic, corresponding to a maximum or minimum.

Finally, we note that the new phenomenon of *zero-dispersion stochastic resonance* (ZDSR), closely associated with the ZDP's, has recently been discovered [32] and discussed [33]. It may be expected that, corresponding to the universality of the ZDP's shape and evolution discussed above, there will be related universal features in ZDSR for small enough Γ. The results, for the case of Eqs. (3) and (5), could be useful in relation to experiments and applications on SQUID's.

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APPENDIX

The dependence of the eigenfrequency upon energy can be calculated numerically for any potential as

$$\omega(E) = \pi \left(\int_{q_{tl}}^{q_{tr}} \frac{dq}{\sqrt{2(E-U(q))}} \right)^{-1}, \tag{A1a}$$

where q_{tl} and q_{tr} are the turning points, i.e. roots of the equation

$$U(q) = E \tag{A1b}$$

[in the particular case of the tilted Duffing oscillator, the integral in (A1a) can be expressed in terms of elliptic functions [33]].

In order to calculate the scaling factors $C_{\rm scale}, \Delta\Omega$, and the parameter γ in (20), we need to know some parameters at $E = E_m$ [see (11b) and (19)]. The quantities ω_m, E_m, and $\omega'' \equiv d^2(E)/dE^2|_{E_m}$ are easily found once $\omega(E)$ (A1a) has been calculated. The partition function Z can be expressed in terms of $\omega(E)$ by use of the canonical transformation [34] to action angle variables I, ψ, and the formula [34]

$$\frac{dI}{dE} = \omega^{-1}(E). \tag{A2}$$

Then

$$Z = 2\pi \int_0^\infty dE \frac{\exp(-E/T)}{\omega(E)}. \tag{A3}$$

Harmonics $q^{(n)}(E_m)$ can be calculated numerically from the definition (9)

$$q^{(n)}(E) = \frac{1}{2\pi} \int_0^{2\pi} d\psi q(E,\psi) e^{in\psi},$$

or, allowing for [cf. (8)],

$$\psi(E,q) = \omega(E) \int_{q_{tl}}^{q} \frac{dq}{\sqrt{2(E-U(q))}}, \tag{A4}$$

we obtain

$$q^{(n)}(E) = \frac{\omega(E)}{4\pi} \int_{q_{tl}}^{q_{tr}} dq \frac{q \cos\left\{ \omega(E) \int_{q_{tl}}^{q} \frac{dq'}{\sqrt{2(E-U(q'))}} \right\}}{\sqrt{2(E-U(q))}}. \tag{A5}$$

The parameter $\overline{p^2}$ can be calculated either directly from the definition (11b) or, equivalently, as [38]

$$\overline{p^2} = \omega(E_m) I(E_m) \equiv \omega_m \int_0^{E_m} dE \frac{1}{\omega(E)}. \tag{A6}$$

Thus, using (A1), (A5), and (A6), we obtain for the models used in our experiments:

(i) TDO model (2) with $A = 2$: $\omega(E)$ as depicted in Fig. 1(b). $E_m = 2.51$; $\omega_m = 1.795$; $\omega'' = 0.062$; $\overline{p^2} = 2.41$; $|q^{(1)}|^2 = 0.355$.

(ii) SQUID model (3) with $B = 0.1$: $\omega(E)$ as depicted in Fig. 2(b). $E_m = 2.72$; $\omega_m = 0.4514$; $\omega'' = -0.119$; $\overline{p^2} = 2.89$; $|q^{(1)}|^2 = 6.61$; $|q^{(3)}|^2 = 0.0462$.

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