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Stochastic resonance for periodically modulated noise intensity

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A different form of stochastic resonance, in which the weak periodic force is applied multiplicatively (rather than additively) in the noise, has been investigated for a Brownian particle moving in a double-well potential. A regular periodic signal whose amplitude increases sharply with increasing noise intensity is shown to arise when the potential is *asymmetric*. The experimental measurements are in good agreement with a theoretical analysis.

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The response of a noisy bistable system to a periodic signal can sometimes be enhanced by the introduction of additional external noise. This highly counterintuitive effect, called *stochastic resonance* (SR), was proposed as a possible explanation of the earth's ice-age cycle [1]. Currently, however, it is being studied mainly for its intrinsic interest as an important scientific phenomenon in its own right. It has already been convincingly demonstrated in bistable lasers [2], in a passive optically bistable system [3], in an electron-spin-resonance (ESR) system [4], and in analog electronic experiments [5–7]; arguably, it is of relevance to the operation of sensory neurons [8]. The phenomenon follows from linear-response theory (LRT) for bistable systems [9] and can be discussed theoretically in terms of a Fokker-Planck analysis [3,10–12], or through the application [6,7] of the fluctuation dissipation theorem.

The overwhelming majority of previous studies of SR have related to the case where the external noise and the weak periodic force are introduced additively. In the present paper, we discuss the rather different situation which arises when the noise and the periodic force are introduced multiplicatively, so that the former is modulated by the latter. Periodically modulated noise is not uncommon and arises, for example, at the output of any amplifier (e.g., in optics or radio astronomy) whose amplification factor varies periodically with time. It is of obvious importance, therefore, to establish whether or not a modulated zero-mean noise can give rise to a periodic signal in the system it is driving. Such an effect would not, of course, occur in a linear system where the signal is directly proportional to the driving noise so that they must both, on average, vanish. We shall show, however, that in a *bistable* system a periodic signal does arise and, furthermore, that in bistable systems there occurs a form of SR for periodically modulated noise. It has some novel features that are strikingly different from those that are now well established for the case of conventional SR.

To demonstrate the onset of SR and to reveal its characteristic features, we treat the simplest nontrivial system: an overdamped Brownian particle moving in an asymmetric bistable potential, with equation of motion

$$\dot{q} + U'(q) = f(t) \equiv [\tfrac{1}{2}A\cos(\Omega t) + 1]\xi(t), \qquad (1)$$
$$U(q) = -\tfrac{1}{2}q^2 + \tfrac{1}{4}q^4 + \lambda q.$$

Here, λ characterizes the asymmetry of the potential. For $-2/(3\sqrt{3}) < \lambda < 2/(3\sqrt{3})$ the potential $U(q)$ has two minima, i.e., the system is bistable. The function $\xi(t)$ represents white Gaussian noise of intensity D, so that

$$\langle f(t)f(t')\rangle = 2D\delta(t-t')\left[1 + A\cos(\Omega t) + \frac{A^2}{8}[1+\cos(2\Omega t)]\right], \qquad (2)$$

i.e., the intensity of the driving force $f(t)$ is periodic in time. In what follows, we assume the modulation to be weak, $A \ll 1$, and neglect the term in A^2 in (2).

For sufficiently weak noise, when D is much less than the depths $\Delta U_{1,2}$ of the potential wells,

$$D \ll \Delta U_1,\ \Delta U_2, \quad \Delta U_n = U(q_s) - U(q_n), \quad n = 1,2$$
$$U'(q_{1,2}) = U'(q_s) = 0, \quad q_1 < q_s < q_2 \qquad (3)$$

the motion of the system consists mostly of small intrawell fluctuations about the equilibrium positions $q_{1,2}$. Occasionally, there will be large fluctuations, sufficient to cause interwell transitions. Periodic modulation of the noise influences both types of fluctuation, and so there are two contributions to the signal $\langle q(t)\rangle$: one from the modulation of the intrawell fluctuations; and the other from the modulation of the populations $w_{1,2}(t)$ of the wells 1, 2

$$\langle q(t)\rangle \simeq \sum_{n=1,2} \langle q(t)\rangle_n w_n(t) \qquad (4)$$

where $\langle\,\rangle_n$ implies averaging over the nth well. In the spirit of LRT, the periodic response to the modulation can be described by a generalized susceptibility $\kappa(\Omega)$

$$\langle q(t)\rangle = \langle q\rangle^{(0)} + A\,\mathrm{Re}\,[\kappa(\Omega)\exp(-i\Omega t)] \qquad (5)$$

where the superscript (0) means that the corresponding quantity refers to the case $A = 0$.

We shall consider the response for the physically important case of low-frequency modulation, $\Omega \ll U''(q_{1,2})$, where the adiabatic approximation holds. Both the intrawell fluctuations and the transition probabilities W_{12}, W_{21} are then the same as they would be for white noise of instantaneous intensity D $(1 + A \cos \Omega t)$. The well populations w_1, w_2 for periodically modulated noise depend on the relationship between Ω and the W_{nm}.

To lowest order in the modulation amplitude A, the probability W_{nm} of an $n \to m$ transition is

$$W_{nm} \equiv W_{nm}(t) \simeq W_{nm}^{(0)} \left(1 + A \frac{\Delta U_n}{D} \cos \Omega t\right) \qquad (6)$$

where $W_{nm}^{(0)} \propto \exp(-\Delta U_n / D)$ is the usual Kramers transition rate. The corresponding periodic modulation of the well populations $w_{1,2}$ is described by the balance equation $\dot{w}_1 = -W_{12} w_1 + W_{21} w_2$. The periodic redistribution over the wells gives a contribution $\kappa_{\rm tr}(\Omega)$ to the susceptibility $\kappa(\Omega)$ of the form

$$\kappa_{\rm tr}(\Omega) = -\frac{1}{D}(q_1 - q_2)(\Delta U_1 - \Delta U_2) w_1^{(0)} w_2^{(0)} \frac{W^{(0)}}{W^{(0)} - i\Omega},$$

$$W^{(0)} = W_{12}^{(0)} + W_{21}^{(0)}, \qquad (7)$$

$$w_1^{(0)} = W_{21}^{(0)} / W^{(0)}, \quad w_2^{(0)} = 1 - w_1^{(0)}.$$

In obtaining (7) from (4)–(6), we have neglected the deviations of $\langle q \rangle_n$ from q_n in comparison with $|q_2 - q_1|$. According to (6) and (7),

$$|\kappa_{\rm tr}(\Omega)| \propto \zeta \exp(-\zeta), \quad \zeta = |\Delta U_1 - \Delta U_2| / D, \qquad (8)$$

i.e., the interwell transitions contribute to $\kappa(\Omega)$ provided that the potential is asymmetric. This is easily understood qualitatively. For a symmetric potential, the wells are equally populated irrespective of noise intensity and so the modulation of the latter does not influence the populations w_1, w_2. For asymmetric potentials, on the other hand, the ratio of the populations $w_1^{(0)}/w_2^{(0)} \propto \exp [(\Delta U_2 - \Delta U_1)/D]$ depends sharply on the noise intensity, and will be strongly influenced by the modulation of D. It is also evident that, for very large ζ, a weak modulation will not result in a substantial redistribution over the wells because the product $w_1 w_2 \propto \exp(-\zeta)$ will remain exponentially small: $|\kappa_{\rm tr}(\Omega)|$ must therefore vary nonmonotonically with $\zeta \propto D^{-1}$, with a maximum at $\zeta = 1$, and increase rapidly with D in the range $\exp(\zeta) \gg 1$. This increase can in itself give rise to stochastic resonance [1], since the periodic signal is rising rapidly with increasing noise intensity.

However, the intrawell fluctuations are also to be considered. Their contribution to the susceptibility $\kappa(\Omega)$ is connected with the local asymmetry of the potential about each of its minima (just as for the zero frequency peaks in the power spectra of single-well underdamped systems [13]). The partial susceptibility for the nth well, $\kappa_n(\Omega)$, can be obtained for small D by expanding $U(q)$ in (1) to second order in $(q - q_n)$ and calculating $(q - q_n)$ formally to second order in $f(t)$. For $\Omega \ll U''(q_n)$ one arrives at the expression

$$\kappa_n(\Omega) = -U'''(q_n) \left[U''(q_n)\right]^{-2} D/2. \qquad (9)$$

The susceptibility $\kappa(\Omega)$ as a whole is then given by the sum of the above contributions

$$\kappa(\Omega) = \sum_{n=1,2} \kappa_n(\Omega) w_n^{(0)} + \kappa_{\rm tr}(\Omega). \qquad (10)$$

Equations (5), (7), (9), and (10) describe completely the periodic response of the system to periodically modulated noise. Following Ref. [2], the influence of the noise intensity on the response can be characterised by a signal-to-noise ratio R equal to the ratio of the δ-like spike in the power spectral density of the fluctuations of the system

$$Q(\omega) = \frac{1}{4\pi\tau} \left| \int_{-\tau}^{\tau} dt\, e^{i\omega t} q(t) \right|^2, \quad \tau \to \infty \qquad (11)$$

at the modulation frequency Ω to the value $Q^{(0)}(\Omega)$ of $Q(\Omega)$ in the absence of modulation. According to (5)

$$R = \tfrac{1}{4} A^2 |\kappa(\Omega)|^2 / Q^{(0)}(\Omega) \qquad (12)$$

[cf. Ref. [6] where a similar equation was given for the case of additive periodic forcing; note, however, that in contrast with Refs. [6,7], the effective susceptibility $\kappa(\Omega)$ is not now given directly by the fluctuation dissipation theorem in terms of $Q^{(0)}(\omega)$].

The most interesting and important situation arises when the main contributions to both $\kappa(\Omega)$ and $Q^{(0)}(\Omega)$ are due to fluctuational interwell transitions. In this case, (12) simplifies and, allowing for the explicit form [6] of $Q^{(0)}(\Omega)$, one obtains

$$R \simeq R_{\rm tr} = \frac{\pi}{4} A^2 \zeta^2 W_{12}^{(0)} W_{21}^{(0)} / (W_{12}^{(0)} + W_{21}^{(0)}). \qquad (13)$$

It can be seen from (6) and (8) that $R_{\rm tr} \propto \zeta^2 \exp(-\Delta U / D)$ where $\Delta U = \max(\Delta U_1, \Delta U_2)$ is the depth of the deeper potential well. For non-equal well depths, it is obvious that $R_{\rm tr}$ increases sharply with increasing D, i.e., stochastic resonance [1] occurs. We emphasize that (13) holds for ζ not too large: this is because the contributions to $\kappa(\Omega), Q^{(0)}(\Omega)$ from the interwell transitions are proportional to $\exp(-\zeta)$ compared to the intrawell contribution [cf. Eqs. (7)–(10)] and, for large ζ, they become small.

The theory has been tested by means of an electronic analog experiment, using a circuit of conventional design [14] to simulate (1). Measurements of signal-to-noise ratio R are shown by the square data points in Fig 1. We note immediately that the existence of stochastic resonance for the case of periodically modulated noise is confirmed by the data. We stress here that the rate of increase of R is faster than D, so that it does not represent merely the proportionality of the modulation to D in Eq. (2). The lower solid line in Fig. 1(a) represents a fit to the experimental data of the theory (9), (10), (12) allowing for the explicit form [7] of $Q^{(0)}(\Omega)$, demonstrating the universal character of the shape of the SR.

It is interesting to compare SR for periodically modulated driving noise with conventional SR (circle data and upper curves in Fig. 1) in periodically driven systems where $f(t)$ in (1) is replaced by

$$\tilde{f}(t) = \xi(t) + A\cos\Omega t. \tag{14}$$

The most substantial difference is that, in the present case, SR occurs for an asymmetric bistable potential with *wells of differing depths* [see (8) and (13)], whereas conventional SR can be regarded [6,9] as a kinetic phase transition phenomenon that is at its most pronounced for equally populated stable states, i.e., *equal well depths* [see circle data points in Fig. 1(b)]. The asymmetry of model (1) is controlled by λ,

$$\zeta \equiv |\Delta U_1 - \Delta U_2|/D \simeq 2|\lambda|/D \tag{15}$$

for $|\lambda| \ll 1$, and R would therefore be expected (13) to increase rapidly with λ; whereas, for additive periodic forcing, R decreases rapidly [6] with increasing $|\lambda|$. These ideas are confirmed directly by the experimental data of Fig. 1(b). According to (13) and (15), $R \propto \lambda^2$ for small $|\lambda|$; but for large $|\lambda|$ the increase saturates because the depth of the deeper well increases, with a corresponding decrease in the contribution to R from interwell transitions. We note that, for periodic forcing of the system described by (1) and (14), R should be larger than for periodic modulation of the noise for the same dimensionless amplitude A, just because of the additional asymmetry factor ζ^2 in (13). It can be seen from Fig. 1 that the above theory (full curves) is in good agreement with the experiment. The results in Fig. 2(b) demonstrate

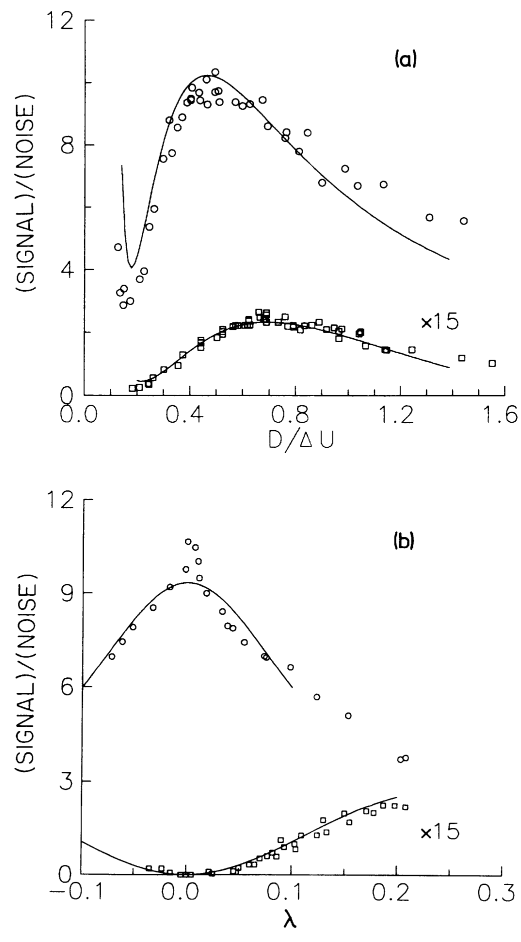

FIG. 1. Measurements (square data points) of the signal-to-noise ratio R ($\times$ 15) for periodically modulated noise compared with theory (lower curves): (a) as a function of reduced noise intensity $D/\Delta U$ with $A = 0.14$, $\lambda = 0.2$, $\Omega = 0.029$; (b) as a function of the asymmetry parameter λ with $A = 0.15$, $(D/\Delta U)_{\lambda=0} = 0.303$, $\Omega = 0.029$. The circle data represent measurements on the same circuit with *additive* periodic forcing (conventional SR) under similar conditions compared with the theory (upper curves) of Ref. [6].

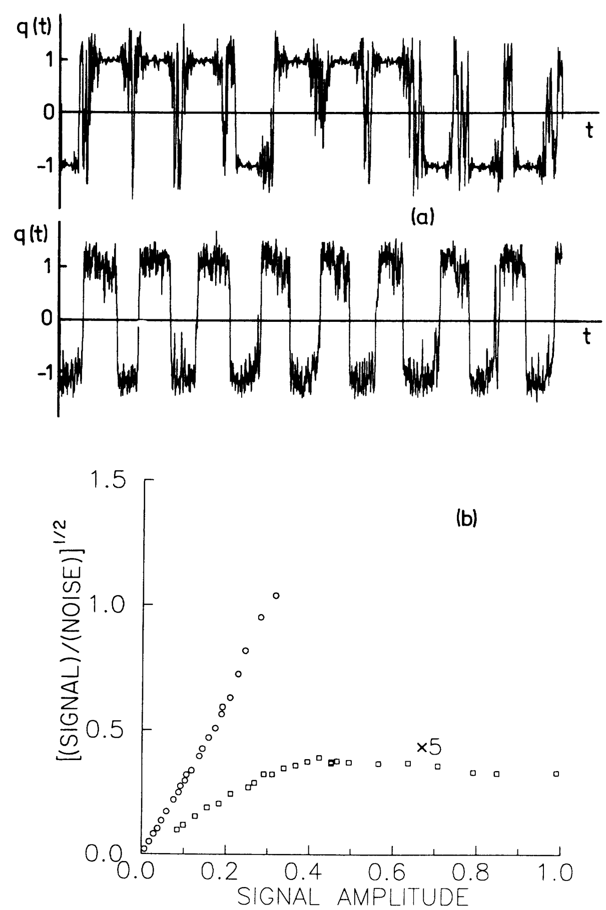

FIG. 2. (a) Digitized time series $q(t)$ from the analog electronic circuit with the asymmetry parameter $\lambda = 0.12$ for periodically modulated noise $f(t)$ in (1) (upper trace) and for additive noise and periodic forcing $\tilde{f}(t)$ (14) in (1) (lower trace). (b) Plot of $[(\text{signal})/(\text{noise})]^{\frac{1}{2}}$ measured as a function of signal amplitude in conventional SR (circles) and ($\times$ 5) in SR with periodically modulated noise (squares) for the same A and D. Note the linear dependence seen for small signals.

that the signal-to-noise ratio saturates with increasing amplitude of the periodic modulation. The effect is easily understood, because the amplitude of the signal due to interwell transitions is effectively limited to one-half of the distance between the attractors. It is more striking than the corresponding saturation effect in conventional SR, for which the additive periodic force also distorts the shape of the potential (cf. Ref. [15] where nonlinear effects for large amplitude modulation in conventional SR are considered).

It is also interesting to note that anisotropy of the potential, shown in the present paper to give rise to SR for periodically modulated noise, can also give rise to SR when the periodic modulation is parametric with the force proportional to the coordinate. This type of SR has been reported very recently for a parametrically driven magnetoelastic ribbon [16]; it would also be anticipated in parametrically driven bistable electronic systems [5(b)] if the potential were to be made asymmetrical.

In conclusion we would comment that, just as in the case of conventional SR, the main features of SR with periodically modulated noise can be well described within the scope of LRT. The marked differences predicted to exist between these two types of SR, and in particular their quite different characteristic variations of R with the asymmetry of the potential, including the rapid decrease of R for additive periodic forcing, have been convincingly confirmed by the analog electronic experiments.

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