Relaxation times of non-Markovian processes

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(Received 17 November 1986)

We consider a general class of non-Markovian processes defined by stochastic differential equations with Ornstein-Uhlenbeck noise. We present a general formalism to evaluate relaxation times associated with correlation functions in the steady state. This formalism is a generalization of a previous approach for Markovian processes. The theoretical results are shown to be in satisfactory agreement both with experimental data for a cubic bistable system and also with a computer simulation of the Stratonovich model. We comment on the dynamical role of the non-Markovianicity in different situations.

I. INTRODUCTION

Stochastic differential equations have been a very useful tool in the study of a wide range of nonequilibrium phenomena. These equations incorporate stochastic linear terms or noises. The origin of the noise can either be intrinsic to the system, or it can be external to it and controlled in the laboratory. It is commonly assumed, at least in a preliminary approach, that the noise is a Gaussian white stochastic process described by a unique parameter D referred to as the noise intensity. This assumption is plausible when the time scale of the noise is several orders of magnitude smaller than the time scale of the system. In this situation the process is Markovian and the steady-state properties of the systems associated with the stationary density P_{st} are well known.[1]

The real world is, however, rather different from this idealization. Recent experiments on optical systems[2] and electronic circuits[3] have related to noise that is definitely nonwhite. The main difficulty in such cases is that, because the process is non-Markovian, the steady-state properties are known exactly only in particular cases. Nonetheless, it is possible to understand in general their main features through the use of approximate approaches.[3–6]

The study of the dynamics of non-Markovian processes is of very recent origin both from the theoretical and the experimental points of view.[2–6] Some attention has, however, already been focussed on relaxation from unstable initial conditions,[7–9] on mean passage times between metastable states,[5,10] and on relaxation times in the steady state.[4(a),11–13] The present paper is devoted to a study of this last quantity. The relaxation time, often called the linear relaxation time, gives dynamical information about the time scale of the evolution of a spontaneous fluctuation in the steady state. The relaxation time associated with the correlation function $c(s)$ is defined by[11,14]

$$T=\int_0^\infty \frac{c(s)}{c(0)}ds \; , \tag{1.1}$$

where

$$c(s)=\lim_{t\to\infty}\langle q(t+s)q(t)\rangle-\langle q\rangle_{st}^2 \tag{1.2}$$

and $q(t)$ represents the stochastic process. It is simple to extend the definition (1.1) to other quantities such as the moments of $q(t)$.

If $q(t)$ is a Markovian process the evaluation of T is a solved question. One can find in the literature several approximate methods[15] and very recently the exact answer.[16] The suitability of these methods in different situations has also been tested by means of electronic circuits.[17]

For non-Markovian processes, however, the present status of the theory cannot be regarded as at all satisfactory,[11] though a number of recent works demonstrate wide interest in this problem.[10,18–24] One of the main aims of this paper is to present an improved theoretical approach, which is outline in Sec. II. It is based on two principal steps: a refined calculation to find the specifically non-Markovian features of the process $q(t)$,[6] and an exact treatment for a quasi-Markovian process. In this sense our approach is a natural extension to non-Markovian processes of the exact method introduced in Refs. 16 and 25 for Markovian ones. Sections III and IV are devoted to specific examples. In Sec. III we study a cubic bistable system and we compare the theoretical results with experimental data. In the second example, Sec. IV, we study the Stratonovich model and we compare the theoretical predictions with numerical data. In Sec. V we comment on the role of the non-Markovianicity in these particular examples and we summarize the main conclusions of this paper.

II. THE THEORY

The process $q(t)$ that we are interested in studying obeys a stochastic differential equation of the general form

$$\dot{q}=v(q)+g(q)\xi(t) , \tag{2.1}$$

where $v(q)$ and $g(q)$ are arbitrary functions of q and $\xi(t)$ is the noise or random force, which is an Ornstein-Uhlenbeck process. It is Gaussian with zero mean and correlation function

$$\langle \xi(t)\xi(t') \rangle = \frac{D}{\tau}\exp\left[-\frac{|t-t'|}{\tau}\right] . \tag{2.2}$$

The noise parameters D and τ measure the noise intensity and correlation time respectively.

Due to the non-white character of the noise $\xi(t)$, the process $q(t)$ is non-Markovian, and the equations satisfied by the probability density $P(q,t)$ and the joint probability density $P(q,t;q',t')$ $(t>t')$ are different.[26] In the limit $t\rightarrow\infty$, $t'\rightarrow\infty$, with $t-t'=s$, and in the weak-noise assumption (small D), it is possible to obtain an equation for the steady-state joint probability density[6,26] of the form

$$\partial_s P_{st}(q,q';s)=[L_q(\tau)+D\exp(-s/\tau)L'_{qq'}(\tau)]P_{st}(q,q';s) , \tag{2.3}$$

where the operators $L_q(\tau)$ and $L'_{qq'}(\tau)$ are given by

$$L_q(\tau)=\partial_q v(q)+D\partial_q g(q)\partial_q H(q) , \tag{2.4}$$

$$L'_{qq'}(\tau)=\partial_q g(q)\partial_{q'}H(q') , \tag{2.5}$$

and

$$H(q)=v(q)[1+\tau v(q)\partial_q]^{-1}\left[\frac{g(q)}{v(q)}\right] . \tag{2.6}$$

The operator $L_q(\tau)$ has a Fokker-Planck-like form and it represents the quasi-Markovian approximation for the process $q(t)$. It determines the evolution of the single probability density, thus it satisfies $L_q(\tau)P_{st}(q)=0$. The second term of (2.3) carries the pure non-Markovian dynamics of the process. It vanishes in the white-noise limit $\tau\rightarrow 0$, whereas $L_q(\tau)$ becomes the well-known Fokker-Planck operator.

Equation (2.3) is of practical interest in the study of correlation functions.[6] By formal integration to first order in D we get

$$P_{st}(q,q';s)=e^{L_q(\tau)s}[1+\partial_q\tilde{H}(q,s)\partial_{q'}H(q')]P_{st}(q,q';0) , \tag{2.7}$$

where $\tilde{H}(q,s)$ is a function defined by the operator relation

$$\partial_q\tilde{H}(q,s)=\int_0^s De^{-s'/\tau}e^{-L_q(\tau)s'}\partial_q g(q)e^{L_q(\tau)s'}ds' . \tag{2.8}$$

An explicit evaluation of $\tilde{H}(q,s)$ is possible in terms of power series of τ (see Ref. 6). To first order in τ it reads

$$\tilde{H}(q,s)=D\tau(1-e^{-s/\tau})g(q)+O(\tau^2) . \tag{2.9}$$

After some algebra, Eq. (2.7) gives rise to a closed expression for the steady-state correlation function

$$\langle \Delta q\,\Delta q(s)\rangle_{st} = \langle \Delta q\,\Delta q(s)\rangle^0_{st} - \left\langle \left[[\partial_q\tilde{H}(q,s)]H(q)+\frac{\tilde{H}(q,s)v(q)}{Dg(q)} \right]\Delta q(s)\right\rangle^0_{st} \tag{2.10}$$

with

$$\langle f_1(q)f_2(q(s))\rangle^0_{st}\equiv\int_a^b dq\, f_2(q)e^{L_q(\tau)s}f_1(q)P_{st}(q) , \tag{2.11}$$

where the interval $[a,b]$ is the domain of definition of $q(t)$.

The usefulness of (2.11) is that it reduces the treatment of $\langle \Delta q\Delta q(s)\rangle_{st}$ to that of generalized correlation functions (2.11) of Markovian processes, characterized by $L_q(\tau)$, for which standard methods are available.[16] A summary of the method followed here for the exact time integration of (2.11) is outlined in Appendix A. However, we cannot apply it yet to the right-hand side (rhs) of (2.10) because of the dependence on s of $\tilde{H}(q,s)$. In order to avoid this difficulty, we take a small-τ approximation, so that (2.10) reduces to[6]

$$\langle \Delta q\,\Delta q(s)\rangle_{st} = \langle \Delta q\,\Delta q(s)\rangle^0_{st} - \tau\langle [Dg(q)g'(q)+v(q)]\Delta q(s)\rangle^0_{st} + O(\tau^2) , \tag{2.12}$$

where we have used (2.9) and $H(q)=g(q)+O(\tau)$. We have also neglected the term $e^{-s/\tau}$ which would give a τ^2 contribution after integration.

In principle, only the second term of (2.10) is involved in the τ expansion although the first one is also dependent on τ via (2.6). This means that it is possible to keep the effect of τ in the quasi-Markovian contribution, whereas the expansion of the second term of (2.10) supplies the pure non-Markovian corrections.

Our goal in this paper is to separate these two types of contributions to T, applying the Jung-Risken method for the exact integration of (2.12). Thus, limiting ourselves to the first-order correction, and as it is explicitly shown in Appendix B, the relaxation time of $c(s)$ takes the form

$$T=T_0(\tau)+\tau T_1 , \tag{2.13}$$

where the first contribution is the corresponding to the effective Markovian problem associated with $L_q(\tau)$ and the second one is the first correction originating from the operator $L'_{qq'}(\tau)$ characterizing pure non-Markovian dynamic effects, not included in any quasi-Markovian approximation.

$T_0(\tau)$ is given by (B1) and is just the Jung-Risken result[16] for a Markovian process defined by (2.4). In practice, however, one has to use a first-order approximation, not only by consistency but in order to avoid unphysical boundaries problems [see (B16) and (B17)].

Now, to evaluate T_1 we take $f_1(q)=-[Dg(q)g'(q)+v(q)]$ and $f_2(q)=q-\langle q\rangle_{st}$ in (A1), and we get straightforwardly (see Appendix B) $T_1\equiv 1$. Therefore, Eq. (2.13) is just

$$T=T_0(\tau)+\tau \tag{2.14}$$

so that the relaxation time T reduces to the value corresponding to an effective Markovian approach plus a contribution τ which is a pure non-Markovian slowing down associated with memory effects, and which is independent of the model and of the intensity of the noise.

A similar result was obtained in Ref. 11, after a decoupling ansatz,

$$T=[\gamma^0(\tau)]^{-1}+\tau\ , \tag{2.15}$$

where $\gamma^0(\tau)^{-1}$ was the first approximation associated with $L_q(\tau)$ in a projector-operator technique.[11] The main advantage of the present approach, as compared to that of Ref. 11, is that in (2.14) the white noise as Markovian case ($\tau=0$) is exactly incorporated. In some sense our approach is a perturbative method in τ where the lowest order is the exact result for $\tau=0$. In Ref. 11 both D and τ are used as perturbative parameters in an uncontrolled way.

III. WELLAND-MOSS MODEL

In this section we study the relaxation time of a cubic bistable system both from a theoretical and an experimental point of view. This model has recently been considered for the particular case where the external noise is a Gaussian white process;[17] our intention here, however, is to try to understand how the non-white characteristics of the actual noise would modify those earlier results. The model is defined by (2.1) where

$$v(q)=-q^3+\lambda q^2-Qq+R\ , \tag{3.1}$$

$$g(q)=q^2\ . \tag{3.2}$$

The external noise has been introduced through the control parameter λ as $\lambda(t)=\lambda+\xi(t)$. In our explicit results we have always chosen $Q=3$ and $R=0.7$. This choice results in a bistable-monostable transition depending on the values of the initial parameter λ as it is explained in Ref. 27. Our interest in this model is to see how the non-Markovian character of the process may modify the slowing-down picture in the bistable region, that was observed in Ref. 17 for the case of its Markovian counterpart.

The theoretical results that will be presented were obtained by numerical integration of $T_0(\tau)$ (B1) following a procedure similar to that explained in Ref. 17.

The relaxation time was also measured experimentally for an electronic circuit that accurately models the system in question (2.1), (3.1), and (3.2). The technique used was essentially the same as described previously[17] except that the external noise applied to the circuit was colored rather than white; that is, its correlation time was no longer very short compared to the characteristic response time of the circuit. The same commercial Gaussian-noise generator was used as before (Wandel and Golterman model RG-1), its output being passed through an active RC filter with component values chosen such that the resultant noisy voltage was exponentially correlated with a relaxation time τ_n that lay in the range $10<\tau_n<1000$ μs. The integrator time constant τ_i of the circuit was 1000 μs. The noise intensity was measured by means of a true-rms-dc converter (AD536A), the value of D being determined from the relation

$$D=V_{\rm rms}^2\frac{\tau_n}{\tau_i}\ .$$

The relaxation time of the circuit was measured for a range of values of λ, D, and τ_n by application of the standard fast-Fourier-transform (FFT) technique already described,[17] and then divided in each case by τ_i to find T.

The measurements were naturally subject to experimental error, arising from several sources. Systematic errors were introduced by the inherently non-ideal character of the electronic components from which the circuit was constructed. These combined to produce a systematic uncertainty in T that we estimate as being no more than $\pm 10\%$. In addition, there was a random error related to the statistics of the averaging process and amounting, typically, to $\pm 5\%$. The effect of drift in component values during the period of acquisition of a correlation function (typically 15 minutes) was also important, and particularly so in the bistable region where small shifts in the preset value of λ could exert a disproportionate influence on T. The drift in λ during acquisition was usually less than 1%. We conclude, therefore, that the measurements of T suffer form a systematic uncertainty no greater than $\pm 10\%$, plus a random uncertainty of about $\pm 5\%$, indicated by their scatter about a smooth curve and that they are subject, in addition, to the uncertainty corresponding to errors of up to $\pm 1\%$ in the set value of λ. It must be emphasized that these estimates refer to the worst conditions encountered in practice, so that they will substantially overestimate the actual errors in most of the measurements.

The measured values of T^{-1} are plotted in Figs. 1 and 2 for comparison with theoretical predictions (curves) based on (2.14). They are seen to be in excellent agreement, and well within the range of experimental uncertainties discussed above. It is immediately clear that the situation has not changed drastically as compared to the white-noise case in the bistable region (the pronounced minimum in Fig. 1), where the dominant dynamical mechanism is the diffusion over the barrier.[17] The most remarkable difference is the shift of the minimum due to the term $T_0(\tau)$ which accounts for an important increase of the relaxation time. In a monostable state, the slowing down produced by the second term in the rhs of (2.14) is more apparent. Figure 2 shows that the extrapolation of the theory to values of τ clearly beyond the domain of re-

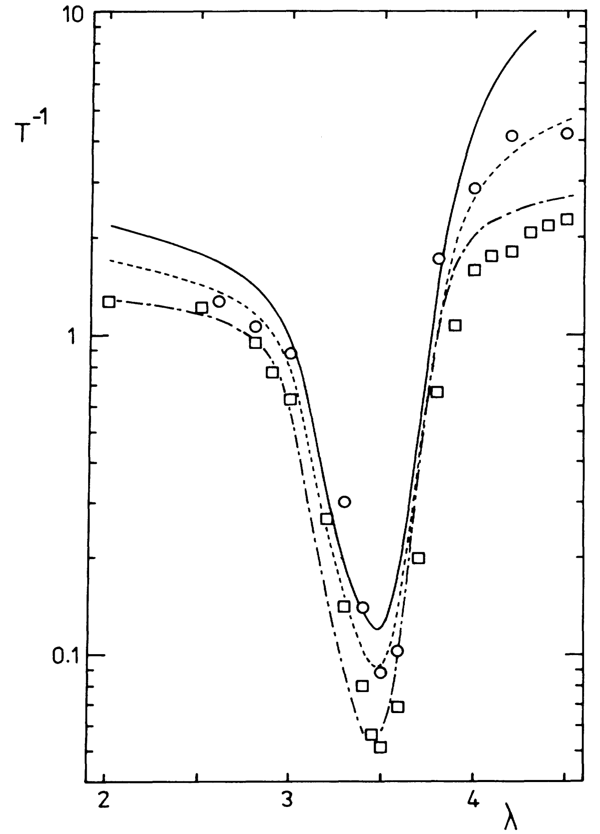

FIG. 1. T^{-1} for the model (3.1) and (3.2) vs λ for $\tau=0$ (solid line), $\tau=0.1$ (dashed line), and $\tau=0.3$ (dash-dotted line). Circles ($\tau=0.1$) and squares ($\tau=0.3$) are the experimental data ($D=0.125$).

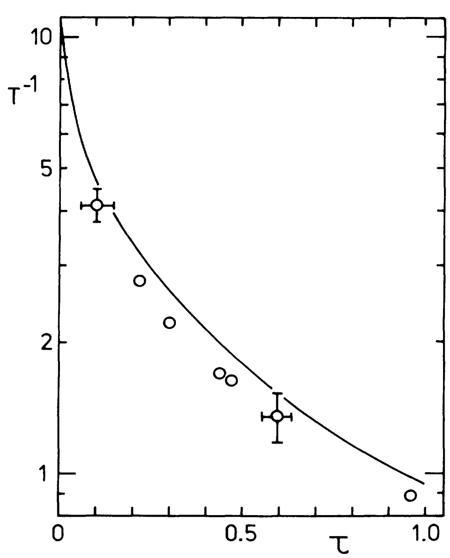

FIG. 2. T^{-1} for the same model as in Fig. 1 vs τ ($\lambda=4.50$ and $D=0.125$). The circles are the experimental data.

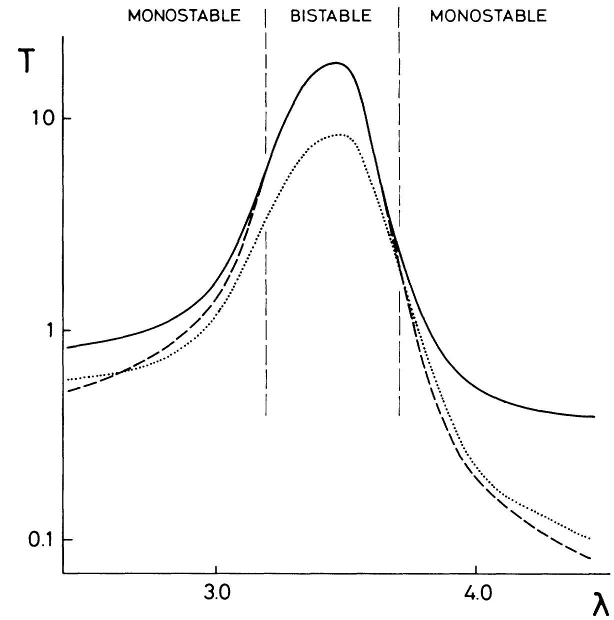

FIG. 3. Different contributions to T: $T_0(\tau)$ (dashed line), $T_0(0)$ (dotted line), and T (solid line), for the model (3.1) and (3.2) ($\tau=0.3$ and $D=0.125$).

liability of a first-order approximation, seems still to be good, at least in this case.

The situation is more clearly displayed in Fig. 3 where the different contributions of (2.14) are compared with the white-noise case. The quasi-Markovian contribution, $T_0(\tau)$, is much more important at the maximum (bistable region) than in monostable situations where the pure non-Markovian term, τ, dominates.

IV. STRATONOVICH MODEL

Now we consider the so-called Stratonovich model which was introduced so far in the context of electronic devices[28] and has been widely studied in many different situations.[4,10–13] The model is defined by (2.1) with

$$v(q)=q-q^3; g(q)=q \; . \tag{4.1}$$

We have computed Eq. (2.14) for different values of noise intensity D and correlation time τ. In Fig. 4 one can see the dependence of T^{-1} versus D for $\tau=0$ and $\tau=\frac{1}{3}$ comparing previous theoretical results of Ref. 11 and new numerical data. The monotonic decrease of T^{-1} with increasing D or τ is immediately evident. This figure clearly demonstrates how the white-noise case is exactly incorporated in (2.14), representing a definite advantage over the previous approach[11] which gives rise to the appearance of an anomalous minimum of $T^{-1}(D)$ and a divergence for $D\to\infty$ when $\tau=0$. The quantitative agreement between the theory (2.14) and the digital simulation is remarkably good. These new data do not present the systematic underestimation of T of Refs. 4a and 11. In Fig. 5 we show the different contributions to T of Eq. (2.14) in

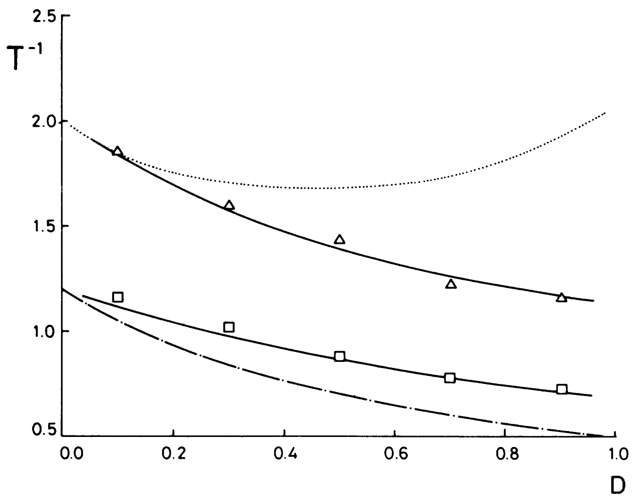

FIG. 4. T^{-1} for the Stratonovich model (4.1) vs D. Solid lines correspond to the expression (2.14) and broken lines are from Ref. 11. Triangles ($\tau=0$) and squares ($\tau=\frac{1}{3}$) are numerical data from a new digital simulation.

order to emphasize the role of the non-Markovianicity in a monostable situation like the present model. Here we will see more clearly one of the effects discussed in Sec. III. In fact, the pure non-Markovian contribution, τ, in (2.14) is the dominant part as can be seen in Fig. 5, but its relative importance decreases with increasing D. This means that there is also a slowing down due to $T_0(\tau)$ (an effective diffusion effect) which is amplified by D, so that for $D \to 0$ it reduces to the white-noise contribution, whereas for D large it becomes dominant. The purely non-Markovian slowing down remains constant, independent of D.

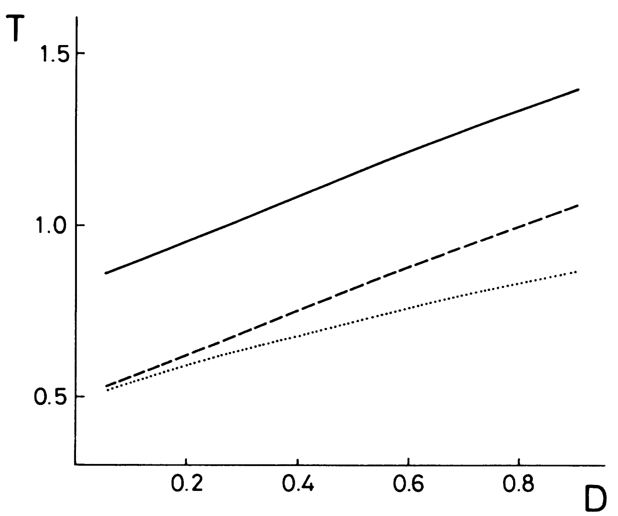

FIG. 5. Different contributions to T: $T_0(\tau)$ (dashed line), $T_0(0)$ (dotted line), and T (solid line), for the Stratonovich model (4.1) ($\tau=\frac{1}{3}$).

V. COMMENTS AND CONCLUSIONS

We have presented a theory to evaluate relaxation times for non-Markovian processes defined by stochastic differential equations with non-white noise. It compares favorably with a previous theory[11] and is a natural extension of the Jung-Risken method[16] deduced for the particular case of Gaussian white noise. Jung and Risken[13] have also presented a theory for non-Markovian processes $q(t)$ but considering the two-variable problem (q,ξ) which is now a Markovian situation. By means of continued matrix fraction expansions they can also get non-Markovian information. This approach has not been applied to the models of Sec. III and IV. The advantage of our treatment is clear for the study of T because our main result (2.14) allows physical interpretation of the changing role of the non-Markovianicity in different situations. In this context Figs. 3 and 5 clarify the differing role of the non-Markovianicity in bistable and in monostable situations. We can conclude that in a bistable situation (Fig. 3) the role of τ is through an effective diffusion (2.4) in the Fokker-Planck equation,[10] which can be interpreted as a quasi-Markovian approximation for a non-Markovian process. This remarkable fact has also appeared in the study of mean-first-passage times for non-Markovian processes.[23,29,30] In monostable situations the role of non-Markovianicity is quite different; the pure non-Markovian effect arising from the memory of the system is relatively much more important when the intensity of the noise D is not too large. The above results are useful in considering the most suitable approximation to use for a description of any given physical situation.

The physical mechanisms that lead to these different behaviors are the following. In a bistable situation the passage time between the two stable states is the dominant time scale. Its magnitude should be related to the Arrhenius Law

$$T \sim \exp\left[\frac{\Delta\phi}{D}\right] . \tag{5.1}$$

From this approximate expression one can clearly see that small changes in the diffusion are magnified by the exponential. As the colored noise reduces the diffusion in the stable states, then T increases exponentially with τ.[10,23,29–31]

In a monostable situation, when $D \ll 1$, the linear or Gaussian approximation is dominant, and the relaxation time is given by[11]

$$T = \text{const} + \tau + O(D) . \tag{5.2}$$

Hence small changes in the diffusion coefficient are not specially relevant whereas the pure non-Markovian effects remain in the term τ.

In conclusion we should like to emphasize that the theory proposed here, although approximate, does nonetheless give reliable results, as has been demonstrated through comparison with experimental and computer-simulation data, over a very wide range of parameter values and in distinctly different physical situations.

ACKNOWLEDGMENTS

This work was supported in part by the Science and Engineering Research Council (United Kingdom), by the U.S.—Spanish Joint Committee for Scientific and Technological Cooperation, by North Atlantic Treaty Organization (NATO) Grant No. RG-85/0770 and by Comisión Asesora de Investigación Científica y Técnica (CAICyT) Project No. 361/84 (Spain).

APPENDIX A: GENERALIZED JUNG-RISKEN METHOD

Here we present a generalization of the scheme introduced in Ref. 16 for the calculation of relaxation times associated with Markovian-like stationary correlation functions defined by the general form

$$\lim_{t\to\infty} \langle f_1(q(t)) f_2(q(t+s))\rangle \equiv C_{12}(s) = \int_a^b dq\, f_2(q) e^{L(q)s} f_1(q) P_{\rm st}(q) , \quad (A1)$$

where $L(q)$ is a Fokker-Planck-like operator with $L(q)P_{\rm st}(q)=0$.

This method is useful for the calculation of the quantity

$$T_{12} = \frac{1}{C_{12}(0)} \int_0^\infty ds\, C_{12}(s) , \quad (A2)$$

which is given exactly up to a quadrature by solving an ordinary differential equation.

The conditions which are implicit in this procedure for all quantities to be well defined are $\langle f_1(q)\rangle_{\rm st} = \langle f_2(q)\rangle_{\rm st}=0$. However, these are not real restrictions given that at least one of them is necessary for the existence of T_{12}; for example, $\langle f_1(q)\rangle_{\rm st}=0$ so that

$$\lim_{s\to\infty} C_{12}(s) \to \langle f_1(q)\rangle_{\rm st} \langle f_2(q)\rangle_{\rm st} = 0$$

and now it is possible to redefine the function $f_2(q)$ as $f_2(q)-\langle f_2(q)\rangle_{\rm st}$ with no change in the resulting correlation function by means of the identity

$$\int_a^b dq (f_2 - \langle f_2\rangle_{\rm st}) e^{Ls} (f_1 - \langle f_1\rangle_{\rm st}) P_{\rm st} = \int_a^b f_2 e^{Ls} f_1 P_{\rm st}\, dq - \langle f_1\rangle_{\rm st} \langle f_2\rangle_{\rm st} . \quad (A3)$$

In these conditions the method follows straightforwardly. Assuming

$$\langle f_1(q)\rangle_{\rm st} = \langle f_2(q)\rangle_{\rm st} = 0$$

in (A1) we define the quantities

$$W(q,s) = e^{L(q)s} f_1(q) P_{\rm st}(q) , \quad (A4)$$

$$\rho(q) = \int_0^\infty ds\, W(q,s) , \quad (A5)$$

and note that $W(q,s)$ satisfies, from its definition, the Fokker-Planck equation defined by the operator $L(q)$. After a formal time integration of this Fokker-Planck equation and taking into account that $W(q,\infty)$ must vanish, we find

$$-f_1(q) P_{\rm st}(q) = L(q)\rho(q) , \quad (A6)$$

where we have used the definitions (A4) and (A5).

Otherwise, inserting (A1) into (A2) and changing the order of integration, the relaxation time T_{12} is given by

$$T_{12} = \frac{1}{C_{12}(0)} \int_a^b dq\, f_2(q)\rho(q) . \quad (A7)$$

Hence, if it is possible to solve (A6), we can substitute $\rho(q)$ into (A7) obtaining an exact expression for T_{12} in terms of a quadrature. When $L(q)$ is a Fokker-Planck-like operator, Eq. (A6) can be solved easily because it reduces to a first-order linear nonhomogeneous differential equation. If the Fokker-Planck operator is modifed with an arbitrary function $\phi(q)$ such as

$$L(q) = -\partial_q v(q) + D\partial_q g(q) \partial_q \phi(q) \quad (A8)$$

the method gives rise to

$$T_{12} = \frac{1}{C_{12}(0)} \int_a^b \frac{F_1(q)F_2(q)}{Dg(q)\phi(q)P_{\rm st}(q)} dq \quad (A9)$$

with

$$F_i(q) = -\int_a^q f_i(q') P_{\rm st}(q') dq' , \quad (A10)$$

which reduces to the Jung-Risken result for the particular case $f_i(q) = q - \langle q\rangle_{\rm st}$ and where the usual diffusion $Dg^2(q)$ has been substituted by an effective one given by $Dg(q)\phi(q)$. The details of the deduction of (A9) and (A10) are the same as explained in Appendix B for the derivation of (B9) and (B10).

APPENDIX B: CALCULATION OF $T_0(\tau)$ AND T_1

In this appendix we give the explicit form of the rhs of Eq. (2.13). The first term $T_0(\tau)$ is easily obtained after the discussion in Appendix A. It reads explicitly

$$T_0(\tau) = \frac{1}{\langle(\Delta q)^2\rangle_{\rm st}} \int_a^b \frac{F^2(q)}{Dg(q)H(q)P_{\rm st}(q)} dq \quad (B1)$$

with

$$F(q) = -\int_a^q (q' - \langle q\rangle_{\rm st}) P_{\rm st}(q') dq' . \quad (B2)$$

Here we treat in more detail the calculation of the second term which accounts for the pure non-Markovian effects.

According to (2.9) and (2.10), we must take in (A1)

$$f_1(q) \equiv -[Dg(q)g'(q) + v(q)] , \quad (B3)$$

$$f_2(q) \equiv q - \langle q\rangle_{\rm st} \equiv \Delta q , \quad (B4)$$

which leads, after a formal integration in both members of (A6) with $L(q)=L_q(\tau)$ of (2.4), to

$$F_1(q) \equiv -\int_a^q f_1(q') P_{\rm st}(q') dq' = [-v(q) + Dg(q)\partial_q H(q)]\rho(q) \quad (B5)$$

given that the associated *probability density current* must vanish at the boundary a,

$$[v(q) - Dg(q)\partial_q H(q)]\rho(q)\,|_{q=a} = 0 . \quad (B6)$$

The general solution of (B5) can be written as

$$\rho(q) = CP_{\rm st}(q) + P_{\rm st}(q) \int_a^q G(q') P_{\rm st}^{-1}(q') dq' , \quad (B7)$$

where

$$G(q)=\frac{F_1(q)}{Dg(q)H(q)} \tag{B8}$$

because the solution of the homogeneous part is just $P_{st}(q)$. However, the first term in the rhs of (B7) does not contribute to the integral (A7) so that, after an integration by parts, we have

$$T_1C(0)=\int_a^b \Delta q\,\rho(q)dq=\int_a^b F_2(q)G(q)P_{st}^{-1}(q)dq\ , \tag{B9}$$

where

$$F_2(q)\equiv-\int_a^q (q'-\langle q\rangle_{st})P_{st}(q')dq'\ . \tag{B10}$$

Otherwise, the relation $L_q(\tau)P_{st}(q)=0$ can be written as

$$\{-\partial_q[v(q)+Dg(q)g'(q)]+D\partial_q^2 g^2(q)\}P_{st}(q)=O(\tau) \tag{B11}$$

and, with vanishing probability density current, (B11) reduces to

$$\{-[v(q)+Dg(q)g'(q)]+D\partial_q g^2(q)\}P_{st}(q)=O(\tau)\ . \tag{B12}$$

From (B12) it is possible to write $f_1(q)P_{st}(q)$ as a total derivative so that $G(q)$ in (B8) reduces to

$$G(q)=P_{st}(q)+O(\tau) \tag{B13}$$

[(B12) also shows that $\langle Dg(q)g'(q)+v(q)\rangle_{st}=O(\tau)$ which enables us to omit it in the definition (B3) because it will not contribute to T_1]. Thus, by inserting (B13) into (B9) we have

$$T_1C(0)=\int_a^b F_2(q)dq+O(\tau)$$
$$=-\int_a^b\left[\int_a^q (q'-\langle q\rangle_{st})P_{st}(q')dq'\right]dq+O(\tau) \tag{B14}$$

and performing an integration by parts, (B14) is nothing but $\langle(\Delta q)^2\rangle_{st}\equiv C(0)$ which implies

$$T_1=1\ , \tag{B15}$$

T_1 being independent of D and of the model.

The effective diffusion we have used in our numerical calculation corresponds to the first-order approximation in τ of Eq. (2.6)

$$D_{eff}(q)\equiv Dg(q)H(q)$$
$$\simeq Dg^2(q)\left[1+\tau g(q)\left[\frac{v(q)}{g(q)}\right]'\right]\ . \tag{B16}$$

As this approximation implies either anomalous boundaries or nonpositive definite diffusion, we have taken in (B1)

$$D_{eff}^{-1}(q)=\frac{1}{Dg^2(q)}\left[1-\tau g(q)\left[\frac{v(q)}{g(q)}\right]'\right]+O(\tau^2)\ , \tag{B17}$$

which avoids these problems in the models studied here. As the same problems appear in the formal expression for $P_{st}(q)$ we have employed the exponentiated form of Ref. 4(a).

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