Scaling Symmetries, Contact Reduction and Poincaré's dream

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We state conditions under which a symplectic Hamiltonian system admitting a certain type of symmetry (a *scaling symmetry*) may be reduced to a type of contact Hamiltonian system, on a space of one less dimension. We observe that such *contact reductions* underly the well-known McGehee blow-up process from classical mechanics. As a consequence of this broader perspective, we associate a type of variational Herglotz principle associated to these classical blow-ups. Moreover, we consider some more flexible situations for certain Hamiltonian systems depending on parameters, to which the contact reduction may be applied to yield contact Hamiltonian systems along with their Herglotz variational counterparts as the underlying systems of the associated scale-invariant dynamics. From a philosophical perspective, one obtains an equivalent description for the same physical phenomenon, but with fewer inputs needed, thus realizing *Poincaré's dream* of a scale-invariant description of the universe.

I. INTRODUCTION

Symmetry plays an important role in physics. When considering observations of measurable quantities, the action of symmetry should be carefully considered in its effect not only on the quantity being measured but also upon the apparatus through which the measurement is made. This is common when considering most physical symmetries. On applying a Galilean transformation to a system – for example transforming reference frames – we argue invariance of observation based on applying the transformation to both the observer and the observed quantity. Absolute motion should not be observable as it is not invariant under Galilean transformations. Physically meaningful quantities and their equations of motion are those invariant under such symmetry transformations.

It has been argued, at least since Poincaré (see e.g. [33] for a detailed discussion), that the same considerations should be applied to the absolute scale of a system. In [44], pg. 94, Poincaré imagines that overnight all dimensions in the universe have been scaled, and argues that there will be no detectable change upon waking due to both the observed system (the universe) and any possible measuring apparatus (e.g. one's height), having undergone a scaling by the same factor. Accordingly, we refer to writing the equations of motion of a system in terms of scale-invariant quantities as *Poincaré's dream* or a *scale reduction* of the system.

The counter to this argument was long held to be the fact that physical constants can be used to give an overall scale based on, for example, the Planck length. However in recent work it has been suggested that if these are included not as given facts about the universe, but rather as physical observations of quantities which must be measured, such a scaling symmetry can be restored [48].

It should be clear that in any scale-invariant description of the physical world one would need one less piece of information with respect to the standard descriptions that include reference to some (unmeasurable) absolute scale, the datum being removed being precisely the one corresponding to the scale.

There have been a large number of works based upon removing a 'scale' or 'size' variable from the equations describing physical systems. For example McGehee's works, eg [38], on blow-ups of classical mechanical systems (see as well [20, 43]), essentially consider coordinates in which the equations of motion are decoupled from the total size of the system. Such blow-ups contain, for instance, valuable properties of such systems near total collapse singularities. The study of singularities by decoupling the equations from a total-size variable has been exploited in various other situations as well, e.g. [35, 39, 41, 46]. Moreover, interesting dynamical properties of these scale-reduced systems, such as a dissipative-like behavior, have been observed, which can provide a natural origin for the observed arrow of time [4, 32, 45]. Remarkably, J. Bryant observed already in [12] and [13]

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that McGehee's blow-ups of n-body systems can be reformulated as a contact Hamiltonian system (we thank A. Albouy for pointing us to this reference).

Despite the large amount of works on this theme, still the treatment of scaling symmetries and of their corresponding reductions so far has been based on a case-by-case study (with the exception of [45]), focused on specific systems or classes of systems. Moreover, in such works, various choices are made for the overall 'size' of the system to determine the resulting scale reduction, and are justified according to the specific goals of the study. While in simple systems such as the 2-body problem the choice of the distance between the two bodies as the overall size is quite natural, when we consider the n-body problem such choice is not so apparent – one could use the mean separation of particles, or the largest separation, or some combination of any variables that scale in a similar way. Would the resulting scale reductions ensuing from these various choices be related somehow? Although it is not difficult to observe in particular cases such reductions are indeed related, this highlights the arbitrary nature of such choices, and begs the question if such reductions are based on a more general principle.

In this work we provide for the first time such a general principle underlying these scaling reductions of Hamiltonian systems. Our main results may be loosely stated as the principle:

Theorem. *A symplectic Hamiltonian system admitting a scaling symmetry (Definition 2 below), may be reduced to a contact Hamiltonian system on a space of one less dimension.*

See Theorems 1 and 2 below for more precise statements. This reduction of a symplectic manifold to a lower-dimensional contact manifold is in a sense inverse to the symplectification – a higher-dimensional symplectic manifold – of a contact manifold, in particular, only involving some fundamental relations between contact and symplectic manifolds. The main thrust of this article is not merely this observation, but also to present examples illustrating the broad scope in which such reductions may be found.

As a consequence, we obtain a great deal of structure underlying these scaling reductions, which appears to have been largely neglected in the previous case-by-case studies. For instance, one obtains associated Herglotz variational characterizations of these scale-reduced systems. In particular, the McGehee blow-up equations of classical mechanics [20, 38, 43] come with a variational characterization (see Proposition 9).

Moreover, we state explicitly the resulting scale-invariant equations of motion (Corollary 2 and Remark 6) resulting from a choice of scaling function or 'size' of the system. Throughout the manuscript we provide a collection of examples to illustrate the process of such reductions.

The paper is laid out as follows: in Section II we give a precise definition of a scaling symmetry and prove that reducing a symplectic system with a scaling symmetry in general yields a 'piecewise' contact system: given by two contact systems on different regions, which coincide exactly when the *degree* of the scaling symmetry is *one* (Theorem 1 and Corollary 1). Besides, in Corollary 2 we prove a general invariance (up to time reparametrizations) of the reduced system with respect to the choice of the scaling function. Then, in Section III we show that Hamiltonian systems which do not exhibit scaling symmetries can be lifted into a broader space of models in which such a symmetry is present, by promoting physical constants (couplings) to observable variables within the system. In this setting one may reduce such systems to a contact Hamiltonian (or Herglotz variational) system on a reduced space (Theorem 2). We proceed in Section IV to present a host of applications of the contact reduction to the case of Hamiltonian systems on cotangent bundles, which, among other things, allows us to re-state our main results in terms of the variational approach of Herglotz to contact systems. A further crucial new result of this section is the derivation of McGehee's blow-up in terms of a contact reduction and, equivalently, as a Herglotz variational system. We conclude in Section V with an outlook on future work and applications. To have the paper as self-contained as possible and in order to further motivate our constructions, in Appendix A we introduce the symplectic and contact Hamiltonian systems that we consider, and briefly summarize some known relationship between them.

All the objects used in this work are assumed to be smooth. Furthermore, we will repeatedly use the following notations: $\mathbb{R}^\times := \mathbb{R}\backslash\{0\}$ and $\mathbb{R}^+ := (0, +\infty)$.

II. SCALING SYMMETRIES, CONTACT REDUCTION AND SYMPLECTIFICATION

In this section we describe how a special type of symmetry, called 'scaling symmetry', provides a new link between symplectic mechanics and contact mechanics (see Appendix A for the relevant definitions and our sign conventions). Such symmetries have been found to be present in cosmology, the Kepler problem and a host of other physical contexts [33, 45–48].

A. Scaling symmetries

We start with a general definition, comprising all those transformations that reparametrize the dynamics without altering the (unparametrized) orbits.

Definition 1. A *dynamical similarity* of a vector field $X \in \mathfrak{X}(M)$ is a vector field $Y \in \mathfrak{X}(M)$ such that $[Y, X] = fX$, for some (in general non-constant) function $f : M \to \mathbb{R}$ (Y is also called a *conformal symmetry*).

In other words, a dynamical similarity is a symmetry of the one-dimensional distribution, or line field, generated by X. When $Y \neq 0$, such symmetries give rise to a 'local reduction of order' [3], pg. 7, in describing integral curves of X.

Among dynamical similarities, there is a special class appearing in Hamiltonian systems that is often found in physical applications, the so-called scaling symmetries.

Definition 2. $\mathbf{D} \in \mathfrak{X}(M)$ is a *scaling symmetry* of *degree* $\Lambda \in \mathbb{R}$ for the Hamiltonian system (M, ω, H) if

i) $L_{\mathbf{D}}\omega = \omega$

ii) $L_{\mathbf{D}}H = \Lambda H$.

Note that with this definition

$$[\mathbf{D}, X_H] = (\Lambda - 1)X_H \tag{1}$$

and therefore scaling symmetries of symplectic Hamiltonian systems are a very special case of dynamical similarities.

Example 1 (Kepler scalings)**.** *Our archetype motivating this definition is the map between solutions to the (planar) Kepler problem, with*

$$M = T^*(\mathbb{C}\backslash\{0\}) = (\mathbb{C}\backslash\{0\}) \times \mathbb{C} \ni (q, p), \qquad \omega = dp \wedge dq = d(p \cdot dq), \qquad \textit{and} \qquad H_K = \frac{|p|^2}{2} - \frac{1}{|q|} \tag{2}$$

where we remark that, in this representation, the usual dot product is

$$x \cdot y = Re(x\bar{y}).$$

In this case it is well-known that the Kepler scalings

$$q \to \lambda^2 q \qquad \textit{and} \qquad p \to \lambda^{-1} p\,,$$

for $\lambda \in \mathbb{R}^\times$, send orbits into orbits. Considering the generator of these scalings in the phase space,

$$\mathbf{D}_K = 2q \cdot \partial_q - p \cdot \partial_p\,, \tag{3}$$

one has $L_{\mathbf{D}_K}\omega = \omega$ and $L_{\mathbf{D}_K}H_K = -2H_K$, that is, $\mathbf{D}_K$ is a scaling symmetry of degree -2. By equation (1), it follows that the time parameter should be rescaled as $t \to \lambda^3 t$ in order to match the corresponding parametrizations ('Kepler's third law').

Are we defining a central dot via $x \cdot y = Re(xy)$?
Dave

usual dot product: $x \cdot y = Re(x\bar{y})$
Connor

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Ale

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Connor

Before exploring further the role of scaling symmetries in Hamiltonian systems, the following two related remarks are at order:

Remark 1. *The set of vector fields satisfying $L_{\mathbf{D}}\omega = \omega$ (the* Liouville vector fields*) form an affine space, directed by the* symplectic vector fields*: $Y \in \mathfrak{X}(M)$ such that $L_Y\omega = 0$ (equivalently $i_Y\omega$ is closed).*

Remark 2. *A scaling symmetry need not be unique, in fact for any two scaling symmetries, $\mathbf{D}, \mathbf{D}'$, of H one has $i_{\mathbf{D}'}\omega = i_{\mathbf{D}}\omega + \alpha$ for some closed 1-form α satisfying $i_{X_H}\alpha = (\Lambda' - \Lambda)H$, where Λ', Λ are the degrees of $\mathbf{D}', \mathbf{D}$. Conversely, any closed 1-form with $i_{X_H}\alpha = (\Lambda' - \Lambda)H$ determines another scaling symmetry of degree Λ' through $i_{\mathbf{D}'}\omega = i_{\mathbf{D}}\omega + \alpha$. For example, given a first integral F of H, and scaling symmetry $\mathbf{D}$, we have that $\mathbf{D} + X_F$ is also a scaling symmetry of the same degree as $\mathbf{D}$. For example, one may always add X_H to $\mathbf{D}$.*

Let us conclude this section by introducing a re-characterization of scaling symmetries that is equivalent to our definition above but will at times be more useful in computations.

Proposition 1. (M, ω, H) admits a scaling symmetry of degree Λ if and only if there exists a primitive, λ, of ω ($d\lambda = \omega$), satisfying

$$i_{X_H}\lambda = \Lambda H. \tag{4}$$

Proof. One takes $\lambda = i_{\mathbf{D}}\omega$ so that $d\lambda = \omega$ is equivalent to $L_{\mathbf{D}}\omega = \omega$. Moreover, one then has the relation $i_{X_H}\lambda = L_{\mathbf{D}}H$ yielding the equivalence of the condition in Eq. (4) to (ii) of Definition 1. ☐

We can now pose our key questions:

1. Given that many symplectic Hamiltonian (resp. variational) systems admit a scaling symmetry, how may one use it in order to *reduce* the system to one defined on a lower-dimensional manifold?

2. Can we do so in such a way that the reduced system has a contact Hamiltonian (resp. variational) structure?

B. Scaling symmetries: from symplectic mechanics to contact mechanics

Once a scaling symmetry $\mathbf{D}$ is in hand, the idea is to quotient by its action and obtain a reduced dynamics on a lower-dimensional manifold.

More precisely, we consider the projections of trajectories of the Hamiltonian system to the quotient of M by $\mathbf{D}$'s flow: $M/\mathbf{D} = M/\sim$ where $m \sim m'$ when m and m' lie on a common integral curve of $\mathbf{D}$. In general this quotient space need not be a manifold. Throughout, we will simply assume that all quotients, such as $M/\mathbf{D}$, are manifolds, e.g. that $\mathbf{D}$'s flow acts freely ($\mathbf{D}$ has no zeroes) and properly, with submersion $M \to M/\mathbf{D}$.

The first step is to describe the additional structure on this reduced space which may be used to characterize these projected trajectories. This is the content of the next results.

Proposition 2. Suppose the flow of $\mathbf{D}$ is complete, acting freely and properly on M so that $C := M/\mathbf{D}$ is a smooth manifold with submersion

$$\pi : M \to C.$$

Set $\lambda := i_{\mathbf{D}}\omega$. Then

(i) C is a contact manifold with contact distribution $\mathscr{D} := \pi_*(\ker \lambda)$,

(ii) on $\mathscr{D}$ there is a conformal symplectic structure,

(iii) there is a local (exact) symplectomorphism from $(M, d\lambda)$ to the symplectification $(\tilde{C}, d\tilde{\alpha})$ of C.

Proof. (i): see e.g. [23], pg. 36 (or Proposition 12 below). Since there are no zeroes of $\mathbf{D}$, we have local coordinates on C by taking a local transverse slice, Σ, to $\mathbf{D}$. Then with $\lambda = i_{\mathbf{D}}\omega$ we have that $\lambda \wedge (d\lambda)^n = \frac{1}{n} i_{\mathbf{D}}\omega^n$, where $2n = \dim M$, is non-degenerate on Σ, thus determining a contact distribution $\ker \lambda|_\Sigma$ on Σ, i.e. $\pi_* \ker \lambda$ on C.

(ii): this is a familiar property of contact manifolds. Namely when a contact distribution, $\mathscr{D}$, is given by the kernel of a 1-form η then $d\eta|_{\mathscr{D}}$ is a symplectic form on $\mathscr{D}$. Changing η to $f\eta$, with $f \neq 0$, modifies $d(f\eta)|_{\mathscr{D}}$ to $f d\eta|_{\mathscr{D}}$, and therefore there is a conformal symplectic structure on $\mathscr{D}$. In our situation, this conformal symplectic structure is represented by $[\omega(\tilde{u}_1, \tilde{u}_2)]$ for $\pi_*\tilde{u}_j = u_j \in \mathscr{D}$.

(iii): recall (see Definition 11) that $\tilde{C} = Ann(\mathscr{D})\backslash C \subset T^*C$, the symplectification of C, is an $\mathbb{R}^\times$-principal bundle over C with exact symplectic structure, $d\tilde{\alpha}$, given by restriction of the standard symplectic structure on T^*C. Consider $\varphi : M \to \tilde{C}$ defined by

$$(\varphi(m), \pi_* v) := \lambda_m(v)$$

for all $v \in T_m M$ and with $(\cdot, \cdot)$ the natural pairing of a vector space and its dual. One computes that

$$((\varphi^*\tilde{\alpha})_m, v) \overset{\text{def of } \varphi^*}{=} (\tilde{\alpha}_{\varphi(m)}, \varphi_* v) \overset{\text{def of } \tilde{\alpha}}{=} (\varphi(m), (\tilde{\pi}_C)_* \varphi_* v) = (\varphi(m), \pi_* v) \overset{\text{def of } \varphi}{=} \lambda_m(v) \quad \forall v \in T_m M\,,$$

that is, $\varphi^*\tilde{\alpha} = \lambda$, and so φ is a (local) exact symplectomorphism. $\square$

By item (ii) in Proposition 2 there is sense in taking orthogonal complements with respect to the conformal symplectic structure in $\mathscr{D}$. We will denote these complements by $^\perp$. As for the projected orbits in C, we have

Proposition 3. The line field $\mathrm{span}(X_H)$ on M determines a line field

$$\ell := \pi_* \mathrm{span}(X_H)$$

on C. Integral curves of ℓ are projections of orbits of the Hamiltonian system to C. Moreover, letting $\Sigma_0 := \pi(H = 0)$, we have

$$\ell|_{\Sigma_0} = (T\Sigma_0 \cap \mathscr{D})^\perp \subset T\Sigma_0 \cap \mathscr{D}.$$

Proof. ℓ is well defined from the scale invariance of $\mathrm{span}(X_H)$: $L_{\mathbf{D}} X_H = (\Lambda - 1) X_H$. Since the conformal symplectic structure of Proposition 2 (ii) is given by $[\omega(\tilde{u}_1, \tilde{u}_2)]$ with $\pi_*\tilde{u}_j = u_j \in \mathscr{D}$, we will show that $\omega(\tilde{u}, X_H) = 0$ for any $\pi_*\tilde{u} \in T\Sigma_0 \cap \mathscr{D}$. First, note that $\mathbf{D}$ is tangent to $H = 0$ and by $i_{X_H} i_{\mathbf{D}}\omega = \Lambda H$, that $X_H|_{H=0} \in \ker \lambda$. Hence $\ell|_{\Sigma_0} \subset \mathscr{D}$. Let $u \in T\Sigma_0 \cap \mathscr{D}$. Any lift, $\pi_*\tilde{u} = u$, of u lies in $T(H^{-1}(0)) \cap \ker \lambda$ and so $\omega(\tilde{u}, X_H) \sim dH(\tilde{u}) = 0$, since $\tilde{u}$ is tangent to $H = 0$. $\square$

Note that for degree zero scaling symmetries, $\Lambda = 0$, then $\mathbf{D}$ is tangent to each energy level, and the last proposition applies to arbitrary projections of energy levels to determine the line field ℓ on C. From here on we will consider the case $\Lambda \neq 0$. As for describing this line field more explicitly in the general case, we first define:

Definition 3. A *scaling function* for $\mathbf{D}$ is a function $\rho : M \to \mathbb{R}$ such that $L_{\mathbf{D}}\rho = \rho$.

Then we have the following properties.

Proposition 4. Given $\mathbf{D}$, the following are equivalent:

(i) the existence of a *global scaling function*, $\rho : M \to \mathbb{R}^+$, for $\mathbf{D}$

(ii) the existence of a global contact 1-form ($\ker\eta = \mathscr{D}$) on C,

(iii) an embedding $\iota : C \to M$ as a slice of the $\mathbf{D}$-action ($\pi \circ \iota = id$).

Proof. The equivalence of (i) and (ii) is through the relation $\pi^*\eta = \lambda/\rho$. (ii) and (iii) is through $\iota^*\lambda = \eta$. (iii) and (i) is through $\iota(x) = m$ s.t. $\rho(m) = 1$. □

Example 2 (Kepler scaling functions)**.** *Consider the Kepler problem from Example 1. Here there are a number of natural choices for scaling functions available, for example any of:*

$$|q|^{1/2}\,, \qquad \frac{1}{|p|}\,, \qquad p\cdot iq\,, \qquad p\cdot q$$

are scaling functions of $\mathbf{D}_K = 2q\cdot\partial_q - p\cdot\partial_p$ (a degree -2 scaling symmetry of the Kepler Hamiltonian, $H_K = \frac{|p|^2}{2} - \frac{1}{|q|}$). Observe that $p\cdot iq$ is the angular momentum and $p\cdot q$ is (half) the rate of change of the moment of inertia, $q\cdot q$. The quotient, $C = M/\mathbf{D}$, may be identified with $S^1\times\mathbb{C}$.

The *choice* of such a scaling function ρ leads to explicit local coordinate expressions for the scale reduced line field over $\rho \neq 0$, see Eqs. (7) below, due to the following relation to contact Hamiltonian flows, generalizing Arnold's description of contact Hamiltonian vector fields (see Proposition 10 in Appendix A 2 and Corollary 1 below).

Theorem 1 (Contact reduction by scaling symmetries: general case)**.** Let $\mathbf{D}$ be a degree Λ scaling symmetry of H on M, with $\rho : M \to \mathbb{R}^+$ a global scaling function, and corresponding contact form $\pi^*\eta = \lambda/\rho$ on $C = M/\mathbf{D}$. Set $\Sigma_0 := \pi(H = 0)$, and $\ell := \pi_*\mathrm{span}X_H$. Then:

(i) the contact Hamiltonian $\pi^*\mathscr{H} := H/\rho^\Lambda$ has contact Hamiltonian vector field spanning ℓ on Σ_0,

(ii) the contact Hamiltonian $|\mathscr{H}|^{1/\Lambda}$ has contact Hamiltonian vector field spanning ℓ on $C\backslash\Sigma_0$.

We call this reduction a *contact reduction by scaling symmetries*, or simply a *contact reduction*.

Proof. Recall from Proposition 10 that Hamiltonian flows of degree one functions F on M ($L_{\mathbf{D}}F = F$) commute with the scaling action of $\mathbf{D}$ and so induce contact Hamiltonian flows associated with η. Item (i) follows by considering the degree one function $H' := \rho^{1-\Lambda}H$ and item (ii) with the degree one function $|H|^{1/\Lambda}$. Note that the Hamiltonian vector fields, X_H and $X_{H'}$, are proportional over $H = 0$, while those of X_H and $X_{|H|^{1/\Lambda}}$ are proportional over $H \neq 0$. □

Before proceeding, some further comments are in order:

Remark 3. *In practice, the reduction may be carried out by embedding C as a hypersurface $\Sigma := \{\rho = 1\}$ transverse to $\mathbf{D}$. Then $\eta = \lambda|_\Sigma$ and $\mathscr{H} = H|_\Sigma$ are obtained simply by restriction. An equivalent way to state Theorem 1 (without cases), is using Definition 13 in Appendix A 2. Then the* Λ-Hamiltonian vector field *of $\mathscr{H}$ spans ℓ on C (see Corollary 2 below).*

Remark 4. *Case (i) in Theorem 1 is exactly the case considered in [45]. In many relevant physical examples the scaling function ρ can be so chosen so that it is non-vanishing globally on M and is physically interpreted as the overall scale of the system, thus being an irrelevant (unmeasurable) degree of freedom for intrinsic observers. Therefore the system can be reduced by eliminating this overall scale and considering only the relational (shape) degrees of freedom (see also [33, 48]).*

Case (ii) in Theorem 1 is also somewhat familiar. Namely, there is always over $C\backslash\Sigma_0$ a contact 1-form and an embedding into M associated to the scaling function $\rho = |H|^{1/\Lambda}$ (the associated embedding being given by restricting to the energy surfaces $H = \pm 1$). Then $\eta = \lambda|_{H=\pm1}$ and ℓ is defined by $i_\ell d\eta = 0$ (i.e. the dynamics may be parametrized as the Reeb flow of $\lambda|_{H=\pm1}$), see [10, 24].

Remark 5. *It may be the case that* $\mathbf{D}$ *does not admit a global (non-vanishing) scaling function on* M*. Indeed, by Proposition 4, the existence of a global scaling function is equivalent to the symplectification of* C *being a trivial* $\mathbb{R}^\times$*-principal bundle, with connected components,* $\tilde{C}_\pm$*, symplectomorphic to* M*, i.e. that* C *admits some global contact one-form. In what follows, we will simply assume that there is a global scaling function, or if the reader prefers that we are working locally over the set* $\rho > 0$ *of* some *scaling function, corresponding to a local trivialization of* $\mathbb{R} \to M \to C$*.*

For comparison to standard constructions in projective geometry and the 's-scalars or vector fields' used in [1], as well as the structure used in [28] and [29], one may state all our results here in terms of appropriate associated bundles over C*, whose sections correspond to degree* α *functions, vector fields, one-forms, etc. on* M *under* $\mathbf{D}$*. Our scaling functions (and their resulting coordinate descriptions) correspond then to working in a local trivialization of these bundles. In what follows however, we will simply use scaling functions on* M *to obtain explicit expressions.*

By Theorem 1, one may treat the general reduction of the symplectic Hamiltonian system to a *contact system* locally as two subcases: with the degree one $|H|^{1/\Lambda}$ over $H \neq 0$, or with $\rho^{1-\Lambda}H$ over $H = 0$, for some scaling function ρ. It is important to stress that for $\Lambda \neq 1$ none of these two cases can be non-trivially extended as contact vector fields (whose flows preserve $\mathscr{D}$) to the other region. While this is obvious for $|H|^{1/\Lambda}$ over $H = 0$, as it is not differentiable in general, it is not so immediate for $\rho^{1-\Lambda}H$ over $H \neq 0$, and this will be explained in more detail shortly (see Corollary 2). Clearly there is a very important special situation in which the two subcases coincide, i.e. when $\Lambda = 1$. In this situation, the two contact Hamiltonian flows in Theorem 1 coincide and we obtain the following reduction:

Corollary 1 (Contact reduction by scaling symmetries: degree one case)**.** Let (M, ω, H) be a Hamiltonian system admitting a scaling symmetry $\mathbf{D}$ of degree one. Then:

i) X_H defines a contact vector field $X = \pi_* X_H$ on C,

ii) for a scaling function ρ, and $\pi^*\eta = \lambda/\rho$, X is the contact Hamiltonian vector field of $\pi^*\mathscr{H} = H/\rho$.

By virtue of the above Corollary, in the case $\Lambda = 1$ one obtains a 'full' contact reduction, meaning that the reduced contact system is completely determined by the original symplectic one (this is precisely the content of Proposition 10 in Appendix A 2, which we have recovered as a particular case). Note also that in this case H itself may be chosen as a scaling function giving the reduced dynamics as a Reeb flow of $\pi^*\eta = \lambda/H$ on $C\backslash\Sigma_0$.

In the general case, the line field we have been considering may in fact be described more explicitly with the use of a scaling function, as detailed in the following

Corollary 2 (Dependence on the scaling function)**.** Let (M, ω, H) be a Hamiltonian system admitting a scaling symmetry $\mathbf{D}$ of degree Λ and ρ a scaling function for $\mathbf{D}$. Then the line field ℓ on $M/\mathbf{D}$ is spanned by

$$X := X_{\mathscr{H}} + (\Lambda - 1)\mathscr{H}\mathscr{R}\,, \tag{5}$$

where $X_{\mathscr{H}}$ is the contact Hamiltonian vector field of $\pi^*\mathscr{H} = H/\rho^\Lambda$ with respect to $\pi^*\eta = \lambda/\rho$ and $\mathscr{R}$ is the Reeb vector field of η on C. Moreover, for any other choice of scaling function, $\tilde{\rho}$, we have

$$\tilde{X} = \left(\frac{\tilde{\rho}}{\rho}\right)^{1-\Lambda} X.$$

Proof. Given a scaling function, ρ, then by Eq. (1), the vector field $\rho^{1-\Lambda}X_H$ is scale invariant and projects to a vector field,

$$X = \pi_* \rho^{1-\Lambda} X_H,$$

on C spanning ℓ. Set $H' := \rho^{1-\Lambda}H$, a degree one Hamiltonian on M with $\pi_* X_{H'} = X_{\mathscr{H}}$ the contact Hamiltonian vector field of $\pi^*\mathscr{H} = H/\rho^\Lambda$ with respect to $\pi^*\eta = \lambda/\rho$. Then

$$dH' = (1 - \Lambda)\rho^{-\Lambda}H \; d\rho + \rho^{1-\Lambda} \; dH$$

so that

$$X_{H'} = (1 - \Lambda)\rho^{-\Lambda}H \; X_\rho + \rho^{1-\Lambda} \; X_H\,. \tag{6}$$

Note that by Corollary 1, the Hamiltonian vector field X_ρ of ρ projects to the Reeb vector field $\mathscr{R} = \pi_* X_\rho$ of η on C. Applying π_* to the last equation we have Eq. (5).

For the last relation, note that for two scaling functions, ρ and $\tilde{\rho}$, we have that $\tilde{\rho}/\rho$ is scale invariant, so there is a function σ on C through $\pi^*\sigma = \tilde{\rho}/\rho$. Then

$$\tilde{X} = \pi_* \tilde{\rho}^{1-\Lambda} X_H = \sigma^{1-\Lambda} X.$$

□

Remark 6. *In Darboux coordinates $\eta = p_a dq^a - dS$ on C, Eq. (5) reads*

$$(q^a)' = \partial_{p_a}\mathscr{H}, \qquad (p_a)' = -\partial_{q^a}\mathscr{H} - p_a\partial_S\mathscr{H}, \qquad S' = p_a\partial_{p_a}\mathscr{H} - \Lambda\mathscr{H}\,. \tag{7}$$

These equations may also be derived directly from Hamilton's equations for H by following Remark 3. Namely, given a scaling function ρ, we take $\Sigma = \{\rho = 1\}$. Then a system of contact coordinates, $\eta = p_a dq^a - dS$ on C, may be extended to symplectic coordinates $P_0 = S, Q_0 = \rho, P = \rho p, Q = q$ on M with $\lambda = \rho\eta = -Q_0 dP_0 + P\cdot dQ$. Then, for $\mathscr{H}(q,p,S)$, we have that

$$H(Q_0, Q, P_0, P) = Q_0^\Lambda \mathscr{H}(Q, P/Q_0, P_0)$$

since $H/\rho^\Lambda = \mathscr{H}$. By chain rule, Hamilton's equations of motion are given as:

$$\rho^{1-\Lambda}\dot{S} = p_a\partial_{p_a}\mathscr{H} - \Lambda\mathscr{H}, \qquad \rho^{1-\Lambda}\dot{\rho} = \rho\partial_S\mathscr{H}, \tag{8}$$
$$\rho^{1-\Lambda}\dot{q}^a = \partial_{p_a}\mathscr{H}, \qquad \rho^{1-\Lambda}\dot{p}_a = -\partial_{q^a}\mathscr{H} - p_a\partial_S\mathscr{H}\,, \tag{9}$$

which project onto $\Sigma = \{\rho = 1\}$ as Eqs. (7). It is also evident from Eqs. (8), (9) of this last computation that, under the reparametrization $\rho^{1-\Lambda}d\tau = dt$, the scale-reduced equations of motion (7) contain a 'blow-up', of Hamilton's equations of motion to $\rho = 0$, as one might expect (compare with [12, 13, 43] for n-body problems). When $\Lambda < 0$, this blow-up is given as the (invariant) set $\Sigma_0 = \{\mathscr{H} = 0\}$.

Remark 7. *The flow of the vector field X in Eq. (5), induced by a scaling function ρ, does not in general preserve $\mathscr{D}$. However restricted to the invariant set $\Sigma_0 = \{\mathscr{H} = 0\}$ it does, being a contact Hamiltonian vector field (case (i) of Theorem 1). In general, one may rescale X away from this set so that its flow does preserve $\mathscr{D}$. Namely, for the scaling function $\tilde{\rho} = |H|^{1/\Lambda}$, we have $\tilde{\mathscr{H}} = 1$, and $\tilde{X} = \Lambda\tilde{\mathscr{R}}$ is a constant multiple of the Reeb vector field of $\tilde{\eta}$ (case (ii) of Theorem 1) so that $|\mathscr{H}|^{1-1/\Lambda}X = \tilde{X}$ preserves $\mathscr{D}$.*

Another interesting aspect about Theorem 1 is the fact that it provides a direct connection with the physics of the problem. Indeed, the function ρ is usually connected with a global scale within the physical description (hence the 'scaling function' name), and therefore by using it in order to reduce the dynamics we automatically obtain a description of the same physical problem in terms of scale-invariant functions only.

While it may seem that the 'ambiguity' in Corollary 2 due to one's choice of scaling function is a defect of the general contact reduction, we posit instead that this reparametrization freedom to choose a scaling function to describe the projected trajectories is in fact an asset of the theory, allowing one to highlight certain aspects of the original systems dynamics by various choices of scaling function corresponding to various reparametrizations. We illustrate the reduction procedure with the Kepler problem in the following example.

Example 3 (Contact-reduced Kepler)**.** *Consider the Kepler problem from Example 1. To illustrate the contact reduction process, one may first consider polar coordinates:*

$$q = re^{i\theta}, \qquad p = \left(p_r + i\frac{p_\theta}{r}\right)e^{i\theta},$$

with scaling functions (Example 2) that we denote:

I think p is the complex conjugate of this
Dave

$$\rho := |q|^{1/2} = r^{1/2}, \qquad J := p\cdot q = rp_r, \qquad G := p\cdot iq = p_\theta,$$

In the coordinates (ρ, θ, J, G) on $M = T^*(\mathbb{C}\backslash\{0\})$, we have

$$p\cdot dq = 2J\frac{d\rho}{\rho} + Gd\theta,$$

it looks ok to me as is (with usual dot product)
Connor

so that

$$\omega = d(p\cdot dq) = \frac{2dJ\wedge d\rho}{\rho} + dG\wedge d\theta, \tag{10}$$

and

$$\mathbf{D}_K = \rho\partial_\rho + J\partial_J + G\partial_G, \qquad \lambda = i_{\mathbf{D}_K}\omega = G\;d\theta - 2J\;d\log\frac{J}{\rho}, \qquad H_K = \frac{J^2+G^2}{2\rho^4} - \frac{1}{\rho^2}\,. \tag{11}$$

The functions J/ρ, G/ρ and θ are $\mathbf{D}_K$-invariant, determining coordinates $(\tilde{J}, \tilde{G}, \theta)$ on the quotient $M/\mathbf{D}_K$ by:

$$\pi^*\tilde{J} = J/\rho, \qquad \pi^*\tilde{G} = G/\rho, \qquad \pi^*\theta = \theta$$

where $\pi : M \to M/\mathbf{D}_K = C$. According to Theorem 1, the scale-reduced orbits may be described on C using:

$$\eta = \tilde{G}\ d\theta - 2d\tilde{J}, \qquad \mathscr{H} = \frac{\tilde{J}^2 + \tilde{G}^2}{2} - 1 \tag{12}$$

with $\pi^*\eta = \lambda/\rho$ and $\pi^*\mathscr{H} = \rho^2 H_K$. More explicitly, by Corollary 2 or Remark 6 with $\Lambda = -2$, the scale-reduced equations of motion may be written:

$$\tilde{J}' = \frac{\tilde{G}^2}{2} + \mathscr{H}, \qquad \tilde{G}' = -\frac{\tilde{G}\tilde{J}}{2}, \qquad \theta' = \tilde{G}. \tag{13}$$

Remark 8. *The scaling symmetry, $\mathbf{D}_K$, of the Kepler problem and its corresponding scale-invariant functions, e.g. $\tilde{J}, \tilde{G}, \mathscr{H}$, have long been known and used in celestial mechanics, in particular their analogues for n-body problems. See for example Section 2.3 of Chenciner's [14], where relations of the scale reduction to McGehee blow-up are explained. On the other hand, the contact structure associated to these contact reductions, although present already in the works of Bryant [12, 13], has been emphasized only more recently [45]. Observe that the contact reduction here is closely related to the structure of 'b-manifolds' in [5]. In particular compare Eq. (10) here to the Darboux normal form (Theorem 2 of [5]) of a 'b-symplectic form' and Example 8.2 of [42] to our construction here of a scale-reduced contact form.*

An important property of the reduction by Theorem 1 is its relation to certain blow-ups as mentioned at the end of Remark 6. We illustrate this relation with the scale-reduced Kepler problem.

Example 4 (Kepler blow-up)**.** *We continue from the previous example with the scale-reduced Kepler problem using the global scaling function $\rho = |q|^{1/2}$. According to Eqs. (8) and (9) with $\Lambda = -2$, the scale-reduced equations of motion on $\pi^*\mathscr{H} = \rho^2 H_K = 0$, represent a blow-up of the dynamics at $\rho = 0$ (collision).*

One may apply Proposition 3 to determine the scale-reduced orbits on $\mathscr{H} = 0$. From Eq. (12), we have:

$$\mathscr{D} = \ker\eta = span\{2\partial_\theta + \tilde{G}\partial_{\tilde{J}}, \partial_{\tilde{G}}\}$$

while $\Sigma = \{\mathscr{H} = 0\}$ is a torus, parametrized by the angle $\theta \in S^1$ and the circle $\tilde{J}^2 + \tilde{G}^2 = 2$. So:

$$T\Sigma_0 = span\{\partial_\theta, \tilde{G}\partial_{\tilde{J}} - \tilde{J}\partial_{\tilde{G}}\}.$$

The collision orbits of the Kepler problem are the homothetic motions. They tend in forwards or backwards time to the fixed points of Eqs. (13), $\tilde{J} = \pm\sqrt{2}, \tilde{G} = 0$. Observe that $\mathscr{D}$ is tangent to Σ_0 exactly at these fixed points. Away from these points, by Proposition 3, the projected Kepler orbits on Σ_0 are integral curves of:

$$\ell|_{\Sigma_0} = \mathscr{D} \cap T\Sigma_0 = span\{2\partial_\theta + \tilde{G}\partial_{\tilde{J}} - \tilde{J}\partial_{\tilde{G}}\}.$$

Taking the ($\mathbf{D}_K$-invariant) angle φ by $p = |p|e^{i\varphi}$, this is the line field:

$$\ell|_{\Sigma_0} = span\{\partial_\varphi + 2\partial_\theta\}.$$

The scale-reduced dynamics on this torus, Σ_0, are a blow-up at collision $\rho = |q|^{1/2} = 0$ of the equations of motion for the Kepler problem. We recover the well-known 'elastic bounce' regularization and collision torus (see §1.3 of [20]), although our path here has been rather different.

In Section IV C we consider some more cases of contact reductions by scaling symmetries in celestial mechanics.

We close this section with an interesting comment about the relationship between Corollary 1 and the contact reduction in [49], which in turn gives yet another proof of Arnold's Proposition 10 in terms of a contact version of symplectic reduction.

The relevant statement from [49] concerns an $\mathbb{R}$-action on a contact manifold N with choice of contact one-form $\hat{\eta}$ by *exact* contactomorphisms: $L_X\hat{\eta} = 0$, where $X \in \mathfrak{X}(N)$ generates the $\mathbb{R}$-action and where $\hat{\mathscr{H}}$ is an $\mathbb{R}$-*invariant* contact Hamiltonian, $X\hat{\mathscr{H}} = 0$, on N. Let $J_0 := (i_X\hat{\eta})^{-1}(0) \subset N$. Then $X, X_{\hat{\mathscr{H}}}$ are tangent to J_0 and the quotient $P_0 := J_0/X$ is a contact manifold with reduced contact dynamics, $(\mathscr{H}, \eta)$, defined through the following diagram:

$$\begin{tikzcd} & N \\ J_0 \arrow[ur, "\iota_0"] \arrow[dr, "\pi_0"'] & \\ & P_0 \end{tikzcd}$$

$$\pi_{0*}X_{\hat{\mathscr{H}}} = X_{\mathscr{H}} \quad \text{for} \quad \pi_0^*\mathscr{H} = \iota_0^*\hat{\mathscr{H}}, \quad \pi_0^*\eta = \iota_0^*\hat{\eta}.$$

Applying this construction, we obtain

Proposition 5. Consider a Hamiltonian system (M, ω, H) admitting $\mathbf{D}$ as a degree one scaling symmetry. For $\lambda = i_{\mathbf{D}}\omega$ and a global scaling function $\rho : M \to \mathbb{R}^+$, let $N := \mathbb{R} \times M$ with $\hat{\eta} := e^{-t}(\lambda + d\rho) - dt$ where t is the coordinate on $\mathbb{R}$. Then

(i) N is a contact manifold and $\hat{\mathbf{D}} = \partial_t + \mathbf{D}$ generates an $\mathbb{R}$-action on N by exact contactomorphisms of $\hat{\eta}$,

(ii) the contact Hamiltonian $\hat{\mathscr{H}} = e^{-t}H$ on N is invariant under the flow of $\hat{\mathbf{D}}$.

(iii) The contact manifolds $C = M/\mathbf{D}$ and $P_0 = J_0/\hat{\mathbf{D}}$ are contactomorphic and moreover the reduced contact dynamics (of [49]) on P_0 corresponds to the contact Hamiltonian dynamics on C (of Theorem 1).

Proof. The corresponding moment map of [49] is $i_{\hat{\mathbf{D}}}\eta = e^{-t}\rho - 1$ and so J_0 is the graph $\{e^t = \rho\} \subset N$, identified with M via $\iota : m \mapsto (\log \rho(m), m)$. Under this identification, we have $\iota_*\mathbf{D} = \partial_t + \mathbf{D} = \hat{\mathbf{D}}|_{J_0}$ so that $C = M/\mathbf{D} \cong P_0 = J_0/\hat{\mathbf{D}}|_{J_0}$. Moreover, $\iota^*\hat{\eta} = \lambda/\rho$ and $\iota_*(e^{-t}H) = H/\rho$. Hence, the contact-reduced flow corresponds to the contact Hamiltonian flow of $\pi^*\mathscr{H} = H/\rho$ with respect to $\pi^*\eta = \lambda/\rho$. □

III. CAN ONE ALWAYS FIND A CONTACT REDUCTION?

Given the above discussion on the general existence of a contact reduction ensuing from Theorem 1 and Corollary 1, we are now interested in the following fundamental question:

given a symplectic Hamiltonian system (M, ω, H) admitting a scaling symmetry $\mathbf{D}$ of degree Λ, can one transform $\mathbf{D}$ to $\tilde{\mathbf{D}}$ such that the new vector field is a scaling symmetry with $\tilde{\Lambda} = 1$?

This will be the content of this section. We will first prove that $\tilde{\mathbf{D}}$ always exists locally and find an explicit algorithm to construct it, and then we will use an example in order to make clear that such a symmetry cannot be found globally in general on M. This will lead us to consider an extension of (M, ω, H) to a 'lifted system', where various parameters are considered as dynamical variables (together with their conjugate momenta). Rather surprisingly, in this space we can prove the existence of a scaling symmetry of degree one, and therefore obtain its corresponding contact reduction, even for cases in which the original Hamiltonian system on M had no (evident) scaling symmetry at all.

A. Locally on M: yes

We begin with the following observation: for any Hamiltonian system in a neighborhood of m with $X_H(m) \neq 0$, there exist Darboux coordinates such that $H = p_1$ and $\omega = dp_a \wedge dq^a$. Then $\tilde{\mathbf{D}} = p_a\partial_{p_a}$ is a degree one scaling symmetry with $\lambda = p_a dq^a = i_{\tilde{\mathbf{D}}}\omega$. We conclude that locally, away from critical points of H, it is always possible to find a scaling symmetry, $\tilde{\mathbf{D}}$, of degree one.

This observation may be described in the following algorithm to find $\tilde{\mathbf{D}}$ locally:

1. fix *some* primitive, λ, of ω and seek a closed 1-form α such that
$$i_{X_H}(\lambda - \alpha) = H \,. \tag{14}$$

2. By the discussion in Section II A (see e.g. Proposition 1), the vector field $\tilde{\mathbf{D}}$ corresponding to $\lambda - \alpha$ via
$$i_{\tilde{\mathbf{D}}}\omega = \lambda - \alpha \tag{15}$$
is the generator of a degree one scaling symmetry.

We conclude that (14) is the equation to be solved (for α) in order to find $\tilde{\mathbf{D}}$.

Notice that, locally at least, $\alpha = dS$ for some function $S : M \to \mathbb{R}$ and then by (15) its associated Hamiltonian vector field X_S is such that $\tilde{\mathbf{D}} = X + X_S$, with X the Liouville vector field satisfying $i_X\omega = \lambda$. Locally therefore, we can make (14) even more explicit. Indeed, we obtain

$$X_H S = i_{X_H}\lambda - H \,. \tag{16}$$

In particular, in local Darboux coordinates with $\lambda = p_a dq^a$, this gives

$$\dot{S} = p_a\dot{q}^a - H = L \,, \tag{17}$$

and therefore S is the action. Remarkably, it is always possible to solve (17) locally away from critical points of H by the choice of a transverse slice Σ to X_H identifying $M \cong \Sigma \times (-\varepsilon, \varepsilon) \ni (\mathbf{s}, t)$ and taking $S(\mathbf{s}, t) := \int_0^t L(\varphi_\tau(\mathbf{s}))\, d\tau$. However, it is clear that globally there may be obstructions to solving (16) – resp. (14) – in general (see e.g. the Kepler case in Section III B). Therefore an alternative route should be put forward, as we will do in Section III C.

In order to illustrate the above procedure, we consider now a very simple case in which the scaling symmetry $\tilde{\mathbf{D}}$ can be found explicitly:

Example 5 (The 2d harmonic oscillator). *We have* $H = \frac{|p|^2 + k|q|^2}{2}$, $\lambda = p \cdot dq$ *and* $\omega = d\lambda$. *Imposing* $i_X \omega = \lambda$ *we get* $X = p \cdot \partial_p$ *which is by definition a Liouville vector field, however not a scaling symmetry of* H.

Now we seek a function S *such that* $\tilde{\mathbf{D}} = X + X_S$ *is a scaling symmetry with* $\Lambda = 1$. *Solving* (16) *we get*

$$S = \frac{p \cdot q}{2} + cp \cdot iq \tag{18}$$

where c *is a constant of integration that we will fix to 0 (this freedom in fixing* $\tilde{\mathbf{D}}$ *comes from the fact that* $p \cdot iq$*, the angular momentum, is an integral of motion, cf. the discussion in Remark 2). This degree one scaling symmetry with* $c = 0$ *is just* $\tilde{\mathbf{D}} = X + X_S = \frac{p \cdot \partial_p + q \cdot \partial_q}{2}$.

Note that in the last example S (and thus $\tilde{\mathbf{D}}$) is globally defined on M. In the next section we will see an example in which a global solution to Eq. (14) on M cannot exist.

B. Globally on M: no

Let us consider the Kepler problem of Example 1. This admits the 'standard scaling symmetry' $\mathbf{D}_K = 2q \cdot \partial_q - p \cdot \partial_p$, which has degree -2. May we apply our algorithm to find a different 'hidden scaling symmetry' of degree 1?

By the discussion in the previous section, this question is equivalent to finding a primitive λ' of ω_K with $i_{X_{H_K}} \lambda' = H_K$. The degree one scaling symmetry is then defined by $i_{\tilde{\mathbf{D}}} \omega_K = \lambda'$. So, let $\lambda = p_a dq^a$ be the canonical 1-form, then all of our options for primitives are given by $\lambda - \alpha$ with α a closed 1-form.

It is instructive to consider first the case in which $\alpha = dS$ is exact. Then we look for a solution of (16). As an immediate observation, we see that for the Kepler problem a solution S fails to exist globally because there are periodic orbits with positive action. Namely, for an elliptic orbit, $\gamma \subset T^*(\mathbb{R}^2 \backslash \{0\})$, with period T and (negative) energy E, a solution S to (16) would satisfy:

$$0 \overset{!}{=} \oint_\gamma dS = S(\mathbf{s}, T) - S(\mathbf{s}, 0) = \int_0^T L(\varphi_\tau(\mathbf{s}))\, d\tau = 3\pi \left(\frac{T}{2\pi}\right)^{1/3} = 3\pi \sqrt{\frac{-1}{2E}} > 0\,. \tag{19}$$

In general, for any closed α as above, consider two periodic elliptic trajectories $\gamma_1, \gamma_2 \subset T^*(\mathbb{R}^2 \backslash \{0\})$ with different values of (negative) energies, $E_1 \neq E_2$ respectively. Such trajectories bound some surface, $\Sigma \subset T^*(\mathbb{R}^2 \backslash \{0\})$, in the phase space. Then, since α is closed, $d\alpha = 0$, we have by Stokes theorem:

$$\oint_{\gamma_1} \alpha = \oint_{\gamma_2} \alpha \tag{20}$$

As we have fixed $\lambda = p_a dq^a$ as the canonical 1-form, Eq. (14) reads

$$i_{X_H} \alpha = p_a \frac{\partial H}{\partial p_a} - H = L\,. \tag{21}$$

Therefore, since γ_1, γ_2 are integral curves of X_H, (20) now reads

$$\int_0^{T_1} L(\gamma_1(t))\, dt = \int_0^{T_2} L(\gamma_2(t))\, dt \tag{22}$$

where $T_1 \neq T_2$ are the periods of γ_1, γ_2. This is impossible, because for the Kepler problem, the total action around an elliptic orbit is a non-constant function of its energy (and period). We conclude that such a α cannot exist in the Kepler case.

In Example 5 for the 2d harmonic oscillator we have used precisely this route to find $\alpha = dS$. Note that in that case we have that the condition (22) is identically satisfied (both sides always being zero).

C. Global contact reductions. The evolution of couplings

Hamiltonian systems, typically, depend on various parameters (e.g. masses, gravitational constants, etc.) whose values are given in certain physical units. The scaling symmetries –acting on positions and momenta– we have been considering, boil down to rescalings of these units. Thus, from a physical point of view, it is natural for such a scaling symmetry to induce an action on these parameters.

We will consider then the following general situation. Let (M, ω, H_A) be a Hamiltonian system depending on some parameters $A \in \mathbb{R}^k$. In place of considering only rescalings on M, we would like to admit the possibility to act on the parameters A. We will state conditions under which such 'extended' scaling symmetries may be used to produce scale-reduced contact Hamiltonian systems depending on parameters.

Remark 9. *Intuitively, the objective is the following: for a symplectic Hamiltonian system, H_A, depending on k parameters on a symplectic manifold of dimension $2n$, one would aim for –as its scale reduction– a contact system on a contact manifold of dimension $2n-1$ depending on k-parameters. Alternately, we may view the symplectic Hamiltonian system with parameters as dynamics on a space of dimension $2n+k$ admitting k first integrals, whose invariant level sets are symplectic manifolds with symplectic Hamiltonian dynamics. Then we aim for –as a scale reduction– a dynamics on a space of dimension $2n-1+k$ admitting k first integrals, whose invariant level sets are contact manifolds with contact Hamiltonian mechanics.*

In many cases, one may simply redefine the parameters so that H_A is degree one in A. For this, it is convenient to introduce the following definition.

Definition 4. *A* system of coupled Hamiltonians *is a system (M, ω, H_A), where*

$$H_A = A_1 H_1 + ... + A_k H_k = A \cdot H \tag{23}$$

with parameters $A_j \in \mathbb{R}$ called the couplings *and $H := (H_1, ..., H_k) : M \to \mathbb{R}^k$. Moreover, to avoid considering repetitive cases, we will always assume in what follows that*

1. $A_j > 0$ for all $j = 1, \dots, k$.
2. There is a Liouville vector field $\mathbf{D}$ on M with $\mathbf{D}H_j = \Lambda_j H_j$.
3. $\Lambda_i \neq \Lambda_j$ for all $i \neq j$ and $\Lambda_k \neq 1$.

Indeed, if some $A_j = 0$, one would simply proceed with less parameters, while if some $A_j < 0$ it would just amount to changing the sign in front of the corresponding term in (23). On the other hand, if all Λ_j were equal, one could proceed by applying Theorem 1. Further, note that one may collect the corresponding terms of (23) for those j's with $\Lambda_j = \Lambda_{j'}$ to take as a new $H_{j'}$ term, so that we assume all Λ_j are distinct. In particular, at least one Λ_j is not equal to 1, which we take without loss of generality as $\Lambda_k \neq 1$.

Our main result (Theorem 2 below), extends Theorem 1 to this situation in line with Remark 9. Our starting point is the following computation:

Proposition 6. Let (M, ω, H_A) be a system of coupled Hamiltonians. Then, taking the *extended space*

$$\hat{M} := M \times \mathbb{R}^k_+ \times \mathbb{R}^k \ni (m, a, b), \qquad \hat{\omega} := \omega + da \wedge db \,,$$

with

$$a_j := A_j^{\frac{1}{1-\Lambda_j}} > 0, \qquad \hat{\mathbf{D}} := \mathbf{D} + a \cdot \partial_a \,,$$

and considering $\hat{H} : \hat{M} \to \mathbb{R}$ given by

$$\hat{H} := a_1^{1-\Lambda_1} H_1 + ... + a_k^{1-\Lambda_k} H_k \,, \tag{24}$$

we have

$$\hat{\mathbf{D}}\hat{H} = \hat{H}, \qquad L_{\hat{\mathbf{D}}}\hat{\omega} = \hat{\omega} \,,$$

that is, $\hat{\mathbf{D}}$ is a degree one scaling symmetry for the symplectic Hamiltonian system $(\hat{M}, \hat{\omega}, \hat{H})$.

Remark 10. *The symmetry of $\hat{\mathbf{D}}$ may be understood as follows. If ψ_s is the flow of $\mathbf{D}$ on M (and $\tilde{\psi}_s$ the induced flow of $\hat{\mathbf{D}}$ on $M \times \mathbb{R}^k \ni (m, a)$) then for $\gamma_a(t) \in M$ a trajectory of X_{H_a}, we have that $\psi_s(\gamma_a(t))$ is a trajectory of $X_{H_{a'}}$ where $a'_j = e^s a_j = \tilde{\psi}_s(a_j)$.*

From the last proposition, we may apply a degree one contact reduction (cf. Corollary 1) on $\hat{M}$ to obtain:

Corollary 3. *The quotient:*

$$\hat{\pi} : \hat{M} \to \hat{C} = \hat{M}/\hat{\mathbf{D}}$$

is a contact manifold, with contact distribution

$$\hat{\mathscr{D}} := \hat{\pi}_* \ker \hat{\lambda}$$

where $\hat{\lambda} = i_{\hat{\mathbf{D}}}\hat{\omega} = \lambda + a \cdot db$ *for* $\lambda = i_{\mathbf{D}}\omega$. *The symplectic Hamiltonian vector field,* $X_{\hat{H}}$ *of* $\hat{H}$ *on* $\hat{M}$ *projects to a contact vector field,* $\hat{X}$*, on* $\hat{C}$*, given by*

$$\hat{X} := \hat{\pi}_* X_{\hat{H}} \, .$$

We are however not yet done since $\hat{C}$ is too large to qualify as a reduced space. Nevertheless, we may still use the following:

Proposition 7. *On* $\hat{C}$ *there is a contact action of* $\mathbb{R}^{k+1}$*, generated by the commuting vector fields*

$$\hat{X} = \hat{\pi}_* X_{\hat{H}}, \quad X_1 = \hat{\pi}_* X_{a_1}, \quad \ldots, \quad X_k = \hat{\pi}_* X_{a_k} \, .$$

As well, we have k *independent first integrals on* $\hat{C}$ *of these vector fields,* $\hat{\alpha}_1, ..., \hat{\alpha}_k$*, given by*

$$\hat{\pi}^* \hat{\alpha}_j = a_j / a_k > 0, \quad j = 1, \ldots, k-1, \qquad \textit{and} \qquad \hat{\pi}^* \hat{\alpha}_k = \hat{H}/a_k \, .$$

Proof. Consider the Hamiltonian vector field $X_{a_j} = \partial_{b_j}$ of a_j on $(\hat{M}, \hat{\omega})$. Since $\hat{\mathbf{D}} a_j = a_j$ we have an induced contact vector field $X_j = \hat{\pi}_* X_{a_j}$ on $\hat{C}$. These vector fields and $\hat{X}$ mutually commute since their lifts, $X_{\hat{H}}, X_{a_j}$ commute. The functions $\hat{\alpha}_1, ..., \hat{\alpha}_k$ are first integrals since their lifts, a_j/a_k and $\hat{H}/a_k$, are first integrals of $X_{\hat{H}}, X_{a_j}$. □

Remark 11. *Given a choice of a contact 1-form* $\hat{\pi}^* \hat{\eta} = \hat{\lambda}/\hat{\rho}$ *on* $\hat{C}$*, with say* $\hat{\rho} > 0$ *some scaling function of* $\hat{\mathbf{D}}$*, the contact vector fields* $X_1, ..., X_k$ *correspond to* k dissipated quantities *of* $\hat{X}$*, given by (see [7, 19])*

$$\hat{\pi}^* \hat{a}_j := \hat{\pi}^* \left(i_{X_j} \hat{\eta} \right) = \frac{a_j}{\hat{\rho}} > 0 \, .$$

Dissipated quantities of a contact Hamiltonian system determine first integrals by considering their ratios. Here, for $\hat{\mathscr{H}} := i_{\hat{X}} \hat{\eta}$ *the contact Hamiltonian of* $\hat{X}$*, these corresponding first integrals may be given as those of the previous proposition:*

$$\hat{\alpha}_j = \frac{\hat{a}_j}{\hat{a}_k} \, , \quad j = 1, \ldots, k-1 \, , \qquad \textit{and} \qquad \hat{\alpha}_k = \frac{\hat{\mathscr{H}}}{\hat{a}_k} \, .$$

From the last proposition, one sees how to produce a full reduction of a system of coupled Hamiltonians (M, ω, H_A). Namely, one restricts to a level set of the first integrals $\hat{\alpha}_1 \ldots, \hat{\alpha}_k$ and passes to its quotient under the $\mathbb{R}^k$-action generated by $X_1, ..., X_k$, obtaining an induced dynamics of $\hat{X}$ on a $(\dim M - 1)$-dimensional space. More precisely, we have the following main result:

Theorem 2 (Contact reduction of systems of coupled Hamiltonians)**.** *Let* (M, ω, H_A) *be a system of coupled Hamiltonians. Consider the resulting contact system* $\hat{C}, \hat{X}$*, as described in Proposition 6, Corollary 3 and Proposition 7 above, with first integrals* $\hat{\alpha}_j$ *and contact symmetries* X_j *generating an* $\mathbb{R}^k$*-action on* $\hat{C}$ *(cf. Remark 11). Then, for a regular level set* $\{\hat{\alpha}_j = const.\}$ *of these first integrals, its quotient:*

$$C_o := \left(\{\hat{\alpha}_j = const.\} \backslash \hat{\Sigma} \right) / \mathbb{R}^k$$

is a contact manifold, where

$$\hat{\Sigma} := \hat{\pi}(\{\mathbf{D}\hat{H} = 0\}) \, .$$

Moreover, the vector field X_o *induced on* C_o *by* $\hat{X}$ *is spanned by a contact vector field.*

Proof. To describe this reduced space, C_o, let us first consider the quotients under the $\mathbb{R}^k$-action generated by $X_j = \hat{\pi}_* \partial_{b_j}$. First, note that the 'upstairs' quotient:

$$\pi_M : \hat{M} \to \hat{M}/\mathbb{R}^k =: \tilde{M} = M \times \mathbb{R}^k \ni (m, a)$$

is just the standard projection $(m, a, b) \mapsto (m, a)$. As well, the scaling vector field $\hat{\mathbf{D}}$ on $\hat{M}$ projects to a scaling vector field, $\tilde{\mathbf{D}} = \mathbf{D} + a \cdot \partial_a$ on $\tilde{M}$. These spaces are related in the diagram:

$$\begin{tikzcd} \hat{M} \arrow[r, "\pi_M"] \arrow[d, "\hat{\pi}"] & \tilde{M} \arrow[d, "\tilde{\pi}"] \\ \hat{C} \arrow[r, "\pi_C"] & \tilde{C} \end{tikzcd}$$

$$\pi_{M*}\hat{\mathbf{D}} = \tilde{\mathbf{D}}\,, \qquad \hat{C} = \hat{M}/\hat{\mathbf{D}}\,, \qquad \tilde{C} = \tilde{M}/\tilde{\mathbf{D}} = \hat{C}/\mathbb{R}^k\,.$$

We may project the $\mathbb{R}^k$-invariant structures on $\hat{M}$ and $\hat{C}$ to their analogues on $\tilde{M}$ and $\tilde{C}$. Thus we have a $\tilde{\mathbf{D}}$-invariant vector field, $\pi_{M*}X_{\hat{H}}$, and $\tilde{\mathbf{D}}$-invariant function, $\pi_M^*\tilde{H} = \hat{H}$ on $\tilde{M}$. The induced scale-reduced dynamics on this quotient, $\tilde{C} = \hat{C}/\mathbb{R}^k$, being given via:

$$\tilde{X} := \pi_{C*}\hat{X} = \tilde{\pi}_*\pi_{M*}X_{\hat{H}},$$

and also admitting k independent first integrals $\tilde{\alpha}_1, ..., \tilde{\alpha}_k$, defined by

$$\pi_C^*\tilde{\alpha}_j := \hat{\alpha}_j.$$

Finally, the contact distribution, $\hat{\mathscr{D}}$ is $\mathbb{R}^k$-invariant, projecting to a hyperplane distribution on $\tilde{C}$:

$$\tilde{\mathscr{D}} := \pi_{C*}\hat{\mathscr{D}} = \tilde{\pi}_* \ker \lambda, \qquad \lambda = i_{\mathbf{D}}\omega.$$

Now, we may form the following explicit description of the quotient $C_o = \{\hat{\alpha}_j = const.\}/\mathbb{R}^k = \{\tilde{\alpha}_j = const.\} \subset \tilde{C}$ by first lifting upstairs to $\tilde{C}_o \subset \tilde{M} = M \times \mathbb{R}^k$, given by

$$\tilde{C}_o \ := \left\{ \frac{a_j}{a_k} = c_j, \quad \frac{\tilde{H}}{a_k} = c_k \right\} = \left\{ a_j = c_j a_k, \quad (c_1 a_k)^{1-\Lambda_1} H_1 + ... + (c_{k-1} a_k)^{1-\Lambda_{k-1}} H_{k-1} + a_k^{1-\Lambda_k} H_k = c_k a_k \right\},$$

where $c_j > 0$, $j = 1, ..., k-1$ and $c_k \in \mathbb{R}$ are constants. Then $C_o = \tilde{C}_o/\tilde{\mathbf{D}}$ is identified with:

$$C_o \cong \tilde{C}_o\Big|_{\{a_k=1\}} = \{C_1 H_1 + ... + C_{k-1} H_{k-1} + H_k = c_k\} \subset M$$

for constants $C_j = c_j^{1-\Lambda_j} > 0$ and $c_k \in \mathbb{R}$.

The induced distribution is determined by the restriction of $\lambda = i_{\mathbf{D}}\omega$ to C_o. This distribution is contact on $C_o \backslash \{\mathbf{D}\tilde{H} = 0\}$, where $\mathbf{D}\tilde{H}|_{C_o} = (C_1\Lambda_1 H_1 + ... + C_{k-1}\Lambda_{k-1}H_{k-1} + \Lambda_k H_k)|_{C_o}$, since here the Liouville vector field $\mathbf{D}$ on M is transverse to C_o, that is, $C_o \backslash \{\mathbf{D}\tilde{H} = 0\}$ is of contact type (see Definition 15). Moreover, the induced vector field, $X_o = \tilde{X}|_{C_o}$ on C_o is given by the restriction of the Hamiltonian vector field of $C_1 H_1 + ... + C_{k-1}H_{k-1} + H_k$ to C_o. Finally, note that $X_o/(\mathbf{D}\tilde{H})|_{C_o}$ is in fact the Reeb field of $\lambda|_{C_o}$. $\square$

Remark 12. *Note that we find that the $\hat{\mathbf{D}}$-invariant region $\{\mathbf{D}\hat{H} = 0\} \subset \hat{M}$ plays an obstructing role to this fully reduced space being a contact manifold with an induced contact dynamics. However, in many examples (see Section III D below), this 'singular' region is in fact empty, and typically it is (at least) codimension one in C_o, since the component functions H_j of the system of coupled Hamiltonians have some independence and, e.g. $\Lambda_k \neq 1$.*

Remark 13. *The space $\tilde{C}$ has dimension $\dim M - 1 + k$ with induced scale-reduced dynamics of $\tilde{X} \in \mathfrak{X}(\tilde{C})$ admitting k first integrals. The level sets of these first integrals are –away from the 'singular' set $\tilde{\Sigma} = \tilde{\pi}(\{\mathbf{D}\hat{H} = 0\})$– contact manifolds on which the restriction of $\tilde{X}$ is proportional to a contact vector field. Thus we realize our initial description outlined in Remark 9.*

Remark 14. *The action of $\mathbb{R}^k$ on $(\hat{M},\hat{\omega})$ is symplectic, and $\tilde{M} = \hat{M}/\mathbb{R}^k$ is its Poisson reduction (ω being a 'pre-symplectic' form on $\tilde{M} = M \times \mathbb{R}^k$). The symplectic reductions of $\hat{M}$ by this $\mathbb{R}^k$-action are just (M,ω) realized as the symplectic leafs $M \times \{a\} \subset \tilde{M}$. One can thus view $\tilde{C} = \hat{C}/\mathbb{R}^k$ as an analogous pre-contact (see e.g. [29]) manifold: having a hyperplane distribution and foliation –away from $\tilde{\Sigma}$– into 'contact leaves' corresponding to contact reductions of $\hat{C}$. In general, relations between symplectic reductions and scaling reductions will be an interesting theme for future work to explore and apply in determining 'full reductions' of explicit examples.*

This type of contact reduction is a slight variation of that from [17]. Namely we do not require the contact action to leave invariant any specific contact 1-form, and as well admit the possibility to obtain 'singular' contact reductions. More precisely, consider a contact action of $\mathbb{R}^{k+1}$ generated by vector fields $X_0, ..., X_k$, on a contact manifold N, with $X_1, ..., X_k$ transverse to the contact distribution, and admitting first integrals:

$$J : N \to \mathbb{RP}^k$$

such that for each choice of 1-form η on N one has

$$J(x) := [i_{X_0}\eta(x) : \cdots : i_{X_k}\eta(x)] \,.$$

Then the space $\bar{N} := \{J = J_o\}/\mathbb{R}^k$ –reduced by the $\mathbb{R}^k$-action generated by $X_1, ..., X_k$– inherits a hyperplane distribution, locally given by the kernel of a 1-form $\bar{\eta}$ which is contact away from $i_{X_0}\bar{\eta} = 0$.

Using this structure, it is not hard to derive some explicit coordinate descriptions for the scale-reduced equations of motion, upon choice of a scaling function. We start with the following explicit expressions.

Corollary 4 (Contact reduction of systems of coupled Hamiltonians with a scaling function)**.** *Let $\tilde{\rho} : \tilde{M} \to \mathbb{R}^+$ be a scaling function of $\tilde{\mathbf{D}} = \mathbf{D} + a \cdot \partial_a$ on $\tilde{M} = M \times \mathbb{R}^k$. Then we have the following functions on $\tilde{C} = \tilde{M}/\tilde{\mathbf{D}}$:*

$$\tilde{\pi}^* \tilde{\mathscr{H}} = \tilde{H}/\tilde{\rho}, \qquad \tilde{\pi}^* \tilde{a}_j = a_j/\tilde{\rho}, \qquad \tilde{\pi}^* \tilde{\sigma} = \mathbf{D}\tilde{H}/\tilde{\rho}$$

as well as the vector field and 1-form:

$$\tilde{X} = \tilde{\pi}_* X_H, \qquad \text{and} \qquad \tilde{\pi}^* \tilde{\eta} = \lambda/\tilde{\rho}, \quad \text{where} \quad \lambda = i_{\mathbf{D}}\omega.$$

Then $\tilde{X}$ admits the first integrals:

$$\tilde{\alpha}_1 = \frac{\tilde{a}_1}{\tilde{a}_k}, \qquad \cdots \qquad \tilde{\alpha}_{k-1} = \frac{\tilde{a}_{k-1}}{\tilde{a}_k}, \qquad \tilde{\alpha}_k = \frac{\tilde{\mathscr{H}}}{\tilde{a}_k}.$$

Let $\tilde{\Sigma} = \{\tilde{\sigma} = 0\} \subset \tilde{C}$. Then a regular level set:

$$C_o := \{\tilde{\alpha}_j = const.\} \backslash \tilde{\Sigma}$$

of these first integrals is a contact manifold, with contact 1-form:

$$\eta_o = \tilde{\eta}|_{C_o}.$$

The restriction of $\tilde{X}$ to C_o is not in general a contact vector field, however it is proportional to a contact vector field. Explicitly, for $X_{\tilde{a}_k}$ the contact Hamiltonian vector field of $\tilde{a}_k|_{C_o}$ with respect to η_o:

$$\tilde{X}|_{C_o} = \frac{\tilde{\sigma}}{\tilde{a}_k} X_{\tilde{a}_k} \,. \tag{25}$$

Equivalently, since $\tilde{a}_k \neq 0$, we have:

$$\tilde{X}|_{C_o} = \frac{\tilde{\sigma}}{\tilde{a}_k} \mathscr{R}$$

where $\mathscr{R}$ is the Reeb vector field of $\frac{\tilde{\eta}}{\tilde{a}_k}|_{C_o}$.

There are a few natural choices of coordinate systems on $\tilde{C}$: one corresponding to a choice of one of the parameters as a scaling function, and the other to a choice of a scaling function of $\mathbf{D}$ on M.

Remark 15 (Contact reduction with scaling function a_k)**.** *For $a_k : \tilde{M} \to \mathbb{R}^+$ as a scaling function of $\tilde{\mathbf{D}}$, we have:*

$$\tilde{C} \cong M \times \mathbb{R}^{k-1} \ni (m, a')$$

where $a' = (a_1, ..., a_{k-1}, 1)$. Let us set as well:

$$A_j = a_j^{1-\Lambda_j}, \qquad A' = (A_1, ..., A_{k-1}, 1)$$

The distribution on $\tilde{C}$ is given by the kernel of the 1-form $\lambda = i_{\mathbf{D}}\omega$, whereas the vector field $\tilde{X}$ is just the Hamiltonian vector field of:

$$H_{A'} = A' \cdot H, \qquad H = (H_1, ..., H_k)$$

determined by $i_{\tilde{X}} d\lambda = -d(H_{A'}|_{A'=cst.})$ and $\tilde{X} A' = 0$. Note that $\tilde{X}$ also satisfies $i_{\tilde{X}}\lambda = \mathbf{D} H_{A'} = \mathbf{\Lambda} A' \cdot H$, where $\mathbf{\Lambda} = diag(\Lambda_1, ..., \Lambda_k)$. The first integrals, $\tilde{\alpha}_j$, are then given by A' and $H_{A'}$.

In the coordinates of Remark 15, it is easy to see Theorem 2 (indeed, the proof of Theorem 2 boiled down to using these coordinates). Namely, the contact reduction of the system is given by fixing the parameters A and an energy level $H_A = c$ in M, which is transverse to the Liouville vector field $\mathbf{D}$ on M away from where $\mathbf{D} H_A = 0$. The Hamiltonian vector field on this level set is then proportional to the Reeb field of $\lambda = i_{\mathbf{D}}\omega$ restricted to the energy level.

Nonetheless, the additional 'lifted' contact structure is still quite useful, not just conceptually, but as well for deriving coordinate expressions more simply (à la Corollary 4).

Remark 16 (Contact reduction with scaling function on M)**.** *For a global scaling function, $\rho : M \to \mathbb{R}^+$, on M of $\mathbf{D}$, we have:*

$$\tilde{C} \cong C \times \mathbb{R}^k$$

where $C = M/\mathbf{D}$ and $\tilde{\pi}^* \tilde{a}_j := a_j/\rho$ are the coordinates on $\mathbb{R}^k$. We set $\tilde{A}_j := \tilde{a}_j^{1-\Lambda_j}$. Moreover, for $\pi^*\eta = i_{\mathbf{D}}\omega/\rho$, a contact 1-form on C, with Reeb field $\mathscr{R}$, the vector field $\tilde{X}$ on $\tilde{C}$ is determined through:

$$i_{\tilde{X}}\eta = \mathbf{\Lambda}\tilde{A} \cdot \mathscr{H}\,, \qquad i_{\tilde{X}} d\eta = -d\tilde{\mathscr{H}} + \mathscr{R}\tilde{\mathscr{H}}\eta\,, \qquad \tilde{X}\tilde{a} = \tilde{a}\;\mathscr{R}\tilde{\mathscr{H}} \tag{26}$$

where $\mathbf{\Lambda} = diag(\Lambda_1, ..., \Lambda_k)$ and $\pi^*\tilde{\mathscr{H}} := \tilde{H}/\rho$ is:

$$\tilde{\mathscr{H}} = \tilde{a}_1^{1-\Lambda_1}\mathscr{H}_1 + ... + \tilde{a}_k^{1-\Lambda_k}\mathscr{H}_k = \tilde{A} \cdot \mathscr{H}$$

for $\tilde{\pi}^*\mathscr{H}_j := H_j/\rho^{\Lambda_j}$, and $\mathscr{H} := (\mathscr{H}_1, ..., \mathscr{H}_k)$. The first integrals are then given by: $\tilde{\alpha}_1 = \frac{\tilde{a}_1}{\tilde{a}_k}, ..., \tilde{\alpha}_{k-1} = \frac{\tilde{a}_{k-1}}{\tilde{a}_k}, \tilde{\alpha}_k = \frac{\tilde{\mathscr{H}}}{\tilde{a}_k}$. Then we have that, away from $\tilde{\Sigma} = \{\mathbf{\Lambda}\tilde{A} \cdot \mathscr{H} = 0\}$, a level set of these first integrals is a contact manifold with contact 1-form given by restriction of η, and moreover that the vector field $\tilde{X}$ restricted to this level set is proportional to a contact vector field (the contact Hamiltonian vector field of $\tilde{a}_k$'s restriction to the level set). In local contact coordinates, $\eta = p \cdot dq - dS$, on C, the equations of motion (26) *take the form:*

$$\dot{q} = \partial_p\tilde{\mathscr{H}}, \qquad \dot{p} = -\partial_q\tilde{\mathscr{H}} - p\,\partial_S\tilde{\mathscr{H}}\,, \qquad \dot{S} = p \cdot \partial_p\tilde{\mathscr{H}} - \mathbf{\Lambda}\tilde{A} \cdot \mathscr{H}\,, \qquad \dot{\tilde{a}} = -\tilde{a}\;\partial_S\tilde{\mathscr{H}}\,. \tag{27}$$

Remark 17. *Some important comments here are in order:*

The first point is the fact that the assumption on H_A in Eq. (23) *is in practice hardly restrictive at all. Indeed, by introducing such parameters, basically any physical Hamiltonian can be written as a sum of pieces that scale differently. This allows one to apply contact reductions to a much more general class of Hamiltonian systems than those which admit a scaling symmetry of some degree Λ, as the examples below shall show.*

The second crucial point (which might have interesting consequences on the way we formulate physical theories) is the fact that while on the lifted symplectic system $\hat{H}$ the dynamical variables a_j are constant, on the corresponding reduced contact system their counterparts, given by $\tilde{\pi}^*\hat{a}_j = a_j/\hat{\rho}$, correspond to dissipated quantities*: $X_{\hat{\mathscr{H}}}\hat{a} = \mathscr{R}(\hat{\mathscr{H}})\hat{a}$ (see Remark 11 and [7, 19, 22] to see that this is the equivalent of conserved quantities for the contact case). However, the global equivalence of the two dynamics is well established by the discussion in Section II. Therefore, we are ready to remark the significance of Theorem 2: as anticipated in the discussion on Poincaré's dream in the Introduction, since the parameters in a Hamiltonian description of the physical reality are* inferred *from measurements based on the observed trajectories and on the specified model, there is no reason a priori to think of them as fixed numbers. Indeed we see from Theorem 2 that if we allow the 'space of physical Hamiltonian (variational) theories' describing a given phenomenon to vary, meaning that the coupling constants*

I think this is what the referee wanted here
Dave

I agree, but I changed a bit the notation *of each theory are different, then what we have just proved means that there always exists in this space a way to reduce the description to an equivalent theory based on a contact Hamiltonian (or Herglotz variational) theory on a reduced space. Despite the dynamical equivalence among all such theories, the contact Hamiltonian one involves fewer elements for its complete description. Following arguments in [34], the extra elements can be considered superfluous structure which should be 'pared down, being careful not to jeopardize their capacity to embed the phenomena'. The embedding of equivalent phenomena is guaranteed by the equivalence of the dynamics, to leave a theory with more 'descriptive power'. See for example [33, 48] and references therein. We call this a* reduction to a more descriptive theory*. We note that the universal dissipative nature of the reduced system provides support for the position that open systems are fundamental to physics [16].*

D. Further examples

We consider some simple examples that illustrate the flexibility of Theorem 2. In particular, such reductions may be applied even when the original system –without any parameters introduced– does not possess any evident type of scaling symmetry of the type in Theorem 1.

Example 6 (Kepler revisited)**.** *Let us revisit the Kepler problem (see Examples 1-3). Now, we would like to use Theorem 2 to reduce by some more general sorts of scaling symmetries of this system. Consider then:*

$$H_A = \underbrace{\frac{|p|^2}{2}}_{H_1} - A \underbrace{\frac{1}{|q|}}_{H_2} .$$

with a parameter $A > 0$. Then, for $\mathbf{D} = \frac{p\cdot\partial_p + q\cdot\partial_q}{2}$, we have:

$$\mathbf{D}H_1 = H_1 , \qquad \mathbf{D}H_2 = -\frac{1}{2}H_2.$$

In particular, here, $\mathbf{D}H_A = \frac{1}{2}\left(|p|^2 + \frac{A}{|q|}\right) > 0$, so that the singular region, $\hat{\Sigma}$, of Theorem 2, is empty.

We thus, according to Proposition 6, consider the scaling by:

$$\tilde{\mathbf{D}} = \mathbf{D} + a\cdot\partial_a , \qquad a = A^{2/3}$$

on $\tilde{M} = M \times \mathbb{R}^+ \ni (m,a)$, where $M = T^*(\mathbb{C}\backslash\{0\}) \ni (q,p)$.

Now Theorem 2 boils down to the observation that for a fixed value of A, an energy level, $H_A = cst.$, of the Kepler problem is a contact manifold with contact 1-form given by the restriction of

$$\lambda = i_{\mathbf{D}}\omega = \frac{p\cdot dq - q\cdot dp}{2}$$

to this energy level. Moreover, the Reeb field of this contact 1-form on the energy level is proportional to the restriction of the Hamiltonian vector field of the Kepler problem.

On the other hand, the overlying structure exposed in the proof of Theorem 2 allows one to determine this contact structure in various coordinates in a systematic *way (see Corollary 4). Consider for example the scaling function $\rho = r^2 = |q|^2$. Then we obtain the contact system on $\hat{C}$ given by restriction of $\hat{\lambda} = \lambda + adb$ and $\hat{H}$ to $\{\rho = 1\} \subset \hat{M}$, namely, the contact Hamiltonian system:*

$$\hat{\eta} = \hat{\lambda}|_{\rho=1} = \tilde{G}d\theta - \frac{d\tilde{J}}{2} + \tilde{a}db , \qquad \hat{\mathscr{H}} = \hat{H}|_{\rho=1} = \frac{\tilde{J}^2 + \tilde{G}^2}{2} - \tilde{A}$$

where $q = re^{i\theta}$, $\tilde{J} = p\cdot q/\rho$, $\tilde{G} = p\cdot iq/\rho$, $\tilde{a} = a/\rho$, and $\tilde{A} = \tilde{a}^{3/2}$. The equations of motion on the quotient $\tilde{C} = \hat{C}/\partial_b$, Eqs. (27), then read

$$\theta' = \tilde{G} , \qquad \tilde{G}' = -2\tilde{J}\tilde{G} , \qquad \tilde{J}' = \tilde{G}^2 - \tilde{J}^2 - \tilde{A} , \qquad \tilde{a}' = -2\tilde{J}\tilde{a} ,$$

with the first integral $\tilde{\alpha}$ given by:

$$\tilde{\alpha} = \tilde{\mathscr{H}}/\tilde{a} = \hat{\mathscr{H}}/\tilde{a}$$

Note that in this case, due to the rotational symmetry, we have a remnant of the angular momentum, corresponding to the additional first integral: $\tilde{G}/\tilde{a}$ of the scale-reduced system.

Moreover, each level set of $\tilde{\alpha}$ is a contact manifold, and on this manifold the restriction of the equations of motion is proportional to a contact vector field. This observation of Theorem 2 is –in these coordinates at least– quite non-trivial to see. For example, thanks to the recipe in Corollary 4 –cf. Eq. (25)*– on the level set $C_o = \{\tilde{\alpha} = 0\}$, one has that $\eta_o = \tilde{G}d\theta - \frac{d\tilde{J}}{2}$ and that the fully-reduced dynamics $\tilde{X}|_{C_o}$ is proportional to the contact Hamiltonian vector field, $X_{\mathscr{F}}$, with*

$$\mathscr{F}(\theta, \tilde{G}, \tilde{J}) := \tilde{a}|_{C_0} = \left(\frac{\tilde{J}^2 + \tilde{G}^2}{2}\right)^{2/3} .$$

Contrary to the Kepler example, in the next case we present a system that does not possess any evident scaling symmetry of degree Λ, but to which we can still apply Theorem 2.

Example 7 (Coupled Kepler-Hooke System)**.** *With M and ω as before, consider now a Hamiltonian which combines the Kepler ($1/r$) and the Hooke (r^2) potential:*

$$H_{KH} := \underbrace{\frac{|p|^2}{2} + \frac{c|q|^2}{2}}_{H_1} \underbrace{-A\ \frac{1}{|q|}}_{H_2} .$$

with parameter $A > 0$. Then, for $\mathbf{D} = \frac{p\cdot\partial_p + q\cdot\partial_q}{2}$, we have:

$$\mathbf{D}H_1 = H_1 , \qquad \mathbf{D}H_2 = -\frac{1}{2}H_2 .$$

Thus we consider the scaling:

$$\tilde{\mathbf{D}} = \mathbf{D} + a \cdot \partial_a , \qquad a = A^{2/3}$$

on $M \times \mathbb{R}^+ \ni (q, p, a)$. Again, we have here that $\mathbf{D}H_{KH} = \frac{1}{2}\left(|p|^2 + c|q|^2 + \frac{A}{|q|}\right) > 0$ so that (when $c \geq 0$, i.e. the two central forces are attractive*) the singular region, $\hat{\Sigma}$, is empty. The reduced equations may be obtained in the same way as the previous example, using the scaling function $\rho = |q|^2$. One obtains the scale-reduced equations:*

$$\theta' = \tilde{G} , \qquad \tilde{G}' = -2\tilde{J}\tilde{G} , \qquad \tilde{J}' = \tilde{G}^2 - \tilde{J}^2 - \tilde{A} - c , \qquad \tilde{a}' = -2\tilde{J}\tilde{a} ,$$

having the first integrals: $\tilde{G}/\tilde{a}$, $\tilde{\alpha} = \tilde{\mathscr{H}}/\tilde{a}$, where $\tilde{\mathscr{H}} = \frac{\tilde{J}^2 + \tilde{G}^2}{2} + c - \tilde{A}$.

We can generalize the last example at once to the case of any potential that can be written as a Laurent series of r as follows.

Example 8 (General Laurent series potentials)**.** *Consider now a Laurent series Hamiltonian:*

$$H_L := \underbrace{\frac{|p|^2}{2}}_{H_{\text{kin}}} + \underbrace{\sum_{j=-n_1}^{n_2} A_j |q|^j}_{\sum_{j=-n_1}^{n_2} A_j H_j} .$$

In this case one does not have an (evident) scaling symmetry of degree Λ. However, for $\mathbf{D} = \frac{p\cdot\partial_p + q\cdot\partial_q}{2}$, we have:

$$L_{\mathbf{D}} H_{\text{kin}} = H_{\text{kin}} \qquad \text{and} \qquad L_{\mathbf{D}} H_j = \frac{j}{2} H_j$$

leading, in the same way as the previous two examples, to the scale-reduced equations of motion:

$$\theta' = \tilde{G} , \qquad \tilde{G}' = -2\tilde{J}\tilde{G} , \qquad \tilde{J}' = \tilde{G}^2 - \tilde{J}^2 - \sum j\tilde{A}_j , \qquad \tilde{A}_j' = (j-2)\tilde{A}_j\tilde{J} .$$

Note that, in case each potential term is attractive *(meaning $sgn(A_j) = sgn(j)$), the singular region $\hat{\Sigma} = \{\mathbf{D}H_L = 0\}$, is empty here as well.*

The next example is somewhat simpler, but it has important physical ramifications in cosmology, and therefore it is worth presenting it at this point.

Example 9 (FLRW Cosmology)**.** *In the field of cosmology, homogeneous, isotropic solutions to Einstein's equations are given by the Friedmann-Lemaître-Robertson-Walker metrics. The symmetries of such space-times allow us to reduce a full field theory to a finite-dimensional 'particle' representation. In these solutions, the space-time manifold is $\mathbb{R} \times \Sigma_k$, in which Σ_k is the three-dimensional spatial slice, consisting of a three sphere ($S^3, k > 0$), Euclidean space ($\mathbb{R}^3, k = 0$) or hyperbolic space ($\mathbb{H}, k < 0$). The metric is determined by the topology of the spatial section:*

$$ds^2 = -dt^2 + v^{\frac{2}{3}}(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right).$$

In the case of a non-compact topology, the volume, v is measured with respect to some fixed, fiducial cell. For minimally coupled matter, the dynamics of which are given in flat space by a Hamiltonian $H_m(p,q)$, we find the equations of motion for the gravitational sector, v, Π, from the Hamiltonian:

$$H = v\left(-\frac{3\Pi^2}{8\pi} + H_m\left(\frac{p}{v}, q\right) - \frac{k}{v^{\frac{2}{3}}}\right), \quad \omega = d\Pi \wedge dv + dp \wedge dq.$$

For flat ($k = 0$) spatial slices the Hamiltonian admits a degree one scaling symmetry $\mathbf{D} = v\partial_v + p\partial_p$ (summed over all matter momenta p if multiple are present), with $i_{\mathbf{D}}\omega = -vd\Pi + pdq$. Therefore in this case the reduction to the contact dynamics on $C = M/\mathbf{D}$ is immediate, and following Corollary 1 with the scaling function $\rho = v$, we have the contact system:

$$\mathscr{H}_0 = -\frac{3\Pi^2}{8\pi} + H_m(p,q) \qquad \text{and} \qquad \eta = pdq - d\Pi.$$

To extend our results to the case in which $k \neq 0$, we follow the process of Theorem 2. Indeed, the Hamiltonian in this case can be split as

$$H = \underbrace{v\left(-\frac{3\Pi^2}{8\pi} + H_m\left(\frac{p}{v}, q\right)\right)}_{H_1} - k\underbrace{v^{1/3}}_{H_2},$$

and for $\mathbf{D} = v\partial_v + p\partial_p$ as before we obtain

$$L_{\mathbf{D}}H_1 = H_1 \qquad \text{and} \qquad L_{\mathbf{D}}H_2 = \frac{1}{3}H_2\,,$$

which, by Proposition 6, leads to the degree one scaling symmetry

$$\hat{\mathbf{D}} = \mathbf{D} + \frac{2}{3}k\partial_k$$

on the extended space and, by Corollary 1, to a scale-reduced contact Hamiltonian dynamics on $\hat{C} = \hat{M}/\hat{\mathbf{D}}$. The scaling function v represents this contact system as:

$$\hat{\mathscr{H}} = -\frac{3\Pi^2}{8\pi} + H_m(p,q) - k\,, \qquad \hat{\eta} = |k|^{3/2}db + pdq - d\Pi$$

with equations of motion on the quotient, $\tilde{C} = \hat{C}/\partial_b$, being given by

$$\dot{q} = \partial_p H_m\,, \qquad \dot{p} = -\partial_q H_m + \frac{3\Pi}{4\pi}p\,, \qquad \dot{\Pi} = p\,\partial_p H_m - H_m + \frac{3\Pi^2}{8\pi} + \frac{k}{3}\,, \qquad \dot{k} = \frac{\Pi}{2\pi}k\,, \tag{28}$$

and admitting the first integral $\tilde{\alpha} = \hat{\mathscr{H}}/|k|^{3/2}$. By Theorem 2, on each level set $C_o = \{\tilde{\alpha} = \tilde{\alpha}_o\}$ we have the contact one form $\eta_o = pdq - d\Pi$ and the scale-reduced dynamics $\tilde{X}|_{C_o}$ is proportional to the contact Hamiltonian vector field $X_{\mathscr{F}}$, with

$$\mathscr{F}(q,p,\Pi) := |k|^{3/2}|_{C_o}\,,$$

given implicitly through:

$$\tilde{\alpha}_0\mathscr{F} = -\frac{3\Pi^2}{8\pi} + H_m(p,q) - sgn(k)\mathscr{F}^{2/3}\,.$$

For instance, when $\tilde{\alpha}_0 = 0$, we have the contact system given by:

$$\mathscr{F}(q,p,\Pi) = \left[sgn(k)\left(H_m(p,q) - \frac{3\Pi^2}{8\pi}\right)\right]^{3/2}, \quad \eta_o = pdq - d\Pi.$$

Remark 18. *There are two interesting facets of cosmological dynamics that are highlighted by this construction. The first is that the apparent 'Hubble friction' arising due to the expansion of the universe can be directly observed since the equations of motion* (28) *of these systems, with the effect of coupling H_m to gravity in the second term acting as apparent friction when compared to the behaviour of matter in a non-expanding background. The second is that both the dynamics of matter in an expanding space, and the expansion of space that results from the presence of matter are described by taking a symplectic matter system $(M, \omega = d\theta, H_m)$ and considering, for $k = 0$, on its contactification $(\tilde{M}, \tilde{\theta}) = (M \times \mathbb{R}, \theta - d\Pi)$ the matter-gravity Hamiltonian $\mathscr{H} = H_m - \frac{3\Pi^2}{8\pi}$. When $k \neq 0$, one may work on the same contactification but using instead,* for total energy zero, *the contact Hamiltonian: $\mathscr{H}_k = (sgn(k)(H_m - \frac{3\Pi^2}{8\pi}))^{3/2}$. In this case, the contact Hamiltonian is equivalent to that obtained from dividing our original Hamiltonian by $v^{1/3}$ and treating the curvature as the new 'total energy' of this system.*

IV. APPLICATIONS

A. Contact reductions of cotangent Hamiltonian systems

The most standard case of Hamiltonian system is dynamics on a cotangent bundle by $H : T^*Q \to \mathbb{R}$, where T^*Q is equipped with its standard (exact) symplectic structure, $\omega = d\lambda_o$. In this case one has more explicit expressions of the contact reduction when the scaling symmetries are lifts of scaling symmetries on the configuration manifold Q. Moreover, this case is of particular interest because in this setting we can develop also the variational counterpart, cf. Section IV B. Therefore we devote this section to consider scaling reductions especially suited to these particular types of systems.

We start by singling out those scaling symmetries characterized by compatibility with the canonical projection

$$T^*Q \to Q.$$

That is, we are interested in scaling symmetries related to actions on the configuration space.

Definition 5. We call a vector field $\mathbf{D}$ on T^*Q a *basic scaling*, when:

1. $L_{\mathbf{D}}\lambda_o = \lambda_o$, for $\lambda_o = p \cdot dq$ the canonical 1-form on T^*Q,
2. the projection, $\bar{\mathbf{D}} \in \mathfrak{X}(Q)$, of $\mathbf{D}$ to the base acts freely and properly on Q.

As well, we call a function $\rho : Q \to \mathbb{R}^+$ a *basic scaling function* when $\bar{\mathbf{D}}\rho = \rho$, and we say $\mathbf{D}$ is a *basic scaling symmetry* of $H : T^*Q \to \mathbb{R}$ (of degree $\Lambda \in \mathbb{R}$) when $\mathbf{D}$ is a basic scaling and $\mathbf{D}H = \Lambda H$.

Remark 19. *Note that the first condition, $L_{\mathbf{D}}\lambda_o = \lambda_o$, of the previous definition implies that $\mathbf{D} \in \mathfrak{X}(T^*Q)$ projects to a vector field $\bar{\mathbf{D}} \in \mathfrak{X}(Q)$ on the base under the canonical projection $T^*Q \to Q$. Equivalently, $\mathbf{D}$ is a basic scaling, when it is of the form:*

$$\mathbf{D} = \bar{\mathbf{D}}^* + \mathbf{D}_o$$

where $\bar{\mathbf{D}}^*$ is the cotangent lift of a vector field $\bar{\mathbf{D}}$ on Q and $\mathbf{D}_o = p \cdot \partial_p$ is the standard Liouville vector field on T^*Q.

Proposition 8. Suppose (T^*Q, ω) admits a basic scaling $\mathbf{D}$, with basic scaling function $\rho : Q \to \mathbb{R}^+$. Then, for

$$\bar{Q} := Q/\bar{\mathbf{D}},$$

one may identify the contact-reduced space

$$C = (T^*Q)/\mathbf{D} \cong T^*\bar{Q} \times \mathbb{R}$$

as the contactification of $T^*\bar{Q}$, that is, having contact 1-form:

$$\eta = \bar{\lambda}_o - dS,$$

where $\bar{\lambda}_o$ is the canonical 1-form on $T^*\bar{Q}$, and S given by $\pi^*S = \frac{i_{\mathbf{D}}\lambda_o}{\rho}$ the coordinate on $\mathbb{R}$.

Proof. First, we will describe the form of the flow, $\psi_s : T^*Q \to T^*Q$, of $\mathbf{D}$ more explicitly. Let $\mathbf{D}_o$ be the Liouville vector field on T^*Q, $i_{\mathbf{D}_o}\omega = \lambda_o$, whose flow ψ^o_s acts by scaling the fibers. Then, since $\psi^*_s\lambda_o = e^s\lambda_o = (\psi^o_s)^*\lambda_o$, we have:

$$(\psi_s\psi^o_{-s})^*\lambda_o = \lambda_o$$

Consequently, $\psi_s\psi^o_{-s}$ is given by the cotangent lift of some diffeomorphism, $\bar{\psi}_s : Q \to Q$ (the flow of $\bar{\mathbf{D}}$), ie:

$$\psi_s(q,p) = (\bar{\psi}_s(q), e^s\bar{\psi}^*_{-s}p).$$

Alternately, we have $\mathbf{D} - \mathbf{D}_o = \bar{\mathbf{D}}^*$ is the cotangent lift of $\bar{\mathbf{D}} \in \mathfrak{X}(Q)$ (the Hamiltonian vector field of $i_{\mathbf{D}}\lambda_o$).

Given our basic scaling function $\rho : Q \to \mathbb{R}^+$, let:

$$\bar{\pi} : Q \to \bar{Q} = Q/\bar{\mathbf{D}}.$$

Then we identify, $\pi : T^*Q \to (T^*Q)/\mathbf{D} = C$, by sending the $\mathbf{D}$ orbit, $[q,p]$ through $(q,p) \in T^*Q$ to:

$$C \ni [q,p] \mapsto (\bar{q}, \bar{p}, S) \in T^*\bar{Q} \times \mathbb{R}$$

where $\bar{q} = \bar{\pi}(q)$, $\pi^*S = \frac{i_{\mathbf{D}}\lambda_o}{\rho}$, i.e. $S = p(\bar{\mathbf{D}})/\rho(q)$, and $\bar{\pi}^*\bar{p} = \frac{p - Sd\rho}{\rho}$, which gives a well defined covector $\bar{p} \in T^*_{\bar{q}}\bar{Q}$, (essentially, $p - Sd\rho$ vanishes on $\bar{\mathbf{D}}$). Moreover, one computes, the canonical 1-form, $\bar{\lambda}_o$, on $T^*\bar{Q}$ pulls back under $\pi : T^*Q \to C \cong T^*\bar{Q} \times \mathbb{R}$ to:

$$\pi^*\bar{\lambda}_o = \frac{\lambda_o - \pi^*Sd\rho}{\rho}. \tag{29}$$

Now we describe the contact 1-form under this identification. Let $\lambda = i_{\mathbf{D}}\omega$, and $\pi^*\eta = \lambda/\rho$ define the contact 1-form, η, on $C \cong T^*\bar{Q} \times \mathbb{R}$. Then:

$$\lambda_o = L_{\mathbf{D}}\lambda_o = \lambda + d(\rho\pi^*S) = \lambda + \rho\pi^*dS + \pi^*Sd\rho$$

so that: $\pi^*\bar{\lambda}_o = \pi^*(\eta + dS)$, ie, $\eta = \bar{\lambda}_o - dS$ where $\bar{\lambda}_o$ is the canonical 1-form on $T^*\bar{Q}$. □

According to Theorem 1 above (see also Remark 3), the projections of trajectories of such Hamiltonian systems admitting a basic scaling symmetry may be described as follows.

Corollary 5. *Suppose the Hamiltonian system (T^*Q, ω, H) admits a basic scaling symmetry, $\mathbf{D}$, of degree Λ with basic scaling function $\rho : Q \to \mathbb{R}^+$. Let $\bar{Q} = Q/\bar{\mathbf{D}} \cong \{\rho = 1\}$. Then its scale-reduced orbits are trajectories of the vector field:*

$$\bar{X} \in \mathfrak{X}(T^*\bar{Q} \times \mathbb{R})$$

determined through:

$$i_{\bar{X}}\eta = \Lambda\mathscr{H}\,, \qquad i_{\bar{X}}d\eta = -d\mathscr{H} + (\mathscr{R}\mathscr{H})\eta$$

where $\eta = \bar{\lambda}_o - dS$, $\pi^*S = \frac{i_{\mathbf{D}}\lambda_o}{\rho}$ and $\pi^*\mathscr{H} = H/\rho^\Lambda$.

Remark 20. *Let $\Sigma := \{\rho = 1\} \subset Q$, with inclusion $\iota : \Sigma \to Q$. Then we have $(T^*Q)/\mathbf{D} \cong \iota^*(T^*Q)$. This contact manifold is then identified with the contactification of $T^*\Sigma$:*

$$T^*\Sigma \times \mathbb{R}, \quad \eta = \lambda_\Sigma - dS.$$

where λ_Σ is the canonical 1-form on $T^*\Sigma$ and $S = i_{\mathbf{D}}\lambda_o|_{\iota^*(T^*Q)}$ the coordinate on $\mathbb{R}$. A degree Λ Hamiltonian then passes to the quotient via restriction: $\mathscr{H} = H|_{\iota^*(T^*Q)}$.

B. Contact reductions of Lagrangian systems

One may obtain 'Lagrangian analogues' of our results above by applying the Legendre transform to Hamiltonian systems and their contact reductions (cf. Appendix A 3). Here we only consider *regular* systems, that is, systems for which Legendre transforms are diffeomorphisms. It is instructive to first consider a local coordinate analogue of Proposition 10, or Corollary 1 (the degree one case).

Example 10. *Consider a Herglotz-Lagrangian system $\mathscr{L}(q, \dot q, S)$ corresponding under Legendre transform to the contact Hamiltonian system $\mathscr{H}(q, p, S)$, $\eta = p \cdot dq - dS$. This contact Hamiltonian system (see, Remark 6 above) is the scale reduction of the Hamiltonian system:*

$$H(\rho, q, S, P) = \rho \mathscr{H}(q, P/\rho, S)$$

on the symplectification, $\omega = d\tilde\alpha = d(P \cdot dq - \rho\ dS)$, by the degree one scaling symmetry $\mathbf{D} = \rho\partial_\rho + P \cdot \partial_P$. We identify this symplectification with the cotangent bundle of $Q \ni (\rho, q)$ by taking $S = p_\rho$, with canonical 1-form:

$$\lambda_o = p_\rho\ d\rho + P \cdot dq = \tilde\alpha + d(\rho p_\rho),$$

for which $S = \frac{i_{\mathbf{D}}\lambda_o}{\rho}$. Now, the Legendre transform of H is the Lagrangian system $L = \dot\rho S + \rho\mathscr{L}$, where

$$\mathscr{L} = \frac{L - \dot\rho S}{\rho}$$

is our original Herglotz-Lagrangian system on $q \in \bar Q = Q/\rho\partial_\rho$, which we consider as the scale reduction of the Lagrangian system L by the scaling symmetry $\mathbf{D}_L = \rho\partial_\rho + \dot\rho\partial_{\dot\rho}$ (corresponding to $\mathbf{D}$ under Legendre transform). We note that we have as well the equivalent coordinate expressions:

$$\mathscr{L} = \partial_\rho L\,, \qquad S = \partial_{\dot\rho} L\,.$$

The description of a scale reduction of a Lagrangian system may be obtained by applying the Legendre transform to a Hamiltonian system and its contact reduction of the form in Proposition 8. The relationship is laid out in the following commutative diagram:

$$\begin{tikzcd}
TQ, L \arrow[r, leftrightarrow, "Legendre"] \arrow[d, "Quotient"'] & T^*Q, H \arrow[d, "Quotient"] \\
T\bar Q \times \mathbb{R}, \mathscr{L} \arrow[r, leftrightarrow, "Legendre"] & T^*\bar Q \times \mathbb{R}, \mathscr{H}
\end{tikzcd}$$

Thus, on the Lagrangian side, we define:

Definition 6. A *basic scaling symmetry* of *degree* Λ for a Lagrangian system (Q, L) is a vector field $\mathbf{D}$ on TQ such that $\mathbf{D}(L) = \Lambda L$, $L_{\mathbf{D}}\lambda_L = \lambda_L$, where $\lambda_L = \partial_{\dot q} L \cdot dq$ is the *Lagrangian 1-form* (the pullback of the canonical 1-form by the Legendre transform), and, moreover, its projection $\bar{\mathbf{D}} \in \mathfrak{X}(Q)$ under $TQ \to Q$ induces a free and proper action on Q.

With this definition, we have:

Theorem 3 (Counterpart to Theorem 1)**.** Let (Q, L) be a regular Lagrangian system, admitting a basic scaling symmetry, $\mathbf{D}$, of degree Λ and a basic scaling function $\rho : Q \to \mathbb{R}^+$. Then one may identify

$$\pi : TQ \to (TQ)/\mathbf{D} \cong T\bar Q \times \mathbb{R} \ni (\bar q, \bar q', S)$$

where $\pi^* S = i_{\mathbf{D}}\lambda_L/\rho$. Let $\delta : TQ \to TQ$ be the reparametrizing dilation: $v \mapsto \rho(q)^{\Lambda-1} v$ for $v \in T_q Q$. Then, extremals of the Lagrangian system project to extremals of the Λ-Herglotz system:

$$\pi^*\mathscr{L} = \left(\frac{L - \dot\rho S}{\rho^\Lambda}\right) \circ \delta.$$

Proof. We sketch the proof, leaving the details to the reader. We consider the same setting of Proposition 8 above, and note that a regular Hamiltonian system, H, on T^*Q, induces a regular contact Hamiltonian $\mathscr{H}$ on $T^*\bar Q \times \mathbb{R}$, as can be seen from Remark 20. This establishes the commutative diagram above relating the quotients and appropriate Legendre transforms. Then the result follows by applying the correspondence between the (reduced) Λ-Hamiltonian and Λ-Herglotz systems via the Legendre transform detailed in Remarks 31 and 32. The relation between the Lagrangian L and its scale-reduced Λ-Herglotz Lagrangian $\mathscr{L}$ is obtained upon using the relation (29) between the canonical one-forms, that $\pi^*\mathscr{H} = H/\rho^\Lambda$, as well as the relations:

$$L\ dt = \lambda_o - H\ dt, \qquad \mathscr{L}\ d\tau = \bar\lambda_o - \mathscr{H}\ d\tau\,,$$

where the reparametrizing factor is by $\rho^{1-\Lambda} d\tau = dt$. □

We are identifying S with $\pi^* S$ here, right?
Ale

yeah, also

Remark 21. *As in the Hamiltonian case (see Remark 6 above), the scale-reduced trajectories are reparametrized according to*

$$\rho^{1-\Lambda} d\tau = dt,$$

and we write $' = \frac{d}{d\tau}$. *Note for* $\Lambda = 1$, *the Lagrangian system scale-reduces to a Herglotz-Lagrangian system, cf. Example 10.*

Example 11 (The 2d harmonic oscillator)**.** *Let us consider the two dimensional harmonic oscillator of Example 5, with Lagrangian*

$$L = \frac{\dot{r}^2 + r^2\dot{\theta}^2 - kr^2}{2}, \tag{30}$$

in polar coordinates. We have the basic scaling symmetry: $\mathbf{D} = \frac{1}{2}(r\partial_r + \dot{r}\partial_{\dot{r}})$, *of degree one, and scaling function* $\rho = r^2$. *Following Theorem 3, we write*

$$S = \frac{i_{\mathbf{D}}\lambda_L}{\rho} = \frac{\dot{\rho}}{4\rho}, \qquad L = \frac{\dot{\rho}S + \rho\dot{\theta}^2 - k\rho}{2}, \tag{31}$$

to obtain the scale-reduced Herglotz Lagrangian system:

$$\mathscr{L} = \frac{L - \dot{\rho}S}{\rho} = \frac{\dot{\theta}^2 - k}{2} - 2S^2 \tag{32}$$

on $TS^1 \times \mathbb{R} \ni (\theta, \dot{\theta}, S)$. *Note that* $\mathscr{L}$ *is the (contact) Legendre transform of* $\mathscr{H} = \frac{p_\theta^2 + k}{2} + 2S^2$ *on* $T^*S^1 \times \mathbb{R}$ *obtained by contact reduction of* $H = \frac{|p|^2 + k|q|^2}{2}$ *by the scaling symmetry* $\frac{q\cdot\partial_q + p\cdot\partial_p}{2}$ *and* $\rho = |q|^2$.

Example 12 (Herglotz-reduced Kepler)**.** *We consider the Kepler system of Example 2, with Lagrangian:*

$$L = \frac{|\dot{q}|^2}{2} + \frac{1}{|q|}$$

and basic scaling symmetry $\mathbf{D}_K = 2q \cdot \partial_q - \dot{q} \cdot \partial_{\dot{q}}$, *of degree* $\Lambda = -2$, *and scaling function* $\rho = |q|^{1/2}$. *Following Theorem 3, we take polar coordinates and write:*

$$S = \frac{i_{\mathbf{D}}\lambda_L}{\rho} = 4\rho^2\dot{\rho}, \qquad L = \frac{\dot{\rho}S + \rho^4\dot{\theta}^2}{2} + \frac{1}{\rho^2}$$

to obtain the scale-reduced -2*-Herglotz system:*

$$\mathscr{L} = \rho^2(L - \dot{\rho}S) \circ \delta = \frac{(\theta')^2}{2} + 1 - \frac{S^2}{8}$$

on $TS^1 \times \mathbb{R} \ni (\theta, \theta', S)$ *(using* $'$ *to denote* $\frac{d}{d\tau}$ *where* $\rho^3 d\tau = dt$*). Again, note that* $\mathscr{L}$ *is related by Legendre transform to the* -2*-Hamiltonian system,* $\mathscr{H}$*, obtained by an analogous contact reduction of the Legendre transform of* L. *Observe that the reparametrization* $\rho^3 d\tau = dt$ *has:*

$$L\ dt = \rho\left(\frac{2\rho'^2}{\rho^2} + \frac{\theta'^2}{2} + 1\right) d\tau = \tilde{L}\ d\tau$$

where $\mathscr{L} = \frac{\tilde{L} - \rho' S}{\rho}$ *gives the scale-reduced* -2*-Herglotz system* ($S = 4\frac{\rho'}{\rho}$)*.*

Finally, we consider a counterpart to Theorem 2 for certain Lagrangian systems depending on parameters, A_j, of the form:

$$L_A = T(q, \dot{q}) + \sum_{j=1}^{k} A_j L_j(q) \tag{33}$$

and for which we have a vector field $\mathbf{D}$ on TQ with

$$\mathbf{D}(T) = T, \qquad \mathbf{D}(L_j) = \Lambda_j L_j, \qquad L_{\mathbf{D}}\lambda_L = \lambda_L\,. \tag{34}$$

Note that the third condition in (34) fixes the parametrization of $\mathbf{D}$, being the precise analogue of the first condition in Definition 2.

Theorem 4 (Counterpart to Theorem 2)**.** Consider a regular Lagrangian system depending on parameters as in Eq. (33), and admitting a vector field $\mathbf{D}$ as in Eq. (34). Then the Lagrangian system

$$\hat{L} := T + \sum_{j=1}^{k} \dot{X}_j \left(\frac{L_j}{\dot{X}_j} \right)^{\frac{1}{\Lambda_j}}$$

on $\hat{Q} := Q \times \mathbb{R}^k \ni (q, X)$ admits $\mathbf{D}$ as a basic degree one scaling symmetry. Moreover, the projection to Q of an extremal of $\hat{L}$ is an extremal of L_A, with parameter values

$$\Lambda_j A_j = \left(\frac{\dot{X}_j}{L_j} \right)^{1-\frac{1}{\Lambda_j}}$$

constant over the extremals of $\hat{L}$.

Remark 22. *More generally, for* $L = \sum A_j L_j$*, one may consider* $\hat{L} = \sum \dot{X}_j \left(\frac{L_j}{\dot{X}_j} \right)^{1/\Lambda_j}$*. Under Legendre transform, this lifted system,* $\hat{L}$*, corresponds to the Hamiltonian system* $\hat{H} = \sum A_j H_j$*, where* $\Lambda_j A_j = \left(\frac{\dot{X}_j}{L_j} \right)^{1-\frac{1}{\Lambda_j}}$ *and* $H_j = \partial_{\dot{q}} L_j \cdot \dot{q} - L_j$ *is the Legendre tranform of* L_j*. A Legendre transform of* $\hat{\mathbf{D}}$ *from Proposition 6 yields a degree one scaling symmetry for this Lagrangian system,* $\hat{L}$ *(which in general is* not $\mathbf{D}$*).*

An alternate possibility –to avoid exponents involving the Λ_j*'s– would be to consider:*

$$\hat{L} := \sum e^{\dot{X}_j + L_j}$$

as the lift of the Lagrangian system, $L_A = \sum A_j L_j$*, where* $e^{\dot{X}_j + L_j} = A_j$ *is constant over extremals of* $\hat{L}$*, with the degree one scaling symmetry* $\hat{\mathbf{D}} = \sum \partial_{\dot{X}_j}$*.*

Example 13 (The Kepler-Hooke potential)**.** *Let us consider the system described in Example 11 and extend the system to include a Kepler potential:*

$$L = \underbrace{\frac{\dot{\rho}^2}{8\rho} + \frac{\rho\dot{\theta}^2}{2} - \frac{k_H \rho}{2}}_{L_1} + \underbrace{\frac{k_K}{\sqrt{\rho}}}_{L_2} \qquad \mathbf{D} = \rho\partial_\rho + \dot{\rho}\partial_{\dot{\rho}} \tag{35}$$

For this system, $\mathbf{D}$ *is not a scaling symmetry, as* $\mathbf{D}(L_1) = L_1$ *but* $\mathbf{D}(L_2) = -\frac{1}{2}L_2$*. However, following Theorem 4 we can consider*

$$\hat{L} = \frac{\dot{\rho}^2}{8\rho} + \frac{\rho\dot{\theta}^2}{2} - \frac{k_H \rho}{2} + \frac{\dot{X}^3 \rho}{k_K^2} \tag{36}$$

for which $\mathbf{D}$ *is a scaling symmetry. We then impose the initial condition* $\dot{X}_0 = \frac{k_K}{(-2)^{\frac{1}{3}} r_0}$*, where* r_0 *is the initial value of* r*. This gives rise to the same equations of motion as the original system. Since this is a Lagrangian system with a scaling symmetry of degree one, we can follow Example 11 and find the equivalent Herglotz Lagrangian system:*

$$\mathscr{L} = -2S^2 + \frac{\dot{\theta}^2}{2} - \frac{k_H}{2} + \frac{\dot{X}^3}{k_K^2}\,. \tag{37}$$

Remark 23. *Note that we could have dropped* k_K *entirely from the description of the system in equations* (36) *and* (37) *by appropriately choosing a boundary condition for* $\dot{X}$*, and hence we have in essence exchanged specifying a coupling constant,* k_K*, for an initial condition.*

C. Blow-ups in celestial mechanics

In this section we remark on the contact reduction by scaling symmetries presented here in relation to some well-known constructions in celestial mechanics (see for example [12–14, 42, 43]). The main observation here is to see how from the results in this work it follows that McGehee's blow-up equations have a variational structure (see Proposition 9 below).

Let us first recall the scale-reduced Kepler problem (Examples 1–4). We denote the scaling functions of Example 2 by

$$\rho := |q|^{1/2}\,, \qquad \kappa := \frac{1}{|p|}\,, \qquad G := p\cdot iq\,, \qquad J := p\cdot q\,.$$

where $\rho^4 = q\cdot q$ is the moment of inertia, and $\frac{1}{\kappa^2} = p\cdot p$ is (twice) the kinetic energy.

Of course, the contact reductions corresponding to each choice of scaling function represent the same curves in $C = M/\mathbf{D}$ obtained by projections of Kepler orbits. They are merely different choices of coordinates for C. However, each choice of coordinates highlights certain properties of these scale-reduced curves. For instance, the scaling function ρ is naturally associated to a blow-up of the collision orbits (Example 4), while the scaling function κ to a blow-up of orbits with $|p|\to\infty$ (as well the collision orbits in the Kepler problem).

To determine some contact reductions (Table I below) for various choices of scaling function, we first note that the scaling functions above are not independent, satisfying the relation $\kappa^2(J^2+G^2) = \rho^4$. One may use any three of these scaling functions, along with a scale-invariant angle, as coordinates on the phase space. There are essentially three (dependent) natural angles, $\theta, \varphi, \delta = \varphi-\theta$, present, where

$$q = \rho^2 e^{i\theta}\,, \qquad p = \frac{1}{\kappa}e^{i\varphi}\,, \qquad \rho^2 e^{i\delta} = \kappa(J+iG) \tag{38}$$

for δ the angle between position and momentum.

To see the equations of motion obtained by these various choices of scaling functions, we will consider (ρ,θ,J,G) for coordinates on the phase space. Then we have (Eq. (11) above):

$$\lambda = G\;d\theta + 2J\;d\log\left(\frac{\rho}{J}\right)\,, \qquad H_K = \frac{1}{2\kappa^2} - \frac{1}{\rho^2}\,, \qquad \kappa^2(J^2+G^2) = \rho^4\,.$$

One may describe the contact reduction for various choices of scaling functions according to Remark 3 by restriction of λ and H_K to a level set of the scaling function, obtaining the following representations of the scale-reduced Kepler problem presented in Table I.

Scaling function	Contact system	Scale invariant equations of motion
1. ρ	$\eta = \tilde G\;d\theta - 2\;d\tilde J,\quad \mathscr{H} = \frac{\tilde J^2+\tilde G^2}{2} - 1$	$\tilde J' = \frac{\tilde G^2}{2} + \mathscr{H},\quad \tilde G' = -\frac{\tilde J\tilde G}{2},\quad \theta' = \tilde G$
2. κ	$\eta_\kappa = G_\kappa\;d\varphi - dJ_\kappa,\quad \mathscr{H}_\kappa = \frac{1}{2} - \frac{1}{R}$	$J_\kappa' = 2\mathscr{H}_\kappa + \frac{G_\kappa^2}{R^3},\quad G_\kappa' = -\frac{G_\kappa J_\kappa}{R^3},\quad \varphi' = \frac{G_\kappa}{R^3}$
3. G	$\eta_G = d\theta - J_G\;d\log\left(\frac{J_G}{\rho_G}\right)^2,\quad \mathscr{H}_G = \frac{J_G^2+1-2\rho_G^2}{2\rho_G^4}$	$J_G' = \frac{1}{\rho_G^2} + 2\mathscr{H}_G,\quad \rho_G' = \frac{J_G}{2\rho_G^3},\quad \theta' = \frac{1}{\rho_G^4}$
4. J	$\eta_J = G_J\;d\theta + d\log\rho_J^2,\quad \mathscr{H}_J = \frac{G_J^2+1-2\rho_J^2}{2\rho_J^4}$	$G_J' = -G_J(2\mathscr{H}_J + \frac{1}{\rho_J^2}),\quad \rho_J' = -\rho_J\mathscr{H}_J - \frac{G_J^2}{2\rho_J^3},\quad \theta' = \frac{G_J}{\rho_J^4}$

TABLE I: Contact reductions of the Kepler problem with various scaling functions. In row 2 we set $R := \sqrt{J_\kappa^2+G_\kappa^2}$.

The contact reduction with ρ (row 1 of Table I) is Example 3 above. In this table we used, for example, in the κ case, the notation:

$$\pi^*J_\kappa = J/\kappa,\quad \pi^*G_\kappa = G/\kappa,\quad \pi^*\varphi = \varphi,\quad \pi^*\mathscr{H}_\kappa = \kappa^2 H_K,\quad \pi^*\eta_\kappa = \lambda/\kappa\,,$$

the subscript notation in the other cases taking the analogous meaning. Moreover, in row 2 we defined the quantity $R := \sqrt{J_\kappa^2+G_\kappa^2}$.

Remark 24. *We have already seen (Example 4), the relation of row 1 in Table I to the blow-up of the collision orbits. Likewise, row 2 yields a blow-up of $|p|\to\infty$ orbits. Here, for the Kepler problem, these are as well the collision orbits. In general n-body problems, choices of various mutual distances as scaling functions or analogues of κ will be interesting to examine.*

We also remark that the use of a first integral as a scaling function, as in row 3, corresponds to a first integral for the scale-invariant curves. Here, this first integral, $\pi^*\mathscr{H}_G = G^2H_K$, is a well-known scale invariant (sometimes referred to as the Dziobek constant). Also observe that the equations of motion for the scaling function J are forms of the well-known Lagrange-Jacobi identity.

The reduction based on the scaling function J of row 4, is used in Barbour et. al's [4] (with n-body problems), where it is referred to as the dilational momentum and used, for instance, to describe long-term 'clustering' behavior of solutions.

Remark 25. *One has an analogous contact reduction for certain inverse power force laws:*

$$H_\alpha = \frac{|p|^2}{2} - \frac{1}{\alpha|q|^\alpha}$$

with $\mathbf{D}_\alpha := \frac{2q\cdot\partial_q - \alpha p\cdot\partial_p}{2-\alpha}$ *a scaling symmetry of degree* $\Lambda = -\frac{2\alpha}{2-\alpha}$ *(when* $\alpha \neq 2$*). Note that* J, G *above are still scaling functions, as is* $\rho_\alpha := |q|^{1-\alpha/2}$*. Carrying out the analogue of Example 4 here gives exactly the blown-up collision tori (see §1.3 of [20]) of these central force problems.*

The contact reduction may be carried out similarly for n-body problems. Indeed, collecting the above structure, we obtain the following result.

Proposition 9 (McGehee's blow-up as a -2-Herglotz system)**.** *McGehee's blow-up equations (see, eg, [43])*

$$s' = y - \tilde{J}s, \qquad y' = \frac{1}{2}\tilde{J}y + \nabla U(s)\,, \tag{39}$$

where: $(s, y) \in S^{nd-1} \times \mathbb{R}^{nd}$, $\tilde{J} = s \cdot y =: S/2$*, for the Newtonian* n*-body problem, may be described variationally as a* -2*-Herglotz system (see Def. 19), on* $TS^{nd-1} \times \mathbb{R}$*, with Herglotz Lagrangian:*

$$\mathscr{L} = \frac{\|s'\|^2}{2} + U(s) - \frac{S^2}{8}. \tag{40}$$

Proof. Consider the n-body Hamiltonian system

$$\omega = dp \wedge dq = d(p \cdot dq), \qquad H = \frac{\|p\|^2}{2} - U(q)$$

with

$$q = (q_1, ..., q_n) \in \mathbb{R}^{nd}, \qquad p = (p_1, ..., p_n) \in \mathbb{R}^{nd}$$

for $q_j, p_j \in \mathbb{R}^d$, and where $U(\lambda q) = \lambda^{-1}U(q)$ is homogeneous of degree -1. For simplicity set the masses to unit (otherwise, one may take, $\|\ \|$, and its associated inner product with suitable mass coefficients). The system admits the degree -2 scaling symmetry:

$$\mathbf{D} = 2q \cdot \partial_q - p \cdot \partial_p.$$

Taking $\rho := \|q\|^{1/2}$ as a scaling function, the quotient $\mathbb{R}^{2nd}/\mathbf{D} \cong S^{nd-1} \times \mathbb{R}^{nd}$, has scale-invariant coordinates:

$$s := q/\rho^2 \in S^{nd-1}, \qquad y := \rho p \in \mathbb{R}^{nd}.$$

Carrying out the cotangent scaling reduction of Proposition 8, we have:

$$S = \frac{i_{\mathbf{D}}(p \cdot dq)}{\rho} = 2\tilde{J}\,, \qquad \tilde{J} = s \cdot y$$

with associated contact 1-form, $\rho\eta = i_{\mathbf{D}}\omega = -2q \cdot dp - p \cdot dq = -\rho(d\tilde{J} + s \cdot dy)$, given as:

$$\eta = \sigma \cdot ds - dS\,, \qquad \sigma := y - \tilde{J}s \in T^*_s S^{nd-1}.$$

That is, the contactification of T^*S^{nd-1}. The associated scale-reduced Hamiltonian, $\mathscr{H} = \rho^2 H$, is then:

$$\mathscr{H} = \frac{\|y\|^2}{2} - U(s) = \frac{\|\sigma\|^2}{2} - U(s) + \frac{S^2}{8}.$$

The corresponding scale-invariant equations of motion (Eqs. (7), with $\Lambda = -2$) are exactly the usual McGehee blow-up equations (39) (e.g. §4.2 of [43]), the collision manifold being given by the invariant level $\mathscr{H} = 0$ (see as well [40]). Recalling the discussion in Section IV B, we conclude that these McGehee blow-up equations are a -2-Herglotz system with Lagrangian function given by (40) via Legendre transform of $\mathscr{H}$. □

Remark 26. *In the coordinates* $TS^{nd-1} \times \mathbb{R} \ni (s, s', S)$ *corresponding to the scale reduction's structure as a contactification, these scale-reduced equations of motion* (39)*, read:*

$$\bar{\nabla}_{s'} s' = \bar{\nabla} U - \frac{S}{4}s'\,, \qquad S' = 2(\|s'\|^2 - U) + \frac{S^2}{4}$$

where $\bar{\nabla}$ *is the Levi-Cevita connection on the sphere* S^{nd-1}*. Note the Herglotz constraint for the curves reads as the equation for* S'*.*

Remark 27. *One may of course apply Proposition 9 to the translation reduction of the n-body system. For example, applied to the planar 3-body problem, one obtains a degree -2 contact system on the contactification:*

$$T^*S^3 \times \mathbb{R}.$$

To fully reduce the problem, one may exploit as well the rotational, S^1, symmetry which descends as contact symmetries of this scale-reduced system. More explicitly, the angular momentum, $G = p \cdot iq$, corresponds to the scale invariant: $\mathscr{G} = G/\rho = \sigma \cdot is$. This $\mathscr{G}$ is no longer a conserved quantity, rather a dissipated one:

$$\mathscr{G}' = -\frac{S}{4}\mathscr{G}, \qquad \mathscr{H}' = \frac{S}{2}\mathscr{H}$$

and associated to the integral ('Dziobek constant'): $\mathscr{G}^2\mathscr{H}$, of the scale-reduced system. Upon fixing this integral and quotienting by the (contact) rotation action, one may obtain –analogously to the coupling constants reduction– a 5-dimensional 'fully reduced' contact system. This full reduction corresponds to that obtained by first performing a symplectic reduction by rotations at a fixed angular momentum value and considering there an energy level of the rotation-reduced (symplectic) Hamiltonian. Describing this structure on the fully reduced n-body systems more explicitly will be developed in future work.

V. CONCLUSIONS

Let us recapitulate the main points of our work. We have shown that a general class of scaling symmetries exist within symplectic systems, and how this symmetry can be used to reduce the system onto its invariants. In Theorem 1 and its Corollary 1 we have shown that the reduction by identifying orbits of the scaling vector field on the symplectic manifold yields a contact manifold, that the Hamiltonian vector field on the symplectic manifold projects onto a (contact) Hamiltonian vector field on the contact manifold, and that the contact Hamiltonian and contact form that generate this vector field are obtained by a simple and explicit procedure. We refer to this procedure as a *contact reduction by scaling symmetries*, or simply *contact reduction*. This process has been further generalized beyond simple systems admitting a scaling symmetry. Indeed, by lifting systems with multiple couplings we find that there is a general process under which these also admit a contact reduction to an equivalent scale-invariant description (Theorem 2), thus realizing Poincaré's dream. The resultant scale-reduced systems, being contact systems, have interesting features not apparent in the unreduced symplectic manifolds, such as measure focusing and dissipation of previously conserved quantities. Such features have important roles in describing physical phenomena such as the arrow of time [33]. Finally, we have presented several examples, especially related to blow-ups in celestial mechanics. Among these, the most striking result is having re-obtained McGehee's blow-up for collisions in n-body systems in terms of a contact reduction and, therefore, as a Λ-Herglotz system (Proposition 9).

For future work, a neat and general framework for the realization of our contact reduction is the one put forward in [28, 29]. Thus we expect to analyze some of the results presented here within this context. Moreover, a most interesting application of the contact reduction is to examine the nature of 'blow-ups' (see Section IV C). For example, when the scale variable in the symplectic system approaches zero, such as at the total collisions of the three-body problem. Many of the problems regarding the continuation of solutions beyond such points arise due to the ill-defined evolution of the scale variable. Such a situation can be avoided in the reduced system as the evolution of the scale is immaterial to the behaviour of the contact system. One can envision situations in which the contact Hamiltonian vector field will remain well-defined on the contact manifold when the symplectic Hamiltonian vector field becomes ill defined, and thus a 'natural' continuation of the symplectic system may be considered as passing to the contact manifold for evolution beyond this point, and symplectifying the resultant evolution. In cosmological systems [35, 39, 46] this has been shown to allow continuation beyond the initial singularity, and in black holes beyond the central singularity [41].

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Appendix A: Symplectic and contact Hamiltonian mechanics

In this appendix we review briefly some of the main definitions and properties of Hamiltonian mechanics on symplectic and contact manifolds, paying special attention to the relationships between the two that will be used in this work in order to have it self-contained. Most of the definitions and results presented here can be found in [2] and [24], to which we refer for more detailed discussions.

1. Symplectic and contact: definitions

We begin with some general definitions that establish the notation and sign conventions.

Definition 7. A *symplectic manifold* is a pair (M, ω), where M is a $2n$-dimensional manifold and ω is a 2-form on M which is closed ($d\omega = 0$) and non-degenerate ($\omega^n \neq 0$). A *symplectic vector field* on M is one whose flow preserves $\omega : L_X\omega = 0$.

Dynamics on a symplectic manifold is usually given by Hamiltonian systems, which are defined as follows.

Definition 8. A (symplectic) *Hamiltonian system* is a triple (M, ω, H), where (M, ω) is a symplectic manifold and $H : M \to \mathbb{R}$ is a differentiable function called the (symplectic) *Hamiltonian.* Given this structure, a unique symplectic vector field X_H, the *(symplectic) Hamiltonian vector field*, is defined on M by the condition

$$i_{X_H}\omega = -dH\,. \tag{A1}$$

By a theorem of Darboux, locally we can always find coordinates (q^a, p_a), $a = 1, \ldots, n$, on a symplectic manifold such that

$$\omega = dp_a \wedge dq^a\,. \tag{A2}$$

These coordinates are called *canonical* or *Darboux* coordinates. It follows directly from (A1) and (A2) that the *trajectories* of X_H – i.e. parametrized curves $\gamma : I \subset \mathbb{R} \to M$ satisfying $\dot\gamma(t) = X_H(\gamma(t))$ – are solutions to the *symplectic Hamiltonian equations*

$$\dot q^a = \frac{\partial H}{\partial p_a} \qquad \text{and} \qquad \dot p_a = -\frac{\partial H}{\partial q^a}\,. \tag{A3}$$

On the contact side, we have the corresponding definitions

Definition 9. A *contact manifold* is a pair $(C, \mathscr{D})$, where C is a $(2n+1)$-dimensional manifold and $\mathscr{D}$ is a maximally non-integrable distribution of hyperplanes on C. Locally at least, there always exists a 1-form η on C such that $\mathscr{D} = \ker\eta$ and the maximal non-integrability condition is expressed as $\eta \wedge (d\eta)^n \neq 0$. Such η is called a *contact 1-form.* A *contact vector field* on C is one whose flow preserves $\mathscr{D} : L_X\eta \sim \eta$.

Note that a contact 1-form η exists globally if and only if $\mathscr{D}$ is co-orientable, meaning that the quotient line bundle $TC/\mathscr{D}$ is trivial (see [24] or Remark 28 below). Furthermore, once we fix a contact 1-form η for $\mathscr{D}$, then any other 1-form of the type $\eta' = f\eta$ with $f : C \to \mathbb{R}^\times$ yields a different contact form for the same $\mathscr{D}$. That is, there is a freedom in choosing contact forms for the same $\mathscr{D}$, up to re-scaling by a nowhere-vanishing function. However, to define certain dynamics, the choice of η is relevant, as the next definition will show.

We recall that on a contact manifold to a choice of η there is associated a special contact vector field, its *Reeb vector field* $\mathscr{R}$, defined by

$$i_{\mathscr{R}}d\eta = 0 \qquad \text{and} \qquad i_{\mathscr{R}}\eta = 1\,. \tag{A4}$$

Now we are ready to introduce the Hamiltonian dynamics on contact manifolds (see as well Proposition 10 and Remark 29 below for some context).

Definition 10. A (contact) *Hamiltonian system* is a triple $(C, \eta, \mathscr{H})$, where (C, η) is a contact manifold with a given choice of contact 1-form and $\mathscr{H} : C \to \mathbb{R}$ is a differentiable function called the (contact) *Hamiltonian.* Given this structure, a unique contact vector field $X_{\mathscr{H}}$, the *contact Hamiltonian vector field*, is defined on C, by the conditions

$$i_{X_{\mathscr{H}}}d\eta = -d\mathscr{H} + \mathscr{R}(\mathscr{H})\eta \qquad \text{and} \qquad i_{X_{\mathscr{H}}}\eta = \mathscr{H}\,. \tag{A5}$$

By a theorem of Darboux, locally we can always find coordinates (q^a, p_a, S) on a contact manifold such that

$$\eta = p_a dq^a - dS\,, \qquad \text{and} \qquad \mathscr{R} = -\frac{\partial}{\partial S}\,. \tag{A6}$$

These coordinates are called *contact* or *Darboux* coordinates. It follows directly from (A5) and (A6) that the trajectories of $X_{\mathscr{H}}$ are solutions to the *contact Hamiltonian equations*

$$\dot{q}^a = \frac{\partial \mathscr{H}}{\partial p_a} \qquad \dot{p}_a = -\frac{\partial \mathscr{H}}{\partial q^a} - p_a \frac{\partial \mathscr{H}}{\partial S} \qquad \dot{S} = p_a \frac{\partial \mathscr{H}}{\partial p_a} - \mathscr{H} \,. \tag{A7}$$

It should be stressed at this point that the dynamical vector fields X_H and $X_{\mathscr{H}}$ can be given variational definitions, provided the corresponding Lagrangians are (hyper-)regular, both in the symplectic and in the contact case. While in the symplectic case Hamilton's variational principle and the corresponding Lagrangian mechanics is well-known, its contact counterpart, known as Herglotz' variational principle, has been far less studied until recently (see [18, 25, 52]). In Appendix A 3 we detail this correspondence, and we re-state the main results of this work from the variational perspective in section IV B.

2. Symplectic and contact: known relationships and new definitions

Now we review some known relationships between symplectic and contact manifolds, together with the respective Hamiltonian dynamics.

We start with a procedure by which there always exists a natural extension of a contact manifold to a symplectic one (see [2], Appendix 4).

Definition 11. Given a contact manifold, $(C, \mathscr{D})$, the *symplectification*, $\tilde{C}$, of C is:

$$\tilde{C} := \{\beta_x \in T^*_x C : \ker \beta_x = \mathscr{D}_x\} \subset T^*C. \tag{A8}$$

Note that $\tilde{C}$ is an $\mathbb{R}^\times$-principal bundle over C with (exact) symplectic structure $d\tilde{\alpha}$ defined by restriction of the canonical symplectic form on T^*C:

$$\tilde{\alpha}_{\tilde{x}}(\xi) := \tilde{x}((\pi_C)_*\xi) \,, \tag{A9}$$

where $\pi_C : \tilde{C} \to C$ is the projection (see also [11]).

Remark 28. *The symplectification of C consists of the non-zero elements, $Ann(\mathscr{D})\backslash C$, of the annihilator of $\mathscr{D}$: $Ann(\mathscr{D}) := \{\beta : \beta|_{\mathscr{D}} \equiv 0\}$ and so is naturally identified with the non-zero elements of $(TC/\mathscr{D})^*$.*

When the contact distribution is co-orientable ($TC/\mathscr{D}$ is a trivial line bundle), then so too is the symplectification a trivial principal bundle: $\tilde{C} \cong C \times \mathbb{R}^\times$, trivialized by a global choice of contact 1-form η. In this case $\tilde{C}$ has two connected components, $\tilde{C}_\pm$ and, when a contact 1-form η has been chosen, it is convenient to refer to the component $\tilde{C}_+ := \{e^s \eta\}$ consisting of positive multiples of η as the symplectification.

To understand further the relationship between $\tilde{C}$ and C, we introduce the following definition.

Definition 12. Let $(M, \omega = d\lambda)$ be an exact symplectic manifold, where a choice for the symplectic potential λ has been made. The vector field $\mathbf{D} \in \mathfrak{X}(M)$ such that

$$i_{\mathbf{D}}\omega = \lambda \tag{A10}$$

is called the *Liouville vector field* of λ.

Then, for $(\tilde{C}, \tilde{\alpha})$ the symplectification of C, the generator, $\mathbf{D}$, of the $\mathbb{R}^\times$-action on $\tilde{C}$'s fibers is the Liouville vector field of $\tilde{\alpha}$. It turns out that there is a very special correspondence between contact vector fields on C and certain Hamiltonian vector fields on $\tilde{C}$. To see this, observe first that symplectic flows of $\tilde{X} \in \mathfrak{X}(\tilde{C})$ commuting with the flow of $\mathbf{D}$ are in fact Hamiltonian, and induce flows of $X = (\pi_C)_*\tilde{X} \in \mathfrak{X}(C)$ preserving $\mathscr{D}$ (contact flows). This leads to the following result (see [2], Appendix 4):

Proposition 10. Every contact vector field on C can be lifted to a symplectic Hamiltonian vector field on $\tilde{C}$ (the symplectification of C) whose Hamiltonian is homogeneous of degree 1 with respect to $\mathbf{D}$, that is, $L_{\mathbf{D}}\tilde{H} = \tilde{H}$. Conversely, every Hamiltonian vector field on $\tilde{C}$ such that $\tilde{H}$ is homogeneous of degree 1 projects onto C as a contact vector field.

Remark 29. *When a contact 1-form, η, of C has been chosen, any contact vector field X on C may be associated with a function $\mathscr{H} = i_X\eta$. The vector field is recovered from the function (and contact 1-form), by the conditions $i_X\eta = \mathscr{H}$, $L_X\eta \sim \eta$, which define a contact Hamiltonian vector field, cf. Eq.* (A5)*. According to Proposition 10, there is a Hamiltonian, $\tilde{H}$, on $\tilde{C}$ determined by X, which we refer to as the* symplectic lift *of $\mathscr{H}$, related to $\mathscr{H}$ by $\iota^*\tilde{H} = \mathscr{H}$, for $\iota : C \to \tilde{C}$ the section associated to η.*

See e.g. [2] and [50, 51] for applications of this correspondence to thermodynamics. Above (Theorem 1 and Corollary 2) we extend this relationship for Hamiltonians, $L_{\mathbf{D}}\tilde{H} = \Lambda\tilde{H}$ of degree Λ, on $\tilde{C}$.

Definition 13. The Λ-*Hamiltonian vector field*, $X^{\Lambda}_{\mathscr{H}}$, of a function $\mathscr{H}$ on the contact manifold C with respect to the contact 1-form η, is determined by the conditions:

$$i_{X^{\Lambda}_{\mathscr{H}}}\eta = \Lambda\mathscr{H}, \qquad i_{X^{\Lambda}_{\mathscr{H}}}d\eta = -d\mathscr{H} + (\mathscr{R}\mathscr{H})\,\eta\,.$$

Using this definition, the results of Section II B above may be stated in a more analogous manner to their degree one counterparts, as detailed in the following observation.

Remark 30. *For $\Lambda = 1$, Definition 13 is exactly the definition of a contact Hamiltonian vector field. In general, the trajectories of a symplectic Hamiltonian system of degree Λ on $\tilde{C}$ project to (reparametrized) trajectories of the Λ-Hamiltonian vector field on C determined by $\mathscr{H} = \iota^*\tilde{H}$, where $\iota : C \to \tilde{C}$ is the section associated to η.*

As a completely analogous construction to the symplectification, if (M, ω) is exact symplectic, there is also a natural procedure to extend it to a contact manifold (see [2], Appendix 4).

Definition 14. Given an exact symplectic manifold $(M, \omega = d\lambda)$, the *contactification* of M is

$$\tilde{M} := M \times \mathbb{R}, \qquad \tilde{\lambda} := \pi_M^*\lambda - dS\,, \tag{A11}$$

where $\pi_M : \tilde{M} \to M$ is the standard projection.

Again, there is an explicit relationship between Hamiltonian vector fields M and certain contact vector fields on $\tilde{M}$, given by

Proposition 11. Every symplectic Hamiltonian vector field on M can be lifted to a contact Hamiltonian vector field on $\tilde{M}$ (the contactification of M) whose Hamiltonian is

$$\tilde{\mathscr{H}} = \pi_M^* H\,. \tag{A12}$$

Conversely, every contact Hamiltonian vector field on $\tilde{M}$ such that $\tilde{\mathscr{R}}\tilde{\mathscr{H}} = 0$ projects onto M as a symplectic Hamiltonian vector field X_H. In this case we call $\tilde{\mathscr{H}}$ the *contact lift of* H.

In Darboux coordinates (q^a, p_a) for M, one has $\tilde{\lambda} = p_a dq^a - dS$ and the integral curves of $X_{\tilde{\mathscr{H}}}$ are given by (A7) by setting $\tilde{\mathscr{H}}(q^a, p_a, S) := H(q^a, p_a)$ (note that in particular $\partial\tilde{\mathscr{H}}/\partial S = 0$).

Note that, by definition, the flow of X_H is just the projection of that of $X_{\tilde{\mathscr{H}}}$. However, on $\tilde{M}$ there can be defined contact Hamiltonians that are not lifts of any symplectic Hamiltonian on M. This has been exploited largely in recent years to describe mechanical systems with dissipation (classical and quantum), thermostatted systems and thermomechanical phenomena in a Hamiltonian framework (see e.g. [6–9, 15, 21, 26, 27, 30, 31]).

Dual to the above constructions, in which we *extended* either a contact or a symplectic manifold, under the appropriate hypotheses one can also *reduce* one of these structures to the other. In particular, we have

Definition 15. Let $(M, \omega = d\lambda)$ be an exact symplectic manifold and $\mathbf{D} \in \mathfrak{X}(M)$ its *Liouville vector field.* Any hypersurface C transverse to $\mathbf{D}$ is called *of contact type.*

Hypersurfaces of contact type on an exact symplectic manifold are naturally endowed with a contact structure as follows.

Proposition 12. Let $(M, \omega = d\lambda)$ be an exact symplectic manifold, $\mathbf{D}$ the Liouville vector field and $\iota_C : C \to M$ a hypersurface of contact type. Then the 1-form $\eta_{\mathbf{D}} := \iota_C^* i_{\mathbf{D}}\omega = \iota_C^*\lambda$ is a contact form on C.

Example 14 (Transverse H levels)**.** *A typical example of a hypersurface of contact type is given by any regular energy level $C_E := H^{-1}(E)$ of a Hamiltonian system on the cotangent bundle $(T^*Q, d\alpha)$, where α is the canonical 1-form, which is transverse to the Liouville vector field. Interestingly, in this case the restriction of the Hamiltonian vector field X_H to C_E is just a reparametrization of the Reeb vector field of $\eta_{\mathbf{X}}$. Indeed, one has $\mathscr{R} = \frac{X_{H|C_E}}{\mathbf{X}(H)_{|C_E}}$, and therefore the* orbits *– the unparametrized trajectories – of the two vector fields are the same.*

Analougously, one can reduce a contact manifold to an exact symplectic one on a proper 'energy hypersurface' of the contact Hamiltonian and find an equivalence between the orbits of the contact Hamiltonian vector field and the Liouville one as follows (see [10]).

Definition 16. Let (C, η) be a contact manifold and $\mathscr{R} \in \mathfrak{X}(C)$ its *Reeb vector field.* Any hypersurface S transverse to $\mathscr{R}$ is called *of symplectic type.*

Hypersurfaces of symplectic type on a contact manifold are naturally endowed with a symplectic structure as follows.

Proposition 13. Let (C, η) be a contact manifold, $\mathscr{R}$ the Reeb vector field and $\iota_S : S \to C$ a hypersurface of symplectic type. Then the 2-form $\Omega = d\theta$, with $\theta = \iota_S^* \eta$ is an exact symplectic form on S.

Example 15 (Transverse $\mathscr{H}$ 0-level)**.** *Let $\mathscr{H} : C \to \mathbb{R}$ be a Hamiltonian function on a contact manifold (C, η) with Reeb vector field $\mathscr{R}$ and assume that $S = \mathscr{H}^{-1}(0) \neq \emptyset$ is transverse to $\mathscr{R}$. Then (S, Ω) is an exact symplectic manifold as above. Moreover, if Δ is the Liouville vector field of the exact symplectic manifold (S, Ω), then $X_{\mathscr{H}}|_S$ is the reparametrization of Δ given by $X_{\mathscr{H}}|_S = -(\mathscr{R}(\mathscr{H}) \circ \iota_S)\Delta$.*

Finally, let us also note that all the above relationships have motivated the definition of a *symplectic sandwich with contact bread*, see [10], where it was proved that the existence of an invariant measure for $X_{\mathscr{H}}|_S$ is equivalent to the existence of a symplectic sandwich with contact bread.

3. Symplectic and contact: variational principles

We conclude this appendix on the known relationships between symplectic and contact systems by recalling some known facts about their variational descriptions. We refer to e.g. [18, 25, 52] for the variational formulation of contact systems and to [36, 37, 53, 54] for the use of variational techniques in order to study dynamical properties and solutions of the related Hamilton-Jacobi equations.

Let us recall that certain Hamiltonian systems admit Lagrangian variational descriptions.

Definition 17. A *Lagrangian system* is a pair (Q, L), where $L : TQ \to \mathbb{R}$. The *action* of a curve $q(t) \in Q$ is $S := \int L(q, \dot{q})\, dt$.

The extremals of the action, with say fixed endpoints, satisfy the well-known *Euler-Lagrange equations*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \tag{A13}$$

The contact analogue of a Lagrangian system is

Definition 18. A *Herglotz Lagrangian system* (also known as a *contact Lagrangian system*) is a pair $(\bar{Q}, \mathscr{L})$, where $\mathscr{L} : T\bar{Q} \times \mathbb{R} \to \mathbb{R}$. The *action* of a curve $q(t) \in \bar{Q}$ is $\Delta S := \int \mathscr{L}(q, \dot{q}, S) dt$, where $S(t)$ solves $\dot{S} = \mathscr{L}(q, \dot{q}, S)$.

This definition extends the scope of Lagrangian systems to allow for dependence on the action. The extremals of this action, with say fixed endpoints and initial condition $S(0) = S_0$, satisfy the *Herglotz-Lagrange equations*:

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}}\right) = \frac{\partial \mathscr{L}}{\partial S}\frac{\partial \mathscr{L}}{\partial \dot{q}} + \frac{\partial \mathscr{L}}{\partial q}, \qquad \dot{S} = \mathscr{L}. \tag{A14}$$

The connection between Herglotz Lagrangian systems and contact Hamiltonian systems runs in parallel to the usual connection between Lagrangian and Hamiltonian systems. Under the *Legendre transform*,

$$Leg : T\bar{Q} \times \mathbb{R} \to T^*\bar{Q} \times \mathbb{R}, \qquad p = \frac{\partial \mathscr{L}}{\partial \dot{q}},$$

the Herglotz-Lagrangian extremals correspond to trajectories of the contact Hamiltonian system

$$C = T^*\bar{Q} \times \mathbb{R}, \qquad \mathscr{H} = p \cdot \dot{q} - \mathscr{L}, \qquad \eta = p \cdot dq - dS. \tag{A15}$$

Remark 31. *Here we consider regular systems, whose Legendre transform is a diffeomorphism. One way to see the connection of a contact Hamiltonian system to its Herglotz variational principle is via the Poincaré-Cartan form: $\alpha := p \cdot dq - \mathscr{H}\ dt$ and $\hat{\eta} := \alpha - dS = \eta - \mathscr{H}\ dt$ on the extended space, $C \times \mathbb{R} \ni (c, t)$. The world lines of trajectories of the contact Hamiltonian system are characterized by lying tangent to the line field $\ker \hat{\eta} \cap \{X : i_X d\alpha \sim \hat{\eta}\}$, or equivalently as extremals of $\gamma \mapsto \int_\gamma \alpha$ among curves satisfying the constraint $\hat{\eta}|_\gamma \equiv 0$.*

As we have observed in Remark 30, the trajectories of a symplectic Hamiltonian system admitting a scaling symmetry of degree Λ scale reduce to trajectories of a Λ-Hamiltonian vector field on C. Such trajectories may also be described variationally.

Definition 19. A Λ-*Herglotz system* is a pair, $(\bar{Q}, \mathscr{L})$, where $\mathscr{L} : T\bar{Q} \times \mathbb{R} \to \mathbb{R}$. The action of a curve $q(t) \in \bar{Q}$ is $S_\Lambda := \int \mathscr{L}(q, q', S)\ dt$, where S solves $S' = \mathscr{L} + (1 - \Lambda)\mathscr{E}$, for $\mathscr{E} := Leg^*\mathscr{H} = \partial_{q'}\mathscr{L} \cdot q' - \mathscr{L}$.

Remark 32. *Under Legendre transform, $\mathscr{H} = p \cdot \dot{q} - \mathscr{L}$, extremals of the Λ-Herglotz system $\mathscr{L}$ correspond to trajectories of the Λ-Hamiltonian vector field of $\mathscr{H}$ (Definition 13). One way to see this connection is via the Poincaré-Cartan form: $\alpha := p \cdot dq - \mathscr{H}\ dt$ and $\hat{\eta}_\Lambda := \eta - \Lambda \mathscr{H}\, dt$ on the extended space, $C \times \mathbb{R} \ni (c, t)$. The world lines of trajectories of the Λ-Hamiltonian vector field are characterized by lying tangent to the line field $\ker \hat{\eta}_\Lambda \cap \{X : i_X d\alpha \sim \hat{\eta}_\Lambda\}$, or equivalently as extremals of $\gamma \mapsto \int_\gamma \alpha$ among curves satisfying the constraint $\hat{\eta}_\Lambda|_\gamma \equiv 0$. Alternately, one may verify the correspondence directly via the equations of motion (Eqs. (7) above).*

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[1] Albouy A. *Projective dynamics and classical gravitation.* Regular and Chaotic Dynamics, 13 : 525–542, 2008.

[2] Arnold V. I. *Mathematical methods of classical mechanics.* Volume 60 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1989.

[3] Arnold V. I. *Geometrical methods in the theory of ordinary differential equations.* Volume 250. Springer Science & Business Media, 2012.

[4] Barbour J., Koslowski T. & Mercati F. *Identification of a gravitational arrow of time.* Physical review letters, 113 : 181101, 2014.

[5] Braddell R., Delshams A., Miranda E., Oms C. & Planas A. *An invitation to singular symplectic geometry.* International Journal of Geometric Methods in Modern Physics, 16, 2019.

[6] Bravetti A. *Contact geometry and thermodynamics.* International Journal of Geometric Methods in Modern Physics, 16 : 1940003, 2019.

[7] Bravetti A. & Garcia-Chung A. *A geometric approach to the generalized Noether theorem.* Journal of Physics A: Mathematical and Theoretical, 54 : 095205, 2021.

[8] Bravetti A. & Tapias D. *Thermostat algorithm for generating target ensembles.* Physical Review E, 93 : 022139, 2016.

[9] Bravetti A., Lopez-Monsalvo C. & Nettel F. *Contact symmetries and Hamiltonian thermodynamics.* Annals of Physics, 361 : 377–400, 2015.

[10] Bravetti A., de León M., Marrero J. C. & Padrón E. *Invariant measures for contact Hamiltonian systems: symplectic sandwiches with contact bread.* Journal of Physics A: Mathematical and Theoretical, 53 : 455205, 2020.

[11] Bruce A. J., Grabowska K., Grabowski J. et al. *Remarks on contact and Jacobi geometry.* SIGMA. Symmetry, Integrability and Geometry: Methods and Applications, 13 : 059, 2017.

[12] Bryant J. *Sur une reductión du problème de n corps mettant en évidence une structure de contact.* C. R. Acad. Sc. Paris, 290 : 673–675, 1980.

[13] Bryant J. *La variété de collision triple comme variété invariante d'un système de contact.* C. R. Acad. Sc. Paris, 291 : 205–208, 1980.

[14] Chenciner A. *A l'infini en temps fini.* Séminaire Bourbaki, 832 : 323–353, 1997.

[15] Ciaglia F. M., Cruz H. & Marmo G. *Contact manifolds and dissipation, classical and quantum.* Annals of Physics, 398 : 159–179, 2018.

[16] Cuffaro M. E. & Hartmann S. *The open systems view.* arXiv:2112.11095, 2021.

[17] de León M. & Lainz Valcázar M. *Contact Hamiltonian systems.* Journal of Mathematical Physics, 60 : 102902, 2019.

[18] De Leon M. & Lainz Valcázar M. *Singular Lagrangians and precontact Hamiltonian systems.* International Journal of Geometric Methods in Modern Physics, 16 : 1950158, 2019.

[19] de León M. & Valcázar M. L. *Infinitesimal symmetries in contact Hamiltonian systems.* Journal of Geometry and Physics : 103651, 2020.

[20] Devaney R. *Singularities in classical mechanical systems.* Ergodic theory and dynamical systems, 1 : 211–333, 1981.

[21] Entov M. & Polterovich L. *Contact topology and non-equilibrium thermodynamics.* arXiv:2101.03770, 2021.

[22] Gaset J., Gràcia X., Muñoz-Lecanda M. C., Rivas X. & Román-Roy N. *New contributions to the Hamiltonian and Lagrangian contact formalisms for dissipative mechanical systems and their symmetries.* International Journal of Geometric Methods in Modern Physics, 17 : 2050090, 2020.

[23] Geiges H. *A brief history of contact geometry and topology.* Expositiones Mathematicae, 19 : 25–53, 2001.

[24] Geiges H. *An introduction to contact topology.* Volume 109 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge. ISBN 978-0-521-86585-2, 2008.

[25] Georgieva B., Guenther R. & Bodurov T. *Generalized variational principle of Herglotz for several independent variables. First Noether-type theorem.* Journal of Mathematical Physics, 44 : 3911–3927, 2003.

[26] Goto S.-i. *Contact geometric descriptions of vector fields on dually flat spaces and their applications in electric circuit models and nonequilibrium statistical mechanics.* Journal of Mathematical Physics, 57 : 102702, 2016.

[27] Goto S.-i. *Nonequilibrium thermodynamic process with hysteresis and metastable states – A contact Hamiltonian with unstable and stable segments of a Legendre submanifold.* Journal of Mathematical Physics, 63 : 053302, 2022.

[28] Grabowska K. & Grabowski J. *A geometric approach to contact Hamiltonians and contact Hamilton–Jacobi theory.* Journal of Physics A: Mathematical and Theoretical, 55 : 435204, 2022.

[29] Grabowska K. & Grabowski J. *Reductions: precontact versus presymplectic.* arXiv:2211.16792, 2022.

[30] Grmela M. *Contact geometry of mesoscopic thermodynamics and dynamics.* Entropy, 16 : 1652–1686, 2014.

[31] Grmela M. & Öttinger H. C. *Dynamics and thermodynamics of complex fluids. I. Development of a general formalism.* Physical Review E, 56 : 6620, 1997.

[32] Gryb S. *New difficulties for the past hypothesis.* Philosophy of Science, 88 : 511–532, 2021.

[33] Gryb S. & Sloan D. *When scale is surplus.* Synthese, 199 : 14769–14820, 2021.

[34] Ismael J. & van Fraassen B. C. *Symmetry as a guide to superfluous theoretical structure.* In Brading K. & Castellani E., editors, *Symmetries in Physics: Philosophical Reflections*, page 371–392. Cambridge University Press, 2003.
[35] Koslowski T. A., Mercati F. & Sloan D. *Through the big bang: Continuing Einstein's equations beyond a cosmological singularity.* Phys. Lett. B, 778 : 339–343, 2018.
[36] Liu Q. & Torres P. J. *Orbital dynamics on invariant sets of contact Hamiltonian systems.* Discrete & Continuous Dynamical Systems-B, 2021.
[37] Liu Q., Torres P. J. & Wang C. *Contact Hamiltonian dynamics: Variational principles, invariants, completeness and periodic behavior.* Annals of Physics, 395 : 26–44, 2018.
[38] McGehee R. *Triple collision in the collinear three-body problem.* Inventiones mathematicae, 27 : 191–227, 1974.
[39] Mercati F. *Through the Big Bang in inflationary cosmology.* JCAP, 10 : 025, 2019.
[40] Mercati F. & Reichert P. *Total collisions in the N-body shape space.* Symmetry, 13 : 1712, 2021.
[41] Mercati F. & Sloan D. *Through a black hole singularity.* arXiv:2109.10753, 2021.
[42] Miranda E. & Oms C. *The geometry and topology of contact structures with singularities.* arXiv:1806.05638, 2018.
[43] Montgomery R. *Geometrical themes inspired by the n-body problem.* Springer : 49–89, 2018.
[44] Poincaré H. & Maitland F. *Science and method.* Courier Corporation, 2003.
[45] Sloan D. *Dynamical similarity.* Phys. Rev. D, 97 : 123541, 2018.
[46] Sloan D. *Scalar fields and the FLRW singularity.* Classical and Quantum Gravity, 36 : 235004, 2019.
[47] Sloan D. *New action for cosmology.* Physical Review D, 103 : 043524, 2021.
[48] Sloan D. *Scale symmetry and friction.* Symmetry, 13 : 1639, 2021.
[49] Valcázar M. L. & de León M. *Contact Hamiltonian Systems.* Journal of Mathematical Physics, 60 : 102902, 2019.
[50] van der Schaft A. *Liouville geometry of classical thermodynamics.* Journal of Geometry and Physics, 170 : 104365, 2021.
[51] van der Schaft A. & Maschke B. *Geometry of thermodynamic processes.* Entropy, 20 : 925, 2018.
[52] Vermeeren M., Bravetti A. & Seri M. *Contact variational integrators.* J. Phys. A: Math. Theor., 52 : 445206, 2019.
[53] Wang K., Wang L. & Yan J. *Implicit variational principle for contact Hamiltonian systems.* Nonlinearity, 30 : 492, 2016.
[54] Wang K., Wang L. & Yan J. *Aubry–Mather theory for contact Hamiltonian systems.* Comm. Math. Phys., 366 : 981–1023, 2019.