# CLIFFORD ALGEBRAS, SYMMETRIC SPACES AND COHOMOLOGY RINGS OF GRASSMANNIANS 

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#### Abstract

We study various kinds of Grassmannians or Lagrangian Grassmannians over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, all of which can be expressed as $\mathbb{G} / \mathbb{P}$ where $\mathbb{G}$ is a classical group and $\mathbb{P}$ is a parabolic subgroup of $\mathbb{G}$ with abelian unipotent radical. The same Grassmannians can also be realized as (classical) compact symmetric spaces $G / K$. We give explicit generators and relations for the de Rham cohomology rings of $\mathbb{G} / \mathbb{P} \cong G / K$. At the same time we describe certain filtered deformations of these rings, related to Clifford algebras and spin modules. While the cohomology rings are of our primary interest, the filtered setting of $K$-invariants in the Clifford algebra actually provides a more conceptual framework for the results we obtain.


## 1. Introduction

This paper is motivated by email correspondence between the third named author and Bert Kostant in 2004 Kos04. In his study of the action of Dirac operators on Harish-Chandra modules attached to a real reductive Lie group $G_{\mathbb{R}}$, the third named author was led to consider the algebra $C(\mathfrak{p})^{K}$. Here $K$ is a maximal compact subgroup of $G_{\mathbb{R}}$ corresponding to a Cartan decomposition $\mathfrak{g}=\mathfrak{k} \oplus \mathfrak{p}$ of the complexified Lie algebra of $G_{\mathbb{R}}$ and $C(\mathfrak{p})$ is the Clifford algebra of $\mathfrak{p}$ with respect to the (extended) Killing form. The third named author remembered hearing Kostant speak about the graded version of $C(\mathfrak{p})^{K},(\bigwedge \mathfrak{p})^{K}$, and its relation to cohomology, so he asked Kostant about the latter algebra expecting the description of the graded version of the algebra will give some information about the algebra $C(\mathfrak{p})^{K}$ he was interested in. Kostant replied as follows:

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From kostant@math.mit.edu Mon Mar 15 21:07:54 2004
    for <pandzic@math.hr>; Mon, 15 Mar 2004 21:07:53 +0100 (MET)
Subject: Re: a question
Dear Pavle
            The following is known. In general (\wedge p)^k is the
cochain complex whose cohomology is H(G/K). In the symmetric case (G/K is a symmetric
space) the cobounbary operator is trivial so that (\wedge p)^k = H(G/K). If in
addition rank K = rank G then all cohomology is even dimension and if e = Euler
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[^0]```
characteristic of \(G / K\) then of course
\(\operatorname{dim}\) ( \(\backslash\) wedge \(p)^{\wedge} k=e\). Also in this case \(\operatorname{dim} p\) is even and \(C(p)=\) End \(S\) where \(S\) is the
spin module. Furthermore under the action of \(K\)
    S=V_1 + . . . . + V_e
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where the V_i are irreducible $K$ modules and all distinct. Thus $C(p)^{\wedge} K$ is an abelian
(semisimple) algebra of dimension e. This is a special case of my result with
Sternberg, Ramond and Gross (GKRS) in the PNAS. It is a nice unsolved problem to
locate the 1 -dimensional idempotents which project onto the various $V_{-}$i. In case $G / K$
is Hermitian symmetric these idempotents I believe correspond to the Schubert classes
in ( $\backslash$ wedge $p)^{\wedge} k$. If so one has some sort of generalization of Schubert classes when
G/K is not Hermitian.
Best regards
Bert

It turns out Kostant was not exactly right in thinking that the idempotents will correspond to the Schubert classes; in fact, they typically all have nonzero top degree term, and one must take their linear combinations in order to get a basis compatible with filtration. His intuition that this question is related to some sort of generalization of Schubert classes when $G / K$ is not Hermitian was however right; as we shall see, these cases correspond to various kinds of real or quaternionic Grassmannians which possess their own Schubert calculus.

Our main goal in this paper is to describe the de Rham cohomology rings of these Grassmannians using their realization as compact symmetric spaces. The main tool is the representations of the Clifford algebras associated with symmetric spaces. When the Grassmannians are complex, the results we obtain here are well known. However, for the real and quaternionic Grassmannians, the results are not widely known. For example, the ring structure of the cohomology of the real Grassmannians was conjectured by Casian and Kodama CK13. Recently, many related papers have appeared (see Rab16, LR22, RSM19, Mat21 for integral cohomology, for example), but these papers treat the Grassmannians individually, and not in a uniform way. See also Che20, CHL20, EH13, HL11 for identifications of compact symmetric spaces and Grassmannians.

We study these cohomology rings systematically. The key ideas are the usage of Clifford algebras as mentioned above and the description of the Grassmannians as flag manifolds corresponding to maximal parabolic subgroups with abelian unipotent radicals. In fact, we start with the Grassmannian $\mathbb{G} / \mathbb{P}$, where $\mathbb{P}$ is a maximal parabolic subgroup with abelian unipotent radical in a reductive Lie group $\mathbb{G}$, then produce symmetric spaces of compact and noncompact type using three involutions $\theta, \sigma$ and $\tau=\theta \sigma$ of $\mathbb{G}$, which are mutually commuting.

Our results describe the de Rham cohomology ring of each of the Grassmannians in our list by explicit generators and relations, and also give an explicit basis consisting of certain monomials. In most cases (including the well known complex Grassmannian cases) we show that our basis can be replaced by a basis consisting of certain Schur polynomials. These have the advantage of a rather well understood multiplication table, related to the Littlewood-Richardson coefficients. However our monomials with their clear structure of generators and relations also lead to an explicit multiplication table, as explained in Section 4 . In this way we get an alternative approach to Schubert calculus on the Grassmannians in question.

The paper is organized as follows.

In Section 2 we describe our Grassmannians $\mathbb{G} / \mathbb{P}$ and give their realizations as compact symmetric spaces. The cases are summarized in the table at the end of the introduction. In each case $\mathbb{P}$ is a maximal parabolic subgroup of $\mathbb{G}$ with abelian unipotent radical. Note that a Grassmannian is the set of certain subspaces of fixed dimension in a vector space $V$. On this set of subspaces, the automorphism group $\mathbb{G}$ of $V$, which typically preserves additional structure (quadratic or symplectic forms), acts transitively. This is justified by the following well known theorem by Witt.

Theorem 1.1. Wit37, Bou74, Ch. 1-2], Die55. Let $V$ be a vector space over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ with a nondegenerate form $\langle$,$\rangle which is either bilinear symmetric, or bilinear skew symmetric,$ or Hermitian, or skew Hermitian. Let $U$ and $W$ be subspaces of $V$ and let $\varphi: U \rightarrow W$ be an isomorphism preserving $\langle$,$\rangle . Then \varphi$ extends to an automorphism of $V$ preserving $\langle$,$\rangle .$

Let $\mathbb{P}$ be the stabilizer in $\mathbb{G}$ of a standard subspace $U$ in our Grassmannian. (In some cases, $U$ is a Lagrangian or isotropic subspace. It depends on the situation.) Then $\mathbb{P}$ is a maximal parabolic subgroup and our Grassmannian is equal to $\mathbb{G} / \mathbb{P}$. So Grassmannians are naturally identified with (partial) flag manifolds $\mathbb{G} / \mathbb{P}$.

As already mentioned, compact symmetric spaces are diffeomorphic to Grassmannians $\mathbb{G} / \mathbb{P}$ where $\mathbb{P}$ has abelian unipotent radical. In fact, these Grassmannians exhaust all such pairs $(\mathbb{G}, \mathbb{P})$ (see Wol76, How95, § 5.5.1], and also [RRS92]). Then, in appropriate realizations, the Grassmannians are varieties of either ordinary subspaces of a vector space, or of Lagrangian subspaces with respect to a certain form.

In this paper, the group $\mathbb{G}$ will be a classical group; in particular, it is a reductive matrix group, and we always consider the standard Cartan involution

$$
\begin{equation*}
\theta(g)=\left(\bar{g}^{t}\right)^{-1}, \quad g \in \mathbb{G} \tag{1.2}
\end{equation*}
$$

The corresponding maximal compact subgroup $\mathbb{G}^{\theta}$ will be denoted by $\mathbb{K}$ and also by $G$.
We will use Proposition 2.4 below to see that $\mathbb{K}$ acts transitively on the Grassmannian, so that $\mathbb{G} / \mathbb{P}$ is diffeomorphic to $\mathbb{K} / \mathbb{P} \cap \mathbb{K}$. This follows easily from the fact that $\mathbb{P}$ contains a minimal parabolic subgroup $\mathbb{P}_{0}=\mathbb{M A} \mathbb{N}_{0}$, and that $\mathbb{G}$ has an Iwasawa decomposition $\mathbb{G}=\mathbb{K} \mathbb{A} \mathbb{N}_{0}$, where $\mathbb{K}$ is as above.

It is known by [TK68, §4] (see also [Kob08, Lemma 7.3.1] and RRS92]) that the following are equivalent:
(a) $\mathbb{P}$ has abelian unipotent radical;
(b) $\mathbb{P}$ has a Levi subgroup $\mathbb{L}$ which is a symmetric subgroup of $\mathbb{G}$;
(c) $K:=\mathbb{P} \cap \mathbb{K}=\mathbb{L} \cap \mathbb{K}$ is a symmetric subgroup of $G=\mathbb{K}$.

We give a short and comprehensive proof of this result in Theorem 2.2 for convenience of the readers.
The involution $\sigma$ of $\mathbb{G}$ mentioned above is related to the above Levi subgroup $\mathbb{L}$ of $\mathbb{P}: \mathbb{L}$ is $\mathbb{G}^{\sigma}$, the subgroup of $\mathbb{G}$ consisting of points fixed by $\sigma$; we will see that also $\mathbb{P}$ itself is $\sigma$-stable. The involution $\sigma$, which we describe explicitly below, commutes with the Cartan involution $\theta$, and hence $K=\mathbb{L}^{\theta}=\mathbb{L} \cap \mathbb{K}$ is a maximal compact subgroup of $\mathbb{L}$.

Let us denote by $\tau=\theta \sigma$ the third involution of $\mathbb{G}$. We denote the fixed point subgroup $\mathbb{G}^{\tau}$ by $G_{\mathbb{R}}$, so that $G_{\mathbb{R}} / K$ is a noncompact Riemannian symmetric space with its compact dual equal to $G / K$. It turns out that in this way we get to cover the full list of compact classical symmetric spaces as listed e.g. in How95, p.69].

We summarize the three involutions and symmetric spaces thus obtained in the following diagram. See also Table 1 at the end of Introduction.


Since $\mathbb{P}$ is block upper triangular with two diagonal blocks, of sizes (say) $p$ and $q, \mathbb{L}$ can be taken as the block diagonal part of $\mathbb{P}$. Now if we denote by $I_{p, q}$ the matrix $\left(\begin{array}{cc}I_{p} & 0 \\ 0 & -I_{q}\end{array}\right)$, then

$$
I_{p, q}\left(\begin{array}{ll}
A & B \\
C & D
\end{array}\right) I_{p, q}=\left(\begin{array}{cc}
A & -B \\
-C & D
\end{array}\right)
$$

and we see that the involution we want is

$$
\begin{equation*}
\sigma(g)=I_{p, q} g I_{p, q} \tag{1.3}
\end{equation*}
$$

It follows that $\mathbb{P}$ is $\sigma$-stable, and it will now be very easy to identify the groups $\mathbb{L}=\mathbb{G}^{\sigma}$. (In fact, we will see in Section 2 that $\sigma$ can be described in terms of $\mathbb{P}$ only; see Theorem 2.2.)

It turns out that the complexifications of these groups are exactly the groups listed in How95, p. 70] as the groups acting in a skew-multiplicity free way on $\mathfrak{p}$, the complexified tangent space of $G / K \simeq \mathbb{G} / \mathbb{P}$ at the base point $e K$. Here "skew-multiplicity free" means that $\mathbb{L}_{\mathbb{C}}$ acts on $\bigwedge \mathfrak{p}$ in a multiplicity free way (note that $\mathfrak{p}$ can also be identified with the complexification of the Lie algebra of $\mathbb{N}$, so that $\mathbb{L}$ acts on it naturally).

We will also describe the groups $G_{\mathbb{R}}=\mathbb{G}^{\tau}$, where $\tau=\sigma \theta$. Note that by (1.2) and (1.3), we get

$$
\begin{equation*}
\sigma \theta(g)=I_{p, q}\left(\bar{g}^{t}\right)^{-1} I_{p, q} . \tag{1.4}
\end{equation*}
$$

The group $G_{\mathbb{R}}$ is a noncompact reductive Lie group with maximal compact subgroup $K=\mathbb{K}^{\sigma}$, and $G / K=\mathbb{K} / \mathbb{K}^{\sigma}$ is the compact dual of the noncompact Riemannian symmetric space $G_{\mathbb{R}} / K$ as explained above. In this way, we get to cover the full list of noncompact classical symmetric spaces. As in the Hermitian symmetric case, the noncompact Riemannian symmetric space $G_{\mathbb{R}} / K$ is embedded into $\mathbb{G} / \mathbb{P}$ as an open subset (a generalization of the Borel embedding).

The realizations of our Grassmannians as symmetric spaces are known in most (or all) cases, but the results are scattered in the literature. We will indicate some references when we get to the case by case analysis. Our view point is to produce the classical Riemannian symmetric spaces, both compact and noncompact ones, in terms of the pairs $(\mathbb{G}, \mathbb{P})$ on our list.

In Section 3 we collect some facts needed for our description of the cohomology rings of compact symmetric spaces $G / K$ as above (and thus also of the corresponding Grassmannians $\mathbb{G} / \mathbb{P}$ ). For $\sigma$ as above, its restriction to $G$ is an involution such that $K=G^{\sigma}$. Let $\mathfrak{g}=\mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of the complexified Lie algebra $\mathfrak{g}$ of $G$ into eigenspaces of $\sigma$. We are assuming $G$ is connected, but $K$ need not be connected.

As noted in Kostant's email message, if $\mathfrak{g}$ and $\mathfrak{k}$ have equal rank, then the algebras $C(\mathfrak{p})^{K}$ and $(\bigwedge \mathfrak{p})^{K}$ can be expressed as

$$
C(\mathfrak{p})^{K}=\operatorname{Pr}(S) ; \quad(\bigwedge \mathfrak{p})^{K}=\operatorname{gr} \operatorname{Pr}(S)
$$

where the algebra $\operatorname{Pr}(S)$ is spanned by the projections of the spin module $S$ to its isotypic components for the pin double cover $\widetilde{K}$ of $K$. As explained in Subsections 3.1 and 3.3 , one can use the natural map $\alpha: U(\mathfrak{k}) \rightarrow C(\mathfrak{p})$ that gives the spin module its $\mathfrak{k}$-module structure, and the fact that the projections are given by the action of the center of $U(\mathfrak{k})$, to express the algebra $\operatorname{Pr}(S)$ as the quotient of $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{K}}$ by the ideal generated by the elements of $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{G}}$ vanishing at $\rho$. Here $\mathfrak{t}$ is the complexification of a Cartan subalgebra of the Lie algebra of $K, W_{K}$ is the Weyl group of $K$ (see (3.30)), $W_{G}=W_{\mathfrak{g}}$ is the Weyl group of $G$ or equivalently the Weyl group of the root system $\Delta(\mathfrak{g}, \mathfrak{t})$, and $\rho$ is the half sum of roots in the (fixed) positive root system $\Delta^{+}(\mathfrak{g}, \mathfrak{t})$. The algebra $\operatorname{gr} \operatorname{Pr}(S)$ attached to the natural filtration of $\operatorname{Pr}(S)$ can be expressed as the quotient of $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{K}}$ by the ideal generated by the elements of $\mathbb{C}\left[\epsilon^{*}\right]^{W_{G}}$ vanishing at 0 .

In the "almost equal rank case" $(\mathfrak{g}, \mathfrak{k})=(\mathfrak{s o}(2 p+2 q+2), \mathfrak{s o}(2 p+1) \times \mathfrak{s o}(2 q+1))$, the algebras $C(\mathfrak{p})^{K}$ and $(\bigwedge \mathfrak{p})^{K}$ are very close to $\operatorname{Pr}(S)$ and $\operatorname{gr} \operatorname{Pr}(S)$; one has to tensor with the Clifford respectively exterior algebra of a certain one-dimensional space. This is explained in Subsections 3.5 and 3.7

On the opposite end are the primary and almost primary cases; in these cases the $\widetilde{K}$-module $S$ has only one isotypic component and the algebra of projections is trivial. These cases are described in Subsections 3.6 and 3.8 The algebra $(\bigwedge \mathfrak{p})^{K}$ is now equal to the exterior algebra of a certain subspace of $(\bigwedge \mathfrak{p})^{K}$ (denoted by $\left.\mathcal{P}_{\wedge}(\mathfrak{p})\right)$, where $\mathcal{P}_{\wedge}(\mathfrak{p})$ is the subspace orthogonal to the square of the augmentation ideal (Definition 3.26). This result is due to Hopf and Samelson Hop41, Sam41 for the group case and Theorem 3.29 for the remaining (almost) primary cases. Similar isomorphisms go way back to Cartan, Chevalley, Koszul and others; see [GHV76, p. 568]. We believe that likewise $C(\mathfrak{p})^{K}$ is the Clifford algebra over $\mathcal{P}_{\wedge}(\mathfrak{p})$, but this is currently known (by the results of Kostant Kos97]) only in the group cases, i.e., when $\mathfrak{g}=\mathfrak{g}_{1} \oplus \mathfrak{g}_{1}$ and $\mathfrak{k} \cong \mathfrak{p} \cong \mathfrak{g}_{1}$.

Finally, in Section 4 we give a precise description, by generators and relations, of the algebras $(\bigwedge \mathfrak{p})^{K}$ in each of the cases. In the equal rank and almost equal rank cases, we also describe the algebras $C(\mathfrak{p})^{K}$, which in these cases amounts to describing the algebras $\operatorname{Pr}(S)$. We also give explicit bases for these algebras. In this way we get to compute the de Rham cohomology of the symmetric spaces on our list, and thus also of the corresponding Grassmannians.

Namely, as mentioned in Kostant's email message, the cohomology of the compact symmetric space $G / K$ is (after complexification) equal to $(\bigwedge \mathfrak{p})^{K}$. This fact is quite well known, but it is not easy to find an appropriate reference. It is proved in Tay18 (unpublished) using Hodge theory, partially proved in Leu16, and proved in GHV76 under the assumption $K$ is connected. It is also mentioned in passing in How95, p. 69], and in BW00, § 1.6]. Borel and Wallach BW00] attribute the result to É. Cartan and de Rham. We start Section 4 by presenting a simple proof of this fact which we learned from Sebastian Goette Goe23.

We are thus led to study the appropriate quotients of the $W_{K}$-invariants in $\mathbb{C}\left[\mathfrak{t}^{*}\right]$ in each of the cases. The results often involve the following algebra.

Definition 1.5. Let $p, q \in \mathbb{Z}$ with $1 \leq p \leq q$, and let $c=\left(c_{1}, \ldots, c_{p+q}\right) \in \mathbb{C}^{p+q}$. We define $\mathfrak{H}(p, q ; c)$ to be the algebra generated by $r_{1}, \ldots, r_{p}$ and $s_{1}, \ldots, s_{q}$ with relations generated by

$$
\sum_{i, j \geq 0 ; i+j=k} r_{i} s_{j}=c_{k}
$$

for $k=1, \ldots, p+q$, where we set $r_{0}=s_{0}=1$ and $r_{i}=0$ if $i>p, s_{j}=0$ if $j>q$.
We can use the first $q$ of the relations to express $s_{1}, \ldots, s_{q}$ in terms of the $r_{i}$, so $\mathfrak{H}(p, q ; c)$ is in fact generated by $r_{1}, \ldots, r_{p}$ only. The remaining relations can be used to obtain the relations among the $r_{i}$, not involving the $s_{j}$. We do that in the proof of Theorem 4.5 the relations among the $r_{i}$ are (4.9) and (4.14) and they form another set of defining relations. From these relations
one can obtain expressions for each monomial in $r_{1}, \ldots, r_{p}$ of degree $q+1$ as a linear combination of lower degree monomials in $r_{1}, \ldots, r_{p}$. We will also see that the monomials in the $r_{i}$ of degree at most $q$ form a basis of the algebra $\mathfrak{H}(p, q ; c)$; so $\mathfrak{H}(p, q ; c)$ can be identified with the space

$$
\mathbb{C}\left[r_{1}, \ldots, r_{p}\right]_{\leq q}
$$

of polynomials in the $r_{i}$ of degree $\leq q$.
We show in Remark 4.16 that the above monomials span the same subspace of $\mathbb{C}\left[r_{1}, \ldots, r_{p}\right]$ as the Schur polynomials $s_{\lambda}$ attached to partitions $\lambda$ with Young diagrams contained in the $p \times q$ box. Moreover, our basis consisting of monomials and the basis consisting of Schur polynomials are connected by a triangular change of basis. In this way we get a connection with the usual Schubert calculus, where the multiplication of the Schur polynomials is given in terms of LittlewoodRichardson coefficients.

For $G / K=\mathrm{U}(p+q) / \mathrm{U}(p) \times \mathrm{U}(q)$, we prove in Theorem 4.5 that $C(\mathfrak{p})^{K}$ is isomorphic to the algebra $\mathfrak{H}(p, q ; c)$, with the $r_{i}$ being the elementary symmetric functions in the first $p$ coordinate functions $x_{1}, \ldots, x_{p}$ on $\mathfrak{t} \cong \mathbb{C}^{p+q}$, the $s_{j}$ being the elementary symmetric functions in the last $q$ coordinate functions $x_{p+1}, \ldots, x_{p+q}$ on $\mathfrak{t}$, and $c=\left(t_{1}(\rho), \ldots, t_{p+q}(\rho)\right)$, where $t_{k}$ are the elementary symmetric functions on $x_{1}, \ldots, x_{p+q}$. The natural filtration on $C(\mathfrak{p})^{K}$ coming from the filtration of $C(\mathfrak{p})$ corresponds to the filtration on $\mathfrak{H}(p, q ; c)$ obtained by setting

$$
\begin{equation*}
\operatorname{deg} r_{i}=2 i, \quad i=1, \ldots, p \tag{1.6}
\end{equation*}
$$

The cohomology ring $(\bigwedge \mathfrak{p})^{K}$ is isomorphic to $\mathfrak{h}(p, q, 0)$, and its natural grading is again obtained by (1.6).

For $G / K=\operatorname{Sp}(p+q) / \operatorname{Sp}(p) \times \operatorname{Sp}(q)$ (Theorem4.19), the algebra $C(\mathfrak{p})^{K}$ is again $\mathfrak{H}(p, q ; c)$, but now the $r_{i}$, the $s_{j}$ and the $t_{k}$ are elementary symmetric functions on the squares of the appropriate variables. The parameter $c$ is again given by evaluating $t_{k}$ at $\rho$. The filtration is now obtained by setting $\operatorname{deg} r_{i}=4 i, i=1, \ldots, p$. The cohomology ring $(\bigwedge \mathfrak{p})^{K}$ is again isomorphic to $\mathfrak{h}(p, q, 0)$, and its natural grading is also obtained by setting $\operatorname{deg} r_{i}=4 i$. For $G / K=\mathrm{SO}(k+m) / S(\mathrm{O}(k) \times \mathrm{O}(m))$ (Theorem 4.21), the algebra $C(\mathfrak{p})^{K}$ is $\mathfrak{H}(p, q ; c)$ if $(k, m)=(2 p, 2 q)$ or $(2 p, 2 q+1)$, with $\left\{r_{i}\right\},\left\{s_{j}\right\}$, $\left\{t_{k}\right\}$ and $c$ defined similarly as above. If $(k, m)=(2 p+1,2 q+1)$ (the almost equal rank case), there is an extra generator $e$, of degree $2 p+2 q+1$, squaring to 1 . The filtration degrees of the $r_{i}$ are again equal to $4 i$. The cohomology ring $(\bigwedge \mathfrak{p})^{K}$ is isomorphic to $\mathfrak{H}(p, q, 0)$ or $\mathfrak{H}(p, q, 0) \oplus \mathfrak{H}(p, q ; 0) e$, and its natural grading is also obtained by setting $\operatorname{deg} r_{i}=4 i$ and $\operatorname{deg} e=2 p+2 q+1$. In this case we get to prove the conjecture of Casian-Kodama CK13.

For $G / K=\mathrm{U}(n) / \mathrm{O}(n)$ (Theorem 3.29), the situation is different: the algebra $(\bigwedge \mathfrak{p})^{K}$ is the exterior algebra on the subspace $\mathcal{P}_{\wedge}(\mathfrak{p})$ (Definition 3.26), and the degrees are given in Table 2,

For $G / K=\operatorname{Sp}(n) / U(n)$ (Theorem 4.25), we are back to elementary symmetric functions: $r_{1}, \ldots, r_{n}$ are the elementary symmetric functions on the coordinate functions $x_{1}, \ldots, x_{n}$ on the Cartan subalgebra $\mathfrak{t} \cong \mathbb{C}^{n}$ of $\mathfrak{k}$, while $t_{1}, \ldots, t_{n}$ are the elementary symmetric functions on the squares of the $x_{i}$. The algebras $C(\mathfrak{p})^{K}$ and $(\bigwedge \mathfrak{p})^{K}$ are generated by the $r_{i}$, but the relations are now different:

$$
r_{k}^{2}=t_{k}+2 r_{k-1} r_{k+1}-2 r_{k-2} r_{k+2}+\ldots, \quad k=1, \ldots, n
$$

where as usual we set $r_{0}=1$ and $r_{i}=0$ for $i>n$ or $i<0$, and where $t_{k}$ should be replaced by $t_{k}(\rho)$ if the algebra is $C(\mathfrak{p})^{K}$ and with 0 if the algebra is $(\bigwedge \mathfrak{p})^{K}$. This time a basis for each of our algebras is given by the monomials

$$
r_{1}^{\varepsilon_{1}} r_{2}^{\varepsilon_{2}} \ldots r_{n}^{\varepsilon_{n}}, \quad \varepsilon_{i} \in\{0,1\}
$$

The filtration degree of $C(\mathfrak{p})^{K}$ inherited from $C(\mathfrak{p})$, and the gradation degree of $(\bigwedge \mathfrak{p})^{K}$ inherited from $\wedge \mathfrak{p}$, are obtained by setting $\operatorname{deg} r_{i}=2 i$ for $i=1, \ldots, n$.

For $G / K=\mathrm{SO}(2 n) / \mathrm{U}(n)$ (Theorem4.28) the situation is entirely analogous to the case $G / K=$ $\operatorname{Sp}(n) / U(n)$, except that we get to eliminate $r_{n}$ from the list of generators.

For the group cases $G \times G / \Delta G \cong G$ where $G$ is $\operatorname{SO}(n), \mathrm{U}(n)$ or $\operatorname{Sp}(n)$ (Theorem 4.30), the algebras $C(\mathfrak{p})^{K} \cong C(\mathfrak{g})^{\mathfrak{g}}$ and $(\bigwedge \mathfrak{p})^{K} \cong(\bigwedge \mathfrak{g})^{\mathfrak{g}}$ are the Clifford respectively exterior algebra of the graded subspace $\mathcal{P}_{\wedge}(\mathfrak{p})$ (Definition 3.26).

For the Clifford algebras, these cases were settled by Kostant in Kos97. For the exterior algebras, the fact that $(\bigwedge \mathfrak{p})^{\mathfrak{k}}$ is isomorphic to a graded subspace goes back to Cartan, Chevalley, Koszul and others GHV76. The degrees are given in Table 2

For the cases $G / K=\mathrm{U}(2 n) / \operatorname{Sp}(n)$ (Theorem4.31) the algebra $(\bigwedge \mathfrak{p})^{K}$ is the exterior algebra of the graded subspace $\mathcal{P}_{\wedge}(\mathfrak{p})$ (Definition 3.26). Again, the degrees are given in Table 2,

Our results are well known for complex Grassmannians, i.e., for $\operatorname{Gr}_{p}\left(\mathbb{C}^{p+q}\right) \cong \mathrm{U}(p+q) / \mathrm{U}(p) \times$ $\mathrm{U}(q), \mathrm{Gr}_{2}\left(\mathbb{R}^{2+q}\right) \cong \mathrm{SO}(2+q) / S(\mathrm{O}(2) \times \mathrm{O}(q)), \operatorname{LGr}\left(\mathbb{C}^{2 n}\right) \cong \mathrm{Sp}(n) / \mathrm{U}(n)$ and $\mathrm{OLGr}^{+}\left(\mathbb{C}^{2 n}\right) \cong$ $\mathrm{SO}(2 n) / \mathrm{U}(n)$. Among many papers dealing with the complex Grassmannians and their Schubert calculus, we mention [Ful97], [FP98], Tam05, Tam01, and [Pra91, PR96, PR03].

Table 1. Table of Grassmannians and corresponding symmetric spaces.

| General Linear $n=p+q$ |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbb{G}$ | $\mathbb{P}$ | $\mathbb{L}=\mathbb{G}^{\sigma}$ | $\mathbb{K}=\mathbb{G}^{\theta}=G$ | $\mathbb{G}^{\sigma \theta}=G_{\mathbb{R}}$ | $\mathbb{K}^{\sigma}=G_{\mathbb{R}}^{\theta}=K$ | $\mathfrak{p}_{0}$ |
| $\mathrm{GL}_{n}(\mathbb{R})$ | $\operatorname{Stab}\left(\mathbb{R}^{p}\right)$ | $\mathrm{GL}_{p}(\mathbb{R}) \times \mathrm{GL}_{q}(\mathbb{R})$ | $\mathrm{U}_{n}(\mathbb{R})$ | $\mathrm{U}_{p, q}(\mathbb{R})$ | $\mathrm{U}_{p}(\mathbb{R}) \times \mathrm{U}_{q}(\mathbb{R})$ | $\operatorname{Mat}_{p, q}(\mathbb{R})$ |
| $\mathrm{GL}_{n}(\mathbb{C})$ | $\operatorname{Stab}\left(\mathbb{C}^{p}\right)$ | $\mathrm{GL}_{p}(\mathbb{C}) \times \mathrm{GL}_{q}(\mathbb{C})$ | $\mathrm{U}_{n}(\mathbb{C})$ | $\mathrm{U}_{p, q}(\mathbb{C})$ | $\mathrm{U}_{p}(\mathbb{C}) \times \mathrm{U}_{q}(\mathbb{C})$ | $\operatorname{Mat}_{p, q}(\mathbb{C})$ |
| $\mathrm{GL}_{n}(\mathbb{H})$ | $\operatorname{Stab}\left(\mathbb{H}^{p}\right)$ | $\mathrm{GL}_{p}(\mathbb{H}) \times \mathrm{GL}_{q}(\mathbb{H})$ | $\mathrm{U}_{n}(\mathbb{H})$ | $\mathrm{U}_{p, q}(\mathbb{H})$ | $\mathrm{U}_{p}(\mathbb{H}) \times \mathrm{U}_{q}(\mathbb{H})$ | $\operatorname{Mat}_{p, q}(\mathbb{H})$ |
| Symplectic$\mathbb{G}$ |  |  |  |  |  |  |
| $\mathrm{Sp}_{2 n}(\mathbb{R})$ | $\operatorname{Stab}\left(L_{0}\right)$ | $\mathrm{GL}_{n}(\mathbb{R})$ | $\mathrm{U}_{n}(\mathbb{C})$ | $\mathrm{GL}_{n}(\mathbb{R})$ | $\mathrm{U}_{n}(\mathbb{R})$ | $\operatorname{Sym}_{n}(\mathbb{R})$ |
| $\mathrm{Sp}_{2 n}(\mathbb{C})$ | $\operatorname{Stab}\left(L_{0}\right)$ | $\mathrm{GL}_{n}(\mathbb{C})$ | $\mathrm{U}_{n}(\mathbb{H})$ | $\mathrm{Sp}_{2 n}(\mathbb{R})$ | $\mathrm{U}_{n}(\mathbb{C})$ | $\operatorname{Sym}_{n}(\mathbb{C})$ |
| Orthogonal <br> G | $\mathbb{P}$ | $\mathbb{L}=\mathbb{G}^{\sigma}$ | $\mathbb{K}=\mathbb{G}^{\theta}=G$ | $\mathbb{G}^{\sigma \theta}=G_{\mathbb{R}}$ | $\mathbb{K}^{\sigma}=G_{\mathbb{R}}^{\theta}=K$ | $\mathfrak{p}_{0}$ |
| $\mathrm{O}_{2 n}(\mathbb{C})$ | $\operatorname{Stab}\left(L_{0}\right)$ | $\mathrm{GL}_{n}(\mathbb{C})$ | $\mathrm{O}_{2 n}(\mathbb{R})$ | $\mathrm{SO}^{*}\left(2 n, j \mathrm{id}_{n}\right)$ | $\mathrm{U}_{n}(\mathbb{C})$ | $\mathrm{Alt}_{n}(\mathbb{C})$ |
| Hermitian $\mathbb{G}$ | $\mathbb{P}$ | $\mathbb{L}=\mathbb{G}^{\sigma}$ | $\mathbb{K}=\mathbb{G}^{\theta}=G$ | $\mathbb{G}^{\sigma \theta}=G_{\mathbb{R}}$ | $\mathbb{K}^{\sigma}=G_{\mathbb{R}}^{\theta}=K$ | $\mathfrak{p}_{0}$ |
| $\mathrm{U}_{n, n}(\mathbb{R})$ | $\operatorname{Stab}\left(L_{0}\right)$ | $\mathrm{GL}_{n}(\mathbb{R})$ | $\mathrm{U}_{n}(\mathbb{R})^{2}$ | $\mathrm{O}_{n}(\mathbb{C})$ | $\Delta\left(\mathrm{U}_{n}(\mathbb{R})\right)$ | $\mathrm{SHer}_{n}(\mathbb{R})$ |
| $\mathrm{U}_{n, n}(\mathbb{C})$ | $\operatorname{Stab}\left(L_{0}\right)$ | $\mathrm{GL}_{n}(\mathbb{C})$ | $\mathrm{U}_{n}(\mathbb{C})^{2}$ | $\mathrm{GL}_{n}(\mathbb{C})$ | $\Delta\left(\mathrm{U}_{n}(\mathbb{C})\right)$ | $\mathrm{SHer}_{n}(\mathbb{C})$ |
| $\mathrm{U}_{n, n}(\mathbb{H})$ | $\operatorname{Stab}\left(L_{0}\right)$ | $\mathrm{GL}_{n}(\mathbb{H})$ | $\mathrm{U}_{n}(\mathbb{H})^{2}$ | $\mathrm{Sp}_{2 n}(\mathbb{C})$ |  | $\mathrm{SHer}_{n}(\mathbb{H})$ |
| Skew Hermitian |  |  |  |  |  |  |
| $\mathbb{G}$ | $\mathbb{P}$ | $\mathbb{L}=\mathbb{G}^{\sigma}$ | $\mathbb{K}=\mathbb{G}^{\theta}=G$ | $\mathbb{G}^{\sigma \theta}=G_{\mathbb{R}}$ | $\mathbb{K}^{\sigma}=G_{\mathbb{R}}^{\theta}=K$ | $p_{0}$ |
| $\mathrm{SO}^{*}(4 n)$ | $\operatorname{Stab}\left(L_{0}\right)$ | $\mathrm{GL}_{n}(\mathbb{H})$ | $\mathrm{U}_{2 n}(\mathbb{C})$ | $\mathrm{GL}_{n}(\mathbb{H})$ | $\mathrm{U}_{n}(\mathbb{H})$ | $\operatorname{Her}_{n}(\mathbb{H})$ |
| Quadric $\mathbb{G}$ | $\mathbb{P}$ | $\mathbb{L}=\mathbb{G}^{\sigma}$ | $\mathbb{K}=\mathbb{G}^{\theta}=G$ | $\mathbb{G}^{\sigma \theta}=G_{\mathbb{R}}$ | $\mathbb{K}^{\sigma}=G_{\mathbb{R}}^{\theta}=K$ | $\mathfrak{p}_{0}$ |
| $\begin{aligned} & \mathrm{SO}_{p+1, q+1}(\mathbb{R})_{e} \\ & \mathrm{SO}_{n+2}(\mathbb{C}) \end{aligned}$ | $P_{1}\left(Q_{n}(\mathbb{R})\right)$ $P_{1}\left(Q_{n}(\mathbb{C})\right)$ | $\mathrm{SO}_{1,1}(\mathbb{R}) \times \mathrm{SO}_{p, q}(\mathbb{R})$ $S\left(\mathrm{O}_{2}(\mathbb{C}) \times \mathrm{O}_{n}(\mathbb{C})\right)$ | $\mathrm{SO}_{p+1}(\mathbb{R}) \times \mathrm{SO}_{q+1}(\mathbb{R})$ $\mathrm{SO}_{n+2}(\mathbb{R})$ | $\mathrm{SO}_{1, p}(\mathbb{R}) \times \mathrm{SO}_{q, 1}(\mathbb{R})$ $\mathrm{SO}_{2, n}(\mathbb{R})$ | $S\left(\mathrm{O}_{p}(\mathbb{R}) \times \mathrm{O}_{q}(\mathbb{R})\right)$ $S\left(\mathrm{O}_{2}(\mathbb{R}) \times \mathrm{O}_{n}(\mathbb{R})\right)$ | $\begin{aligned} & \mathbb{R}^{p-1} \oplus \mathbb{R}^{q-1} \\ & \mathbb{C}^{n}=\operatorname{Mat}_{2, n}(\mathbb{R}) \end{aligned}$ |

Key: $\quad n=p+q$,
$\mathbb{P}$ : maximal parabolic subgroup with abelian nilpotent radical, $\mathbb{P}=\mathbb{L} \ltimes \mathbb{N}$ : Levi decomposition, $\mathbb{L}=\mathbb{G}^{\sigma}$ : Levi of $\mathbb{P}, \quad G=\mathbb{K}=\mathbb{G}^{\theta} \subset \mathbb{G}:$ maximal compact, $\quad G_{\mathbb{R}}=\mathbb{G}^{\sigma \theta}$ : noncompact real group, $K=G_{\mathbb{R}}^{\theta}=\mathbb{G}^{\sigma, \theta} \subset G_{\mathbb{R}}$ : maximal compact subgroup, $\operatorname{Lie}(\mathbb{N}) \simeq \mathfrak{p}_{0}=\operatorname{Lie}(G)^{-\sigma}$, $K=\mathbb{K} \cap \mathbb{P}\left(\right.$ except for $\mathbb{G}=\mathrm{SO}_{n+2}(\mathbb{C})$ when $K_{e}=\mathbb{K} \cap \mathbb{P}$ and $\left.\mathbb{G} / \mathbb{P} \simeq G / K_{e}\right)$, Compact symmetric space: $G / K=\mathbb{K} / K \simeq \mathbb{G} / \mathbb{P}$ Grassmannian,
$L_{0}$ : maximal Lagrangian subspace, $\quad P_{1}\left(Q_{n}(\mathbb{F})\right)$ : stabiliser of a point in quadric,
$\operatorname{Her}_{n}(\mathbb{F})$ : Hermitian matrices, $\operatorname{SHer}_{n}(\mathbb{F})$ : Skew Hermitian matrices,
$\mathrm{U}_{p, q}(\mathbb{F})=\mathrm{O}(p, q)$ if $\mathbb{F}=\mathbb{R}, \mathrm{U}(p, q)$ if $\mathbb{F}=\mathbb{C}, \operatorname{Sp}(p, q)$ if $\mathbb{F}=\mathbb{H}$.

## 2. Realization of certain Grassmannians as compact symmetric spaces

2.1. Some general facts. Let $\mathbb{G}$ be one of the groups in Table 1; note that the corresponding symmetric spaces $G / K$ and $G_{\mathbb{R}} / K$ exhaust the list of classical symmetric spaces given in Hel79, Ch.9, Sec.4]. Let $\mathbb{P}$ be the parabolic subgroup of $\mathbb{G}$ described in Table 1. Then $\mathbb{P}$ has a Levi decomposition $\mathbb{P}=\mathbb{L} \mathbb{N}$ specified in Table 1 . (As we shall see in the case by case analysis in the subsequent subsections, $\mathbb{P}$ consists of the block upper triangular matrices in $\mathbb{G}$ with two diagonal blocks, while $\mathbb{L}$ consists of the block diagonal matrices in $\mathbb{P}$.) Let $\mathfrak{P}=\mathfrak{L} \oplus \mathfrak{N}$ be the corresponding decomposition of the Lie algebra of $\mathbb{P}$. The opposite parabolic subalgebra is $\mathfrak{P}^{-}=\mathfrak{L} \oplus \mathfrak{N}^{-}=\mathfrak{L} \oplus \theta \mathfrak{N}$, where the differential of $\theta$ is still denoted by $\theta$.

The parabolic subgroup $\mathbb{P}$ contains a minimal parabolic subgroup $\mathbb{P}_{0}=\mathbb{M} \mathbb{N}_{0}$ corresponding to an Iwasawa decomposition $\mathbb{G}=\mathbb{K} \mathbb{A} \mathbb{N}_{0}$. The Levi subgroup $\mathbb{L}$ of $\mathbb{P}$ contains $\mathbb{M} \mathbb{A}$, while the unipotent radical $\mathbb{N}$ of $\mathbb{P}$ is contained in $\mathbb{N}_{0}$. Let $\mathfrak{G}, \mathfrak{A}, \mathfrak{M}$ be the Lie algebras of $\mathbb{G}, \mathbb{A}, \mathbb{M}$. Recall that $\mathfrak{P}$ can be constructed by taking a subset of simple $(\mathfrak{G}, \mathfrak{A})$ roots. Then one generates a root subsystem by these simple roots, which defines $\mathfrak{L}$ as the span of $\mathfrak{M} \oplus \mathfrak{A}$ and the root spaces for the roots in this subsystem, and defines $\mathfrak{N}$ to be the span of the root spaces for the remaining positive roots.

Lemma 2.1. (1) With the above notation, suppose that $\gamma, \delta$ are roots of $\mathfrak{N}$ such that $\gamma+\delta$ is a root (hence a root of $\mathfrak{N}$ ). Then $\left[\mathfrak{G}_{\gamma}, \mathfrak{G}_{\delta}\right] \neq 0$.
(2) Suppose $\mathfrak{N}$ is abelian. Then $\left[\mathfrak{N}, \mathfrak{N}^{-}\right]=[\mathfrak{N}, \theta \mathfrak{N}]$ is contained in $\mathfrak{L}$.

Proof. (1) Let $\delta-k \gamma, \ldots, \delta, \delta+\gamma, \ldots, \delta+n \gamma$ be the $\gamma$-string of roots through $\delta$, with $k \geq 0$ and $n \geq 1$. Let $e \in \mathfrak{G}_{\gamma}$ be nonzero. By [Kna96, Proposition 6.52] there is an $\mathfrak{s l}_{2}$-triple $e, h, f$ with $h \in \mathfrak{A}$ and $f \in \mathfrak{G}_{-\gamma}$. Now $\mathfrak{G}_{\delta-k \gamma} \oplus \cdots \oplus \mathfrak{G}_{\delta} \oplus \mathfrak{G}_{\delta+\gamma} \oplus \cdots \oplus \mathfrak{G}_{\delta+n \gamma}$ is a representation of the $\mathfrak{s l}_{2}$ spanned by $e, h, f$, and since $\mathfrak{G}_{\delta}$ and $\mathfrak{G}_{\delta+\gamma}$ are both nonzero, the action of $e$ between them can not be 0 . This implies (1).
(2) Assume that $[\mathfrak{N}, \theta \mathfrak{N}]$ is not contained in $\mathfrak{L}$. Then there are root vectors $x \in \mathfrak{G}_{\alpha}, y \in \mathfrak{G}_{\beta}$ in $\mathfrak{N}$ such that $[x, \theta y] \notin \mathfrak{L}$. Since $[x, \theta y] \in \mathfrak{G}_{\alpha-\beta}$, it follows that $\alpha-\beta$ is a root either of $\mathfrak{N}$ or of $\mathfrak{N}^{-}$. If $\alpha-\beta$ is a root of $\mathfrak{N}$, then since $\alpha=(\alpha-\beta)+\beta$ is a root, $(1)$ implies that $\left[\mathfrak{G}_{\alpha-\beta}, \mathfrak{G}_{\beta}\right] \neq 0$, so $[\mathfrak{N}, \mathfrak{N}] \neq 0$ and $\mathfrak{N}$ is not abelian. If $\alpha-\beta$ is a root of $\mathfrak{N}^{-}$, then $\beta-\alpha$ is a root of $\mathfrak{N}$, so $\beta=(\beta-\alpha)+\alpha$ again implies that $[\mathfrak{N}, \mathfrak{N}] \neq 0$ and so $\mathfrak{N}$ is not abelian.

The following theorem was proved in TK68, §4]. See also Kob08, Lemma 7.3.1] and [RRS92]. We present a short proof for the convenience of the reader.

Theorem 2.2. TK68. Let $\mathbb{G}, \mathbb{K}$ and $\mathbb{P}=\mathbb{L} \mathbb{N}$ be as above (i.e., as in Table 1). Then the following statements are equivalent:
(a) $\mathbb{N}$ (or equivalently $\mathfrak{N}$ ) is abelian;
(b) $\mathbb{L}$ is a symmetric subgroup of $\mathbb{G}$;
(c) $\mathbb{P} \cap \mathbb{K}=\mathbb{L} \cap \mathbb{K}$ is a symmetric subgroup of $\mathbb{K}$.

Proof. $(\mathbf{a}) \Rightarrow(\mathbf{b})$. It is enough to show that in the decomposition $\mathfrak{G}=\mathfrak{L} \oplus(\mathfrak{N} \oplus \theta \mathfrak{N})$ we have $[\mathfrak{N} \oplus \theta \mathfrak{N}, \mathfrak{N} \oplus \theta \mathfrak{N}] \subseteq \mathfrak{L}$. But Lemma [2.1(2) implies that

$$
[\mathfrak{N} \oplus \theta \mathfrak{N}, \mathfrak{N} \oplus \theta \mathfrak{N}]=[\mathfrak{N}, \mathfrak{N}]+[\mathfrak{N}, \theta \mathfrak{N}]+[\theta \mathfrak{N}, \mathfrak{N}]+[\theta \mathfrak{N}, \theta \mathfrak{N}]=0+[\mathfrak{N}, \theta \mathfrak{N}]+0 \subseteq \mathfrak{L}
$$

Note that the associated involution $\sigma$ is defined to be +1 on $\mathfrak{L}$ and $(-1)$ on $\mathfrak{N} \oplus \theta \mathfrak{N}$. In particular $\mathbb{P}$ is $\sigma$-stable.
$(\mathbf{b}) \Rightarrow(\mathbf{c})$. Let $\sigma$ be an involution of $\mathbb{G}$ such that $\mathbb{G}_{e}^{\sigma} \subseteq \mathbb{L} \subseteq \mathbb{G}^{\sigma}$, where $\mathbb{G}_{e}^{\sigma}$ denotes the connected component of $\mathbb{G}^{\sigma}$. Since $\mathbb{P}$ is standard, we may assume $\sigma$ commutes with $\theta$. Then the restriction
of $\sigma$ to $\mathbb{K}$ is an involution, and $\mathbb{K}_{e}^{\sigma} \subseteq \mathbb{L} \cap \mathbb{K} \subseteq \mathbb{K}^{\sigma}$. So $\mathbb{L} \cap \mathbb{K}$ is a symmetric subgroup of $\mathbb{K}$. (We remark that in all the examples we consider we will have $\mathbb{L}=\mathbb{G}^{\sigma}$ and $\mathbb{L} \cap \mathbb{K}=\mathbb{K}^{\sigma}$.)
$(\mathbf{c}) \Rightarrow(\mathbf{a})$. Suppose that $\mathfrak{N}$ is not abelian. Then there are roots $\alpha, \beta$ of $\mathfrak{N}$ and $x \in \mathfrak{G}_{\alpha}, y \in \mathfrak{G}_{\beta}$ such that $[x, y] \neq 0$. Then $x+\theta x, y+\theta y \in(\mathfrak{N} \oplus \theta \mathfrak{N})^{\theta}$, and we have

$$
[x+\theta x, y+\theta y]=[x, y]+[x, \theta y]+[\theta x, y]+[\theta x, \theta y]
$$

with

$$
[x, y] \in \mathfrak{G}_{\alpha+\beta},[x, \theta y] \in \mathfrak{G}_{\alpha-\beta},[\theta x, y] \in \mathfrak{G}_{-\alpha+\beta},[\theta x, \theta y] \in \mathfrak{G}_{-\alpha-\beta}
$$

Since the root $\alpha+\beta$ is strictly greater than $\alpha-\beta,-\alpha+\beta$ and $-\alpha-\beta$ (in the usual lexicographical order), we see that $[x, y] \in \mathfrak{N} \backslash 0$ implies $[x+\theta x, y+\theta y] \notin \mathfrak{L} \cap \mathfrak{K}$. It follows that $\mathbb{L} \cap \mathbb{K}$ is not a symmetric subgroup of $\mathbb{K}$.

Remark 2.3. Suppose that $[\mathfrak{G}, \mathfrak{G}]$ is simple and that $\mathfrak{P}=\mathfrak{L} \oplus \mathfrak{N}$ is a standard parabolic subalgebra as above. If $\mathfrak{N}$ is abelian, then $\mathfrak{P}$ is a maximal parabolic subalgebra. See RRS92, Lemma 2.2, p.651].

Proposition 2.4. Let $\mathbb{G}, \mathbb{K}$ and $\mathbb{P}=\mathbb{L} \mathbb{N}$ be as above (i.e., as in Table 1). Then $\mathbb{K}$ acts transitively on $\mathbb{G} / \mathbb{P}$, and therefore $\mathbb{G} / \mathbb{P}$ is diffeomorphic to $\mathbb{K} / \mathbb{P} \cap \mathbb{K}=\mathbb{K} / \mathbb{L} \cap \mathbb{K}$. In particular, $\mathbb{G} / \mathbb{P}$ is diffeomorphic to a symmetric space.

Proof. It is clear from the Iwasawa decomposition that $\mathbb{K}$ acts transitively on $\mathbb{G} / \mathbb{P}_{0}$, where $\mathbb{P}_{0}$ is a minimal parabolic subgroup of $\mathbb{G}$ contained in $\mathbb{P}$. Since $\mathbb{P} \supseteq \mathbb{P}_{0}$, there is a natural projection from $\mathbb{G} / \mathbb{P}_{0}$ to $\mathbb{G} / \mathbb{P}$, sending $g \mathbb{P}_{0}$ to $g \mathbb{P}$. This projection intertwines the $\mathbb{G}$-actions, hence also the $\mathbb{K}$-actions. It follows that $\mathbb{K}$ acts transitively on $\mathbb{G} / \mathbb{P}$. Indeed, if $g \mathbb{P} \in \mathbb{G} / \mathbb{P}$, let $k \in \mathbb{K}$ be such that $k \mathbb{P}_{0}=g \mathbb{P}_{0}$. Taking the projection we see that $k \mathbb{P}=g \mathbb{P}$, which implies transitivity of the $\mathbb{K}$-action.
2.2. Ordinary Grassmannians. Let $\mathbb{F}$ be $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Let $\operatorname{Gr}_{p}\left(\mathbb{F}^{p+q}\right)$ be the Grassmannian of $p$ dimensional subspaces of the vector space $\mathbb{F}^{p+q}$. The group $\mathbb{G}=\mathrm{GL}(p+q, \mathbb{F})$ clearly acts transitively on $\operatorname{Gr}_{p}\left(\mathbb{F}^{p+q}\right)$, so $\operatorname{Gr}_{p}\left(\mathbb{F}^{p+q}\right)=\mathbb{G} / \mathbb{P}$, where $\mathbb{P}$ is the stabilizer in $\mathbb{G}$ of the standard $p$-dimensional subspace

$$
\begin{equation*}
\mathbb{F}^{p}=\left\{\left(x_{1}, \ldots, x_{p}, 0, \ldots, 0\right) \mid x_{1}, \ldots, x_{p} \in \mathbb{F}\right\} \subseteq \mathbb{F}^{p+q} \tag{2.5}
\end{equation*}
$$

In other words,

$$
\mathbb{P}=\left\{\left.\left(\begin{array}{cc}
A & B \\
0 & C
\end{array}\right) \right\rvert\, A \in \mathrm{GL}(p, \mathbb{F}), C \in \mathrm{GL}(q, \mathbb{F}), B \in M_{p q}(\mathbb{F})\right\}
$$

Let $\sigma$ be the involution of $\mathbb{G}$ defined as in (1.3), i.e., $\sigma(g)=I_{p, q} g I_{p, q}$ where $I_{p, q}=\left(\begin{array}{cc}I_{p} & 0 \\ 0 & -I_{q}\end{array}\right)$. Then $\mathbb{G}^{\sigma}$ is equal to the Levi subgroup $\mathbb{L}$ of $\mathbb{P}$ and it consists of block diagonal matrices in $\mathbb{G}$, i.e.,

$$
\mathbb{G}^{\sigma}=\mathbb{L}=\operatorname{GL}(p, \mathbb{F}) \times \operatorname{GL}(q, \mathbb{F})
$$

The maximal compact subgroup $\mathbb{K}$ of $\mathbb{G}$ is the unitary group of $\mathbb{F}^{p+q}$ with respect to the standard inner product, denoted as $\mathrm{U}(p+q, \mathbb{F})$. In other words, $\mathbb{K}$ is $\mathrm{O}(p+q)$ if $\mathbb{F}=\mathbb{R}, \mathrm{U}(p+q)$ if $\mathbb{F}=\mathbb{C}$, and $\operatorname{Sp}(p+q)$ if $\mathbb{F}=\mathbb{H} . \mathbb{P} \cap \mathbb{K}=\mathbb{K}^{\sigma}$ is $\mathrm{U}(p, \mathbb{F}) \times \mathrm{U}(q, \mathbb{F})$, embedded block diagonally. In other words, $\mathbb{K}^{\sigma}$ is $\mathrm{O}(p) \times \mathrm{O}(q)$ if $\mathbb{F}=\mathbb{R}, \mathrm{U}(p) \times \mathrm{U}(q)$ if $\mathbb{F}=\mathbb{C}$, and $\mathrm{Sp}(p) \times \mathrm{Sp}(q)$ if $\mathbb{F}=\mathbb{H}$.

Now Proposition 2.4 implies the following well known result which can be found for example in Oni94, Ch. I, §4].

Proposition 2.6. $\operatorname{Gr}_{p}\left(\mathbb{F}^{p+q}\right)=\mathbb{G} / \mathbb{P}$ is diffeomorphic to $\mathrm{U}(p+q, \mathbb{F}) / \mathrm{U}(p, \mathbb{F}) \times \mathrm{U}(q, \mathbb{F})$. In other words, $\operatorname{Gr}_{p}\left(\mathbb{R}^{p+q}\right)$ is diffeomorphic to $\mathrm{O}(p+q) / \mathrm{O}(p) \times \mathrm{O}(q), \operatorname{Gr}_{p}\left(\mathbb{C}^{p+q}\right)$ is diffeomorphic to $\mathrm{U}(p+$ $q) / \mathrm{U}(p) \times \mathrm{U}(q)$, and $\operatorname{Gr}_{p}\left(\mathbb{H}^{p+q}\right)$ is diffeomorphic to $\operatorname{Sp}(p+q) / \operatorname{Sp}(p) \times \operatorname{Sp}(q)$

For cohomology computation we need $G=\mathbb{K}$ to be connected, and it is connected if $\mathbb{F}$ is $\mathbb{C}$ or $\mathbb{H}$. If $\mathbb{F}=\mathbb{R}$, we note that

$$
\mathbb{G} / \mathbb{P} \cong \mathrm{SO}(p+q) / S(\mathrm{O}(p) \times \mathrm{O}(q))
$$

This follows immediately from the fact that $\operatorname{SL}(p+q, \mathbb{R})$ acts transitively on $\operatorname{Gr}_{p}\left(\mathbb{R}^{p+q}\right)$. We can now conclude

Corollary 2.7. The cohomology ring (with complex coefficients) of the Grassmannian $\operatorname{Gr}_{p}\left(\mathbb{F}^{p+q}\right)$ is described by: Theorem 4.21 if $\mathbb{F}=\mathbb{R}$; Theorem 4.5 if $\mathbb{F}=\mathbb{C}$; Theorem 4.19 if $\mathbb{F}=\mathbb{H}$.

Since the involution $\sigma \theta$ of $\mathbb{G}$ is given by (1.4), i.e., by $\sigma \theta(g)=I_{p, q}\left(\bar{g}^{t}\right)^{-1} I_{p, q}$, the group $G_{\mathbb{R}}=\mathbb{G}^{\sigma \theta}$ is equal to $\mathrm{U}(p, q ; \mathbb{F})$. In other words, $G_{\mathbb{R}}$ is $\mathrm{O}(p, q)$ if $\mathbb{F}=\mathbb{R} ; \mathrm{U}(p, q)$ if $\mathbb{F}=\mathbb{C}$; and $\operatorname{Sp}(p, q)$ if $\mathbb{F}=\mathbb{H}$.
2.3. The symplectic Lagrangian Grassmannians. Let $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, and let $\operatorname{LGr}\left(\mathbb{R}^{2 n}\right)$ be the (symplectic) Lagrangian Grassmannian, i.e., the manifold of all Lagrangian subspaces of $\mathbb{F}^{2 n}$ with respect to the standard symplectic form $\langle$,$\rangle given by$

$$
\langle x, y\rangle=x^{t} J_{n} y, \quad \text { where } \quad J_{n}=\left(\begin{array}{cc}
0 & I_{n} \\
-I_{n} & 0
\end{array}\right) .
$$

Let $\mathbb{G}$ be the group $\operatorname{Sp}(2 n, \mathbb{F})$ of $2 n \times 2 n$ matrices over $\mathbb{F}$ preserving the form $\langle$, $\rangle$, i.e., satisfying $g^{t} J_{n} g=J_{n}$. Then $\mathbb{G}$ acts on $\operatorname{LGr}\left(\mathbb{F}^{2 n}\right)$, and this action is transitive by Witt's Theorem 1.1 Thus $\operatorname{LGr}\left(\mathbb{F}^{2 n}\right)=\mathbb{G} / \mathbb{P}$, where $\mathbb{P}$ is the (Siegel) parabolic subgroup of $\operatorname{Sp}(2 n, \mathbb{F})$, defined as the stabilizer of the standard Lagrangian subspace

$$
L_{0}=\left\{\left(x_{1}, \ldots, x_{n}, 0, \ldots, 0\right) \mid x_{1}, \ldots, x_{n} \in \mathbb{F}\right\} \subseteq \mathbb{F}^{2 n}
$$

Writing $g \in \mathbb{G}$ as a block matrix $(\underset{C}{A} \underset{D}{B})$ with $n \times n$ blocks, the condition $g^{t} J_{n} g=J_{n}$ implies

$$
\mathbb{G}=\operatorname{Sp}(2 n, \mathbb{F})=\left\{\left.\left(\begin{array}{ll}
A & B  \tag{2.8}\\
C & D
\end{array}\right) \in \mathrm{GL}(2 n, \mathbb{F}) \right\rvert\, A^{t} C=C^{t} A, B^{t} D=D^{t} B, A^{t} D-C^{t} B=I_{n}\right\}
$$

It follows that

$$
\mathbb{P}=\left\{\left.\left(\begin{array}{cc}
A & B  \tag{2.9}\\
0 & D
\end{array}\right) \in \mathrm{GL}(2 n, \mathbb{F}) \right\rvert\, B^{t} D=D^{t} B, A^{t} D=I_{n}\right\}
$$

Let $\sigma$ be the involution of $\mathbb{G}$ given by (1.3), i.e., by $\sigma(g)=I_{n, n} g I_{n, n}$. Then $\mathbb{G}^{\sigma}$ is equal to the Levi subgroup $\mathbb{L}$ of $\mathbb{P}$ and it consists of block diagonal matrices in $\mathbb{G}$, i.e.,

$$
\mathbb{G}^{\sigma}=\mathbb{L}=\left\{\left.\left(\begin{array}{cc}
A & 0 \\
0 & \left(A^{t}\right)^{-1}
\end{array}\right) \right\rvert\, A \in \mathrm{GL}(n, \mathbb{F})\right\} \cong \mathrm{GL}(n, \mathbb{F}) \quad \text { via }\left(\begin{array}{cc}
A & 0 \\
0 & \left(A^{t}\right)^{-1}
\end{array}\right) \leftrightarrow A
$$

Since $\mathbb{K}$ consists of the fixed points of the Cartan involution $\theta(g)=\left(\bar{g}^{t}\right)^{-1}$, and since $g \in \mathbb{G}$ is equivalent to $\left(g^{t}\right)^{-1}=J_{n} g J_{n}^{-1}$, we see that $\theta(g)=g$ is equivalent to $J_{n} \bar{g}=g J_{n}$. This implies

$$
\mathbb{K}=\left\{\left.\left(\begin{array}{cc}
A & -\bar{C}  \tag{2.10}\\
C & \bar{A}
\end{array}\right) \right\rvert\, A^{t} C=C^{t} A, \bar{A}^{t} A+\bar{C}^{t} C=I_{n}\right\}
$$

If $\mathbb{F}=\mathbb{R}$ (so the bars can be omitted), this is exactly the standard description of $\mathrm{U}(n)$ inside $\mathrm{GL}(2 n, \mathbb{R})$; more precisely, $\mathbb{K} \cong \mathrm{U}(n)$ via $\left(\begin{array}{cc}A & -C \\ C & A\end{array}\right) \leftrightarrow A+i C$.

If $\mathbb{F}=\mathbb{C}$, then $(\sqrt{2.10)}$ is exactly the standard description of $\operatorname{Sp}(n)$ inside $\mathrm{GL}(2 n, \mathbb{C})$.

We now also see that

$$
\mathbb{P} \cap \mathbb{K}=\mathbb{K}^{\sigma}=\left\{\left.\left(\begin{array}{cc}
A & 0 \\
0 & \bar{A}
\end{array}\right) \right\rvert\, A^{t} A=I\right\} \cong \mathrm{U}(n, \mathbb{F}) \quad \text { via }\left(\begin{array}{cc}
A & 0 \\
0 & \bar{A}
\end{array}\right) \leftrightarrow A .
$$

In other words, if $\mathbb{F}=\mathbb{R}$, then $\mathbb{K}^{\sigma}=\mathrm{O}(n)$ via $\left(\begin{array}{cc}A & 0 \\ 0 & A\end{array}\right) \leftrightarrow A$, and if $\mathbb{F}=\mathbb{C}$, then $\mathbb{K}^{\sigma}=\mathrm{U}(n)$ via $\left(\begin{array}{cc}A & 0 \\ 0 & A\end{array}\right) \leftrightarrow A$.

Now Proposition 2.4 implies
Proposition 2.11. The real symplectic Lagrangian Grassmannian $\operatorname{LGr}\left(\mathbb{R}^{2 n}\right)$ is diffeomorphic to $\mathrm{U}(n) / \mathrm{O}(n)$, while the complex symplectic Lagrangian Grassmannian $\operatorname{LGr}\left(\mathbb{C}^{2 n}\right)$ is diffeomorphic to $\mathrm{Sp}(n) / \mathrm{U}(n)$.

Corollary 2.12. The cohomology ring (with complex coefficients) of the symplectic Lagrangian Grassmannian $\operatorname{LGr}\left(\mathbb{F}^{2 n}\right)$ is described by: Theorem 3.29 if $\mathbb{F}=\mathbb{R}$; Theorem 4.25 if $\mathbb{F}=\mathbb{C}$.

Finally, we describe the group $\mathbb{G}^{\sigma \theta}$. Let first $\mathbb{F}=\mathbb{R}$. Then since $\sigma \theta(g)=I_{n, n}\left(g^{t}\right)^{-1} I_{n, n}$, and since any $g \in \mathbb{G}$ satisfies $\left(g^{t}\right)^{-1}=J_{n} g J_{n}^{-1}, \sigma \theta(g)=g$ is equivalent to $g D_{n}=D_{n} g$ where $D_{n}=\left(\begin{array}{cc}0 & I_{n} \\ I_{n} & 0\end{array}\right)$. It follows that

$$
\mathbb{G}^{\sigma \theta}=\left\{\left.g=\left(\begin{array}{ll}
A & B \\
B & A
\end{array}\right) \right\rvert\, A^{t} B=B^{t} A, A^{t} A-B^{t} B=I_{n}\right\} .
$$

Conjugating $\left(\begin{array}{cc}A & B \\ B & A\end{array}\right)$ by $\left(\begin{array}{c}I_{n} \\ I_{n} \\ I_{n} \\ I_{n}\end{array}\right)$ we get the matrix $\left(\begin{array}{cc}A+B & 0 \\ 0 & A-B\end{array}\right)$, and the conditions $A^{t} B=$ $B^{t} A, A^{t} A-B^{t} B=I_{n}$ imply $(A+B)^{t}(A-B)=I_{n}$, so $A-B=\left((A+B)^{t}\right)^{-1}$. Conversely, starting from the matrix $\left(\begin{array}{cc}Z & 0 \\ 0 & \left(Z^{t}\right)^{-1}\end{array}\right)$ and setting $A=\frac{1}{2}\left(Z+\left(Z^{t}\right)^{-1}\right), B=\frac{1}{2}\left(Z-\left(Z^{t}\right)^{-1}\right)$, we get $A^{t} B=B^{t} A, A^{t} A-B^{t} B=I_{n}$. Thus

$$
\mathbb{G}^{\sigma \theta} \cong\left\{\left.\left(\begin{array}{cc}
Z & 0 \\
0 & \left(Z^{t}\right)^{-1}
\end{array}\right) \right\rvert\, Z \in \mathrm{GL}(n, \mathbb{R})\right\} \cong \mathrm{GL}(n, \mathbb{R}) \text { via }\left(\begin{array}{cc}
Z & 0 \\
0 & \left(Z^{t}\right)^{-1}
\end{array}\right) \leftrightarrow Z .
$$

Now let $\mathbb{F}=\mathbb{C}$. Since $\sigma \theta(g)=I_{n, n}\left(\bar{g}^{t}\right)^{-1} I_{n, n}$, and since any $g \in \mathbb{G}$ satisfies $\left(g^{t}\right)^{-1}=J_{n} g J_{n}^{-1}$, we see that $\sigma \theta(g)=D_{n} \bar{g} D_{n}$. We claim that $\mathbb{G}^{\sigma \theta} \cong \operatorname{Sp}(2 n, \mathbb{R})$. To see this, we note that

$$
C_{n}=\frac{1}{2}\left(\begin{array}{ll}
(1+i) I_{n} & (1-i) I_{n}  \tag{2.13}\\
(1-i) I_{n} & (1+i) I_{n}
\end{array}\right) \quad \text { implies } \quad C_{n}^{2}=D_{n} \text { and } C_{n}^{-1}=\bar{C}_{n} .
$$

This implies that

$$
g \in \mathbb{G}^{\sigma \theta} \quad \text { if and only if } \quad \overline{C_{n} g C_{n}^{-1}}=C_{n} g C_{n}^{-1},
$$

i.e., that $\mathbb{G}^{\sigma \theta}=C_{n}^{-1} \operatorname{Sp}(2 n, \mathbb{R}) C_{n} \cong \operatorname{Sp}(2 n, \mathbb{R})$.
2.4. Orthogonal Lagrangian Grassmannians. Let $\operatorname{OLGr}\left(\mathbb{C}^{2 n}\right)$ be the (complex) orthogonal Lagrangian Grassmannian. In other words, $\operatorname{OLGr}\left(\mathbb{C}^{2 n}\right)$ is the manifold of all Lagrangian subspaces of $\mathbb{C}^{2 n}$ with respect to the symmetric bilinear form $\langle$,$\rangle defined by$

$$
\langle x, y\rangle=\sum_{r=1}^{n} x_{r} y_{n+r}+\sum_{r=1}^{n} x_{n+r} y_{r}=x^{t} D_{n} y,
$$

where as before, $D_{n}=\left(\begin{array}{cc}0 & I_{n} \\ I_{n} & 0\end{array}\right)$.
Let $\mathbb{G}=\mathrm{O}(2 n, \mathbb{C})$ be the group of $2 n \times 2 n$ complex matrices preserving the form $\langle$, , , i.e., satisfying $g^{t} D_{n} g=D_{n}$. Writing $g=\left(\begin{array}{cc}A \\ C & B \\ D\end{array}\right)$ we see that

$$
\mathbb{G}=\mathrm{O}(2 n, \mathbb{C})=\left\{\left.\left(\begin{array}{ll}
A & B  \tag{2.14}\\
C & D
\end{array}\right) \in \mathrm{GL}(2 n, \mathbb{C}) \right\rvert\, C^{t} A=-A^{t} C, D^{t} B=-B^{t} D, A^{t} D+C^{t} B=I_{n}\right\} .
$$

The group $\mathbb{G}$ acts on $\operatorname{OLGr}\left(\mathbb{C}^{2 n}\right)$, and this action is transitive by Witt's Theorem 1.1 Thus $\operatorname{OLGr}\left(\mathbb{C}^{2 n}\right)=\mathbb{G} / \mathbb{P}$, where $\mathbb{P}$ is the parabolic subgroup of $\mathbb{G}$, defined as the stabilizer of the standard Lagrangian subspace

$$
L_{0}=\left\{\left(x_{1}, \ldots, x_{n}, 0, \ldots, 0\right) \mid x_{1}, \ldots, x_{n} \in \mathbb{C}\right\} \subset \mathbb{C}^{2 n}
$$

An element $g=\left(\begin{array}{ll}A & B \\ C & D\end{array}\right)$ of $\mathbb{G}$ stabilizes $L_{0}$ if and only if $C=0$, so

$$
\mathbb{P}=\left\{\left.\left(\begin{array}{cc}
A & B  \tag{2.15}\\
0 & D
\end{array}\right) \in \mathrm{GL}(2 n, \mathbb{C}) \right\rvert\, D^{t} B=-B^{t} D, A^{t} D=I_{n}\right\}
$$

Let $\sigma$ be the involution of $\mathbb{G}$ defined by (1.3), i.e., by $\sigma(g)=I_{n, n} g I_{n, n}$. Then $\mathbb{G}^{\sigma}$ is equal to the Levi subgroup $\mathbb{L}$ of $\mathbb{P}$ and it consists of block diagonal matrices in $\mathbb{G}$, i.e.,

$$
\mathbb{G}^{\sigma}=\mathbb{L}=\left\{\left.\left(\begin{array}{cc}
A & 0 \\
0 & \left(A^{t}\right)^{-1}
\end{array}\right) \right\rvert\, A \in \mathrm{GL}(n, \mathbb{C})\right\} \cong \mathrm{GL}(n, \mathbb{C}) \quad \text { via }\left(\begin{array}{cc}
A & 0 \\
0 & \left(A^{t}\right)^{-1}
\end{array}\right) \leftrightarrow A
$$

Since $\theta(g)=\left(\bar{g}^{t}\right)^{-1}$ and since $g \in \mathbb{G}$ is equivalent to $\left(g^{t}\right)^{-1}=D_{n} g D_{n}$, we see that $\theta(g)=g$ is equivalent to $D_{n} \bar{g} D_{n}=g$, or $D_{n} \bar{g}=g D_{n}$. Thus

$$
\mathbb{K}=\left\{\left.\left(\begin{array}{ll}
A & \bar{C}  \tag{2.16}\\
C & \bar{A}
\end{array}\right) \in \mathrm{GL}(2 n, \mathbb{C}) \right\rvert\, C^{t} A=-A^{t} C, \bar{A}^{t} A+\bar{C}^{t} C=I_{n}\right\}
$$

To identify this subgroup, we connect our $\mathbb{G}$ with the more usual group $\mathbb{G}^{\prime}=\mathrm{O}(2 n, \mathbb{C})^{\prime}$ given by $g^{t} g=I_{2 n}$. The maximal compact subgroup $\mathbb{K}^{\prime}$ of $\mathbb{G}^{\prime}$ is given by the condition $\left(\bar{g}^{t}\right)^{-1}=g$, or equivalently $\bar{g}=g$, so $\mathbb{K}^{\prime}=\mathrm{O}(2 n)$. Since $\mathbb{G}$ and $\mathbb{G}^{\prime}$ are isomorphic, $\mathbb{K} \cong \mathrm{O}(2 n)$. In fact, explicit isomorphisms $\mathbb{G} \cong \mathbb{G}^{\prime}$ and $\mathbb{K} \cong \mathbb{K}^{\prime}$ are given by conjugation by the matrix $C_{n}$ of (2.13).

We also see that

$$
\mathbb{P} \cap \mathbb{K}=\mathbb{K}^{\sigma}=\left\{\left.\left(\begin{array}{cc}
A & 0 \\
0 & \bar{A}
\end{array}\right) \right\rvert\, \bar{A}^{t} A=I_{n}\right\} \cong \mathrm{U}(n) \quad \text { via } \quad\left(\begin{array}{cc}
A & 0 \\
0 & \bar{A}
\end{array}\right) \leftrightarrow A
$$

Now Proposition 2.4 implies
Proposition 2.17. The orthogonal Lagrangian Grassmannian $\operatorname{OLGr}\left(\mathbb{C}^{2 n}\right)$ is diffeomorphic to $\mathrm{O}(2 n) / \mathrm{U}(n)$.

Since our computation of cohomology of a compact symmetric space $G / K$ requires $G$ to be connected, we replace $\mathrm{O}(2 n) / \mathrm{U}(n)$ by $\mathrm{SO}(2 n) / \mathrm{U}(n)$. The orbit of $\mathrm{SO}(2 n)$ on $\operatorname{OLGr}\left(\mathbb{C}^{2 n}\right)$ is one of the two components of $\operatorname{OLGr}\left(\mathbb{C}^{2 n}\right)$, which we denote by $\mathrm{OLGr}^{+}\left(\mathbb{C}^{2 n}\right)$, and still call it the orthogonal Lagrangian Grassmannian. The other component of $\operatorname{OLGr}\left(\mathbb{C}^{2 n}\right)$ is diffeomorphic to $\operatorname{OLGr}^{+}\left(\mathbb{C}^{2 n}\right)$ and thus has the same cohomology.
Corollary 2.18. The cohomology ring (with complex coefficients) of the orthogonal Lagrangian Grassmannian $\mathrm{OLGr}^{+}\left(\mathbb{C}^{2 n}\right) \cong \mathrm{SO}(2 n) / \mathrm{U}(n)$ is described by Theorem 4.28.

Finally, we describe the group $G_{\mathbb{R}}=\mathbb{G}^{\sigma \theta}$ in case $\mathbb{G}=\operatorname{SO}(2 n, \mathbb{C})$. Since $\sigma \theta(g)=I_{n, n}\left(\bar{g}^{t}\right)^{-1} I_{n, n}$, we see that $\mathbb{G}^{\sigma \theta}=\mathrm{O}(2 n, \mathbb{C}) \cap \mathrm{U}(n, n)=\mathrm{SO}(2 n, \mathbb{C}) \cap \mathrm{SU}(n, n)$, and this is exactly the description of $\mathrm{SO}^{*}(2 n)$ given in Kna96, 1.141].

Remark 2.19. One could define the group $\mathrm{O}^{*}(2 n)$ as $\mathrm{O}(2 n, \mathbb{C}) \cap U(n, n)$, or alternatively, as the group of automorphisms of $\mathbb{H}^{n}$ preserving a skew Hermitian form; see Section 2.6. Conceivably, an element of this group could have determinant equal to $\pm 1$. However, we prove in Section 2.6 that the maximal compact subgroup of this group is $\mathrm{U}(n)$, so it follows that the group is connected and the determinant must be 1 . In other words, $\mathrm{O}^{*}(2 n)=\mathrm{SO}^{*}(2 n)$.
2.5. The Hermitian Lagrangian Grassmannians. Let $\mathbb{F}$ be $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and let $\operatorname{HLGr}\left(\mathbb{F}^{2 n}\right)$ be the Hermitian Lagrangian Grassmannian. In other words, $\operatorname{HLGr}\left(\mathbb{F}^{2 n}\right)$ is the manifold of all Lagrangian subspaces of $\mathbb{F}^{2 n}$ with respect to the Hermitian form $\langle$,$\rangle of signature (n, n)$, defined by

$$
\langle x, y\rangle=\sum_{r=1}^{n} \bar{x}_{r} y_{n+r}+\sum_{r=1}^{n} \bar{x}_{n+r} y_{r}=\bar{x}^{t} D_{n} y
$$

where as before, $D_{n}=\left(\begin{array}{cc}0 & I_{n} \\ I_{n} & 0\end{array}\right)$.
Let $\mathbb{G}=\mathrm{U}(n, n ; \mathbb{F})$ be the group of $2 n \times 2 n$ matrices over $\mathbb{F}$ preserving the form $\langle$,$\rangle , i.e., satisfying$ $\bar{g}^{t} D_{n} g=D_{n}$. So if $\mathbb{F}=\mathbb{R}, \mathbb{G}=\mathrm{O}(n, n)$; if $\mathbb{F}=\mathbb{C}, \mathbb{G}=\mathrm{U}(n, n)$; and if $\mathbb{F}=\mathbb{H}, \mathbb{G}=\operatorname{Sp}(n, n)$.

Writing $g=\left(\begin{array}{ll}A & B \\ C & D\end{array}\right)$ with $n \times n$ blocks, we see that

$$
\mathbb{G}=\mathrm{U}(n, n ; \mathbb{F})=\left\{\left.\left(\begin{array}{ll}
A & B  \tag{2.20}\\
C & D
\end{array}\right) \in \mathrm{GL}(2 n, \mathbb{F}) \right\rvert\, \bar{C}^{t} A=-\bar{A}^{t} C, \bar{D}^{t} B=-\bar{B}^{t} D, \bar{A}^{t} D+\bar{C}^{t} B=I_{n}\right\}
$$

The group $\mathbb{G}$ acts on $\operatorname{HLGr}\left(\mathbb{F}^{2 n}\right)$, and this action is transitive by Witt's Theorem 1.1. Thus $\operatorname{HLGr}\left(\mathbb{F}^{2 n}\right)=\mathbb{G} / \mathbb{P}$, where $\mathbb{P}$ is the parabolic subgroup of $\mathbb{G}$ defined as the stabilizer of the standard Lagrangian subspace

$$
L_{0}=\left\{\left(x_{1}, \ldots, x_{n}, 0, \ldots, 0\right) \mid x_{1}, \ldots, x_{n} \in \mathbb{F}\right\} \subset \mathbb{F}^{2 n}
$$

It follows that

$$
\mathbb{P}=\left\{\left.\left(\begin{array}{cc}
A & B  \tag{2.21}\\
0 & D
\end{array}\right) \in \mathrm{GL}(2 n, \mathbb{F}) \right\rvert\, \bar{D}^{t} B=-\bar{B}^{t} D, \bar{A}^{t} D=I_{n}\right\}
$$

Let $\sigma$ be an involution of $\mathbb{G}$ defined by (1.3), i.e., by $\sigma(g)=I_{n, n} g I_{n, n}$. Then $\mathbb{G}^{\sigma}$ is equal to the Levi subgroup $\mathbb{L}$ of $\mathbb{P}$ and it consists of block diagonal matrices in $\mathbb{G}$, i.e.,

$$
\mathbb{G}^{\sigma}=\mathbb{L}=\left\{\left.\left(\begin{array}{cc}
A & 0 \\
0 & \left(\bar{A}^{t}\right)^{-1}
\end{array}\right) \right\rvert\, A \in \mathrm{GL}(n, \mathbb{F})\right\} \cong \mathrm{GL}(n, \mathbb{F}) \quad \text { via }\left(\begin{array}{cc}
A & 0 \\
0 & \left(\bar{A}^{t}\right)^{-1}
\end{array}\right) \leftrightarrow A
$$

Since the Cartan involution is $\theta(g)=\left(\bar{g}^{t}\right)^{-1}$ and since $g \in \mathbb{G}$ is equivalent to $\left(\bar{g}^{t}\right)^{-1}=D_{n} g D_{n}$, we see that $\theta(g)=g$ is equivalent to $D_{n} g D_{n}=g$, or $D_{n} g=g D_{n}$. It follows that

$$
\mathbb{K}=\left\{\left.\left(\begin{array}{ll}
A & B  \tag{2.22}\\
B & A
\end{array}\right) \in \mathrm{GL}(2 n, \mathbb{F}) \right\rvert\, \bar{B}^{t} A=-\bar{A}^{t} B, \bar{A}^{t} A+\bar{B}^{t} B=I_{n}\right\}
$$

To identify this subgroup, we conjugate the matrix $\left(\begin{array}{cc}A & B \\ B & A\end{array}\right)$ by $\left(\begin{array}{cc}I_{n} & I_{n} \\ I_{n} & -I_{n}\end{array}\right)$ and get the matrix $\left(\begin{array}{cc}A+B & 0 \\ 0 & A-B\end{array}\right)$. The conditions $\bar{B}^{t} A=-\bar{A}^{t} B, \bar{A}^{t} A+\bar{B}^{t} B=I_{n}$ imply $\overline{(A+B)}^{t}(A+B)=I_{n}$ and $\overline{(A-B)}^{t}(A-B)=I_{n}$, so $A+B$ and $A-B$ are in $\mathrm{U}(n, \mathbb{F})$ (i.e., in $\mathrm{O}(n)$ if $\mathbb{F}=\mathbb{R}$; in $\mathrm{U}(n)$ if $\mathbb{F}=\mathbb{C}$; and in $\operatorname{Sp}(n)$ if $\mathbb{F}=\mathbb{H})$. Conversely, starting from matrices $Z$ and $W$ in $\mathrm{U}(n, \mathbb{F})$, we can reconstruct $A$ and $B$ as $A=\frac{1}{2}(Z+W), B=\frac{1}{2}(Z-W)$, and the matrix $\left(\begin{array}{ll}A & B \\ B & A\end{array}\right)$ will satisfy the conditions $\bar{B}^{t} A=-\bar{A}^{t} B, \bar{A}^{t} A+\bar{B}^{t} B=I_{n}$. So we found an explicit isomorphism $\mathbb{K} \cong \mathrm{U}(n, \mathbb{F}) \times \mathrm{U}(n, \mathbb{F})$. Under this isomorphism, the subgroup $\mathbb{P} \cap \mathbb{K}=\mathbb{K}^{\sigma}$ corresponds to the diagonal $\Delta \mathrm{U}(n, \mathbb{F}) \subset \mathrm{U}(n, \mathbb{F}) \times \mathrm{U}(n, \mathbb{F})$.

Now Proposition 2.4 implies
Proposition 2.23. The Hermitian Lagrangian Grassmannian $\operatorname{HLGr}\left(\mathbb{F}^{2 n}\right)$ is diffeomorphic to $\mathrm{U}(n, \mathbb{F}) \times \mathrm{U}(n, \mathbb{F}) / \Delta \mathrm{U}(n, \mathbb{F})$. In other words, $\operatorname{HLGr}\left(\mathbb{F}^{2 n}\right)$ is diffeomorphic to $\mathrm{O}(n) \times \mathrm{O}(n) / \Delta \mathrm{O}(n)$ if $\mathbb{F}=\mathbb{R}$; to $\mathrm{U}(n) \times \mathrm{U}(n) / \Delta \mathrm{U}(n)$ if $\mathbb{F}=\mathbb{C}$; and to $\operatorname{Sp}(n) \times \operatorname{Sp}(n) / \Delta \operatorname{Sp}(n)$ if $\mathbb{F}=\mathbb{H}$.

To compute the cohomology of $G / K$ we need $G$ to be connected, and in case $\mathbb{F}=\mathbb{R}$ the group $G=\mathbb{K}=\mathrm{O}(n) \times \mathrm{O}(n)$ is not connected. Thus in the real case we replace $\mathrm{O}(n) \times \mathrm{O}(n) / \Delta \mathrm{O}(n)$ by $\mathrm{SO}(n) \times \mathrm{SO}(n) / \Delta \mathrm{SO}(n)$. This amounts to replacing $\operatorname{HLGr}\left(\mathbb{R}^{2 n}\right)$ by the orbit $\operatorname{HLGr}^{+}\left(\mathbb{R}^{2 n}\right)$ of $\mathrm{SO}(n) \times \mathrm{SO}(n)$ which is one of the two connected components of $\operatorname{HLGr}\left(\mathbb{R}^{2 n}\right)$.

Corollary 2.24. The cohomology rings (with complex coefficients) of the Hermitian Lagrangian Grassmannians $\operatorname{HLGr}^{+}\left(\mathbb{R}^{2 n}\right)$, $\operatorname{HLGr}\left(\mathbb{C}^{2 n}\right)$, and $\operatorname{HLGr}\left(\mathbb{H}^{2 n}\right)$, are described by Theorem 4.30.

Finally, we describe the group $G_{\mathbb{R}}=\mathbb{G}^{\sigma \theta}$. Since $\sigma \theta(g)=I_{n, n}\left(\bar{g}^{t}\right)^{-1} I_{n, n}$ and since any $g \in \mathbb{G}$ satisfies $\left(\bar{g}^{t}\right)^{-1}=D_{n} g D_{n}$, we see that

$$
\sigma \theta(g)=I_{n, n} D_{n} g D_{n} I_{n, n}=J_{n} g J_{n}^{-1}
$$

Thus $\sigma \theta(g)=g$ is equivalent to $J_{n} g=g J_{n}$, and it follows that

$$
\mathbb{G}^{\sigma \theta}=\left\{\left.\left(\begin{array}{cc}
A & -C  \tag{2.25}\\
C & A
\end{array}\right) \right\rvert\, \bar{C}^{t} A=-\bar{A}^{t} C, \bar{A}^{t} A-\bar{C}^{t} C=I_{n}\right\}
$$

If $\mathbb{F}=\mathbb{R}$, recall that $A+i C \mapsto\left(\begin{array}{cc}A & -C \\ C & A\end{array}\right)$ is the standard embedding of $\operatorname{GL}(n, \mathbb{C})$ into $\operatorname{GL}(2 n, \mathbb{R})$, and note that the conditions $C^{t} A=-A^{t} C, A^{t} A-C^{t} C=I_{n}$ correspond to $(A+i C)^{t}(A+i C)=I_{n}$. It follows that $\mathbb{G}^{\sigma \theta}=\mathrm{O}(n, \mathbb{C})$.

If $\mathbb{F}=\mathbb{C}$, we conjugate $\left(\begin{array}{cc}A & -C \\ C & A\end{array}\right)$ by $\left(\begin{array}{cc}1 & i \\ 1 & -i\end{array}\right)$ and get $\left(\begin{array}{cc}A+i C & 0 \\ 0 & A-i C\end{array}\right)$. The conditions $\bar{C}^{t} A=$ $-\bar{A}^{t} C, \bar{A}^{t} A-\bar{C}^{t} C=I_{n}$ imply $\left.\overline{(A+i C}\right)^{t}(A-i C)=I_{n}$. Conversely, given $\left(\begin{array}{cc}Z & \bar{Z}^{0} \\ 0 & \left(\bar{Z}^{t}\right)^{-1}\end{array}\right)$ we can reconstruct $A$ and $C$ as $A=\frac{1}{2}\left(Z+\left(\bar{Z}^{t}\right)^{-1}\right)$ and $C=\frac{1}{2 i}\left(Z-\left(\bar{Z}^{t}\right)^{-1}\right)$ and get $\left(\begin{array}{cc}A & -C \\ C & A\end{array}\right) \in \mathbb{G}^{\sigma}$. It follows that

$$
\mathbb{G}^{\sigma \theta} \cong \mathrm{GL}(n, \mathbb{C}), \quad \text { via }\left(\begin{array}{cc}
A & -C \\
C & A
\end{array}\right) \leftrightarrow\left(\begin{array}{cc}
A+i C & 0 \\
0 & A-i C
\end{array}\right) \leftrightarrow A+i C
$$

If $\mathbb{F}=\mathbb{H}$, we claim that $\mathbb{G}^{\sigma \theta}$ is isomorphic to $\operatorname{Sp}(2 n, \mathbb{C})$. To see this, we consider the map

$$
\left(\begin{array}{cc}
A & -C \\
C & A
\end{array}\right)=\left(\begin{array}{cc}
A_{1}+j A_{2} & -C_{1}-j C_{2} \\
C_{1}+j C_{2} & A_{1}+j A_{2}
\end{array}\right) \mapsto\left(\begin{array}{cc}
A_{1}+i C_{1} & -\bar{A}_{2}-i \bar{C}_{2} \\
A_{2}+i C_{2} & \bar{A}_{1}+i \bar{C}_{1}
\end{array}\right)
$$

It is a tedious but straightforward computation to check that the conditions (2.25) imply the conditions in (2.8), so our map sends $\mathbb{G}^{\sigma \theta}$ into $\operatorname{Sp}(2 n, \mathbb{C})$. Conversely, if $(\underset{Z}{X} \underset{T}{Y}) \in \operatorname{Sp}(2 n, \mathbb{C})$, then we can reconstruct an element of $\mathbb{G}^{\sigma \theta}$ mapping to $\binom{X}{Z}$; it is given by

$$
A_{1}=\frac{1}{2}(X+\bar{T}), \quad A_{2}=\frac{1}{2}(Z-\bar{Y}), \quad C_{1}=\frac{1}{2 i}(X-\bar{T}), \quad C_{2}=\frac{1}{2 i}(Z+\bar{Y})
$$

(Another tedious computation shows that this element does satisfy the conditions of (2.25).)
2.6. The skew Hermitian quaternionic Lagrangian Grassmannians. Consider the skew Hermitian form on $\mathbb{H}^{2 n}$ given by

$$
\begin{equation*}
(x \mid y)=\bar{x}^{t} J_{n} y=\sum_{r=1}^{n} \bar{x}_{r} y_{n+r}-\sum_{r=1}^{n} \bar{x}_{n+r} y_{r} \tag{2.26}
\end{equation*}
$$

where as before, $J_{n}=\left(\begin{array}{cc}0 & I_{n} \\ -I_{n} & 0\end{array}\right)$. (Note that the form $(\mid)$ is different from, but equivalent to, the form considered in Ros02. Also, recall that $\mathbb{H}$ acts on $\mathbb{H}^{2 n}$ by right scalar multiplication, and that the form ( $\mid$ ) satisfies the condition $(x \alpha \mid y \beta)=\bar{\alpha}(x \mid y) \beta$ for $x, y \in \mathbb{H}^{2 n}$ and $\alpha, \beta \in \mathbb{H}$.)

The group $\mathbb{G}=\mathrm{SO}^{*}(4 n)$ is the group of automorphisms of $\mathbb{H}^{2 n}$ preserving the form (|), i.e.,

$$
\mathbb{G}=\left\{g \in \mathrm{GL}(2 n, \mathbb{H}) \mid \bar{g}^{t} J_{n} g=J_{n}\right\}
$$

The operations bar and transpose are defined by passing to the complex matrices: if $X$ is any $2 n \times 2 n$ quaternionic matrix, we write it as $X=U+j V$ with $U, V$ complex and identify $X$ with the $4 n \times 4 n$ complex matrix $\left(\begin{array}{cc}U & -\bar{V} \\ V & \bar{U}\end{array}\right)$. Then

$$
\bar{X}=\overline{\left(\begin{array}{cc}
U & -\bar{V} \\
V & \bar{U}
\end{array}\right)}=\bar{U}+j \bar{V} ; X^{t}=\left(\begin{array}{cc}
U & -\bar{V} \\
V & \bar{U}
\end{array}\right)^{t}=U^{t}-j \bar{V}^{t}
$$

Then one has $\overline{X Y}=\bar{X} \bar{Y}$ and $(X Y)^{t}=Y^{t} X^{t}$. The reader is cautioned that the operations bar and transpose cannot be performed directly on the quaternionic matrix in the usual way, but their composition can, since

$$
\overline{(U+j V)}^{t}=(\bar{U}+j \bar{V})^{t}=\bar{U}^{t}-j V^{t}=\bar{U}^{t}+\bar{V}^{t} \bar{j} .
$$

Upon writing $g \in \mathbb{G}$ as $\left(\begin{array}{cc}A & B \\ C & D\end{array}\right)$ with $n \times n$ (quaternionic) blocks and writing out the condition $\bar{g}^{t} J_{n} g=J_{n}$, we see

$$
\mathbb{G}=\mathrm{SO}^{*}(4 n)=\left\{\left.\left(\begin{array}{ll}
A & B  \tag{2.27}\\
C & D
\end{array}\right) \right\rvert\, \bar{A}^{t} C=\bar{C}^{t} A, \bar{B}^{t} D=\bar{D}^{t} B, \bar{A}^{t} D-\bar{C}^{t} B=I_{n}\right\}
$$

Clearly, $\mathbb{G}$ acts on the skew Hermitian quaternionic Lagrangian Grassmannian $\operatorname{LGr}^{*}\left(\mathbb{H}^{2 n}\right)$, the manifold of all Lagrangian subspaces of $\mathbb{H}^{2 n}$ with respect to $(\mid)$, and this action is transitive by Witt's Theorem 1.1. Thus $\operatorname{LGr}^{*}\left(\mathbb{H}^{2 n}\right)=\mathbb{G} / \mathbb{P}$, where $\mathbb{P}$ is the stabilizer in $\mathbb{G}$ of the standard Lagrangian subspace

$$
L_{0}=\left\{\left(x_{1}, \ldots, x_{n}, 0, \ldots, 0\right) \mid x_{1}, \ldots, x_{n} \in \mathbb{H}\right\} \subset \mathbb{H}^{2 n}
$$

It follows that

$$
\mathbb{P}=\left\{\left.\left(\begin{array}{ll}
A & B  \tag{2.28}\\
0 & D
\end{array}\right) \in \mathrm{GL}(2 n, \mathbb{H}) \right\rvert\, \bar{B}^{t} D=\bar{D}^{t} B, \bar{A}^{t} D=I_{n}\right\}
$$

Let $\sigma$ be an involution of $\mathbb{G}$ given by (1.3), i.e., by $\sigma(g)=I_{n, n} g I_{n, n}$. Then $\mathbb{G}^{\sigma}$ is equal to the Levi subgroup $\mathbb{L}$ of $\mathbb{P}$ and it consists of block diagonal matrices in $\mathbb{G}$, i.e.,

$$
\mathbb{G}^{\sigma}=\mathbb{L}=\left\{\left.\left(\begin{array}{cc}
A & 0 \\
0 & \left(\bar{A}^{t}\right)^{-1}
\end{array}\right) \right\rvert\, A \in \mathrm{GL}(n, \mathbb{H})\right\} \cong \mathrm{GL}(n, \mathbb{H}) \quad \text { via }\left(\begin{array}{cc}
A & 0 \\
0 & \left(\bar{A}^{t}\right)^{-1}
\end{array}\right) \leftrightarrow A .
$$

Since $\mathbb{K}$ consists of the fixed points of the Cartan involution $\theta(g)=\left(\bar{g}^{t}\right)^{-1}$, and since $g \in \mathbb{G}$ is equivalent to $\left(\bar{g}^{t}\right)^{-1}=J_{n} g J_{n}^{-1}$, we see that $g \in \mathbb{K}$ if and only if $g J_{n}=J_{n} g$. This implies

$$
\mathbb{K}=\left\{\left.\left(\begin{array}{cc}
A & -C  \tag{2.29}\\
C & A
\end{array}\right) \in \mathrm{GL}(2 n, \mathbb{H}) \right\rvert\, \bar{A}^{t} C=\bar{C}^{t} A, \bar{A}^{t} A+\bar{C}^{t} C=I_{n}\right\}
$$

We now also see that

$$
\mathbb{P} \cap \mathbb{K}=\mathbb{K}^{\sigma}=\left\{\left.\left(\begin{array}{cc}
A & 0 \\
0 & A
\end{array}\right) \in \operatorname{GL}(2 n, \mathbb{H}) \right\rvert\, \bar{A}^{t} A=I_{n}\right\} \cong \operatorname{Sp}(n) \quad \text { via }\left(\begin{array}{cc}
A & 0 \\
0 & A
\end{array}\right) \leftrightarrow A
$$

We claim that $\mathbb{K} \cong \mathrm{U}(2 n)$. To see this, we consider a different copy $\mathbb{G}^{\prime}$ of $\mathrm{SO}^{*}(4 n)$ inside $\mathrm{GL}(2 n, \mathbb{H})$, the one preserving the skew Hermitian form

$$
\langle x, y\rangle=\bar{x}^{t} i y=\sum_{r=1}^{2 n} \bar{x}_{r} i y_{r} .
$$

So $\mathbb{G}^{\prime}$ is the subgroup of $\mathrm{GL}(2 n, \mathbb{H})$ consisting of matrices $g$ such that $\bar{g}^{t} i g=i I_{2 n}$, and upon writing $g=U+j V$ with $U, V$ complex, we see

$$
\begin{equation*}
\mathbb{G}^{\prime}=\left\{U+j V \in \mathrm{GL}(2 n, \mathbb{H}) \mid \bar{U}^{t} U+\bar{V}^{t} V=I_{2 n}, U^{t} V=V^{t} U\right\} \tag{2.30}
\end{equation*}
$$

Since $g \in \mathbb{G}^{\prime}$ is equivalent to $\left(\bar{g}^{t}\right)^{-1}=-i g i, \theta g=g$ is equivalent to $i g=g i$. Writing $g=U+j V$ with $U, V$ complex, we see

$$
\begin{equation*}
\mathbb{K}^{\prime}=\left\{U+j 0 \in \mathrm{GL}(2 n, \mathbb{H}) \mid \bar{U}^{t} U=I_{2 n}\right\} \tag{2.31}
\end{equation*}
$$

So $\mathbb{K}^{\prime}$ is the usual $\mathrm{U}(2 n)$, embedded into $\mathrm{GL}(2 n, \mathbb{H})$ as $U(n)+j 0$.
To show an explicit connection between $\mathbb{G}$ and $\mathbb{G}^{\prime}$ and also between $\mathbb{K}$ and $\mathbb{K}^{\prime}$, we note that the matrix

$$
T_{n}=\frac{1}{\sqrt{2}}\left(\begin{array}{cc}
I_{n} & -i I_{n} \\
j I_{n} & -k I_{n}
\end{array}\right)
$$

satisfies $\bar{T}_{n}^{t} i T_{n}=J_{n}$. It follows that

$$
T_{n} \mathbb{G} T_{n}^{-1}=\mathbb{G}^{\prime}
$$

and since $\theta T_{n}=T_{n}$, also $T_{n} \mathbb{K} T_{n}^{-1}=\mathbb{K}^{\prime}=\mathrm{U}(2 n)$. Moreover, $T_{n}(\mathbb{P} \cap \mathbb{K}) T_{n}^{-1}$ is the standard $\operatorname{Sp}(n)$ inside $\mathrm{U}(2 n)$, embedded as matrices of the form $\left(\begin{array}{cc}U & -\bar{V} \\ V & \bar{U}\end{array}\right)$.

Now Proposition 2.4 implies the following result, which can also be found in CN06.
Proposition 2.32. The skew Hermitian quaternionic Lagrangian Grassmannian $\mathbb{G} / \mathbb{P}=$ $\mathrm{LGr}^{*}\left(\mathbb{H}^{2 n}\right)$ is diffeomorphic to $\mathrm{U}(2 n) / \operatorname{Sp}(n)$.

Corollary 2.33. The cohomology ring (with complex coefficients) of the skew Hermitian quaternionic Lagrangian Grassmannian $\operatorname{LGr}^{*}\left(\mathbb{H}^{2 n}\right) \cong \mathrm{U}(2 n) / \operatorname{Sp}(n)$ is described by Theorem 4.31.

Finally, we describe the group $G_{\mathbb{R}}=\mathbb{G}^{\sigma \theta}$. Since $\sigma \theta(g)=I_{n, n}\left(\bar{g}^{t}\right)^{-1} I_{n, n}$ and since $g \in \mathbb{G}$ is equivalent to $\left(\bar{g}^{t}\right)^{-1}=J_{n} g J_{n}^{-1}$,

$$
\sigma \theta(g)=I_{n, n} J_{n} g J_{n}^{-1} I_{n, n}=D_{n} g D_{n}, \quad g \in \mathbb{G}
$$

Thus $\sigma \theta(g)=g$ is equivalent to $g D_{n}=D_{n} g$. It follows that

$$
\mathbb{G}^{\sigma \theta}=\left\{\left.g=\left(\begin{array}{ll}
A & B \\
B & A
\end{array}\right) \right\rvert\, \bar{A}^{t} B=\bar{B}^{t} A, \bar{A}^{t} A-\bar{B}^{t} B=I_{n}\right\}
$$

Conjugating $\left(\begin{array}{ll}A & B \\ B & A\end{array}\right)$ by $\left(\begin{array}{cc}I_{n} & I_{n} \\ I_{n} & -I_{n}\end{array}\right)$ we get the matrix $\left(\begin{array}{cc}A+B & 0 \\ 0 & A-B\end{array}\right)$, and the conditions $\bar{A}^{t} B=$ $\bar{B}^{t} A, \bar{A}^{t} A-\bar{B}^{t} B=I_{n}$ imply $\overline{(A+B)^{t}}(A-B)=I_{n}$, so $A-B=\left(\overline{(A+B)^{t}}\right)^{-1}$. Conversely, starting from the matrix $\left(\begin{array}{cc}Z & 0 \\ 0 & \left(\bar{Z}^{t}\right)^{-1}\end{array}\right)$ and setting $A=\frac{1}{2}\left(Z+\left(\bar{Z}^{t}\right)^{-1}\right), B=\frac{1}{2}\left(Z-\left(\bar{Z}^{t}\right)^{-1}\right)$, we get $\bar{A}^{t} B=\bar{B}^{t} A, \bar{A}^{t} A-\bar{B}^{t} B=I_{n}$. Thus

$$
\mathbb{G}^{\sigma \theta} \cong\left\{\left.\left(\begin{array}{cc}
Z & 0 \\
0 & \left(\bar{Z}^{t}\right)^{-1}
\end{array}\right) \right\rvert\, Z \in \mathrm{GL}(n, \mathbb{H})\right\} \cong \mathrm{GL}(n, \mathbb{H}) \quad \text { via }\left(\begin{array}{cc}
Z & 0 \\
0 & \left(\bar{Z}^{t}\right)^{-1}
\end{array}\right) \leftrightarrow Z
$$

2.7. The quadric cases. In this subsection $\mathbb{F}$ is equal to $\mathbb{R}$ or $\mathbb{C}$. The following is taken from Tho16, Chapter 4.4]

Let $f$ be a nondegenerate symmetric bilinear form on $\mathbb{F}^{n+2}$. The quadric $Q_{f}(\mathbb{F})$ is defined to be the subset of the projective space $P^{n+1}(\mathbb{F})$ :

$$
Q_{f}(\mathbb{F})=\left\{x=\left(x_{1}: \ldots: x_{n+2}\right) \in P^{n+1}(\mathbb{F}) \mid f(x, x)=0\right\}
$$

If $\mathbb{F}=\mathbb{R}$ then $f$ has normal form with matrix $I_{p+1, q+1}$ (if the form is definite, $Q_{f}(\mathbb{R})$ does not contain projective lines) and we denote $Q_{f}(\mathbb{R})$ by $Q_{p, q}(\mathbb{R})$. If $\mathbb{F}=\mathbb{C}$ then all $f$ are equivalent and
we denote $Q_{f}(\mathbb{C})$ by $Q_{n}(\mathbb{C})$. The group $\mathbb{G}=\mathrm{SO}(p+1, q+1)_{e}$ acts transitively on the quadric $Q_{p, q}(\mathbb{R})$ with parabolic $\mathbb{P}=\operatorname{Stab}(1: 0: \ldots: 1: 0: \ldots: 0)$. The maximal compact subgroup of $\mathbb{G}$ is $G=\mathbb{K}=\mathrm{SO}(p+1) \times \mathrm{SO}(q+1)$ and $K=\mathbb{K} \cap \mathbb{P}=S(\mathrm{O}(p) \times \mathrm{O}(q))$. In this setting $Q_{p, q}(\mathbb{R})$ is equal to the symmetric space

$$
G / K=\mathrm{SO}(p+1) \times \mathrm{SO}(q+1) / S(\mathrm{O}(p) \times \mathrm{O}(q)), \quad p+q>2 .
$$

The symmetric space is not an irreducible symmetric space, it has a double cover by spheres $S^{p} \times S^{q}$, however it is indecomposable. When $\mathbb{F}=\mathbb{C}$ then $\mathrm{SO}_{n+2}(\mathbb{C})$ acts transitively on $Q_{n+2}(\mathbb{C})$ and the parabolic $\mathbb{P}=\operatorname{Stab}(1: i: \ldots: 0)$ has abelian unipotent radical, furthermore

$$
Q_{n}(\mathbb{C})=\mathrm{SO}_{n+2}(\mathbb{C}) / \operatorname{Stab}(1: i: \ldots: 0) \simeq \mathrm{SO}(n+2) / \mathrm{SO}(n) \times \mathrm{SO}(2) \quad n \geq 3
$$

As a compact symmetric space, $Q_{n}(\mathbb{C})$ coincides with $\mathrm{SO}(n+2) / \mathrm{SO}(n) \times \mathrm{SO}(2)$, which is equal to a double cover of the Grassmannian of 2 -planes in $\mathbb{R}^{n+2}$. We leave the calculation of the cohomology of double covers of Grassmannians and these quadrics to future work.

## 3. The structure of $C(\mathfrak{p})^{K}$ and $(\bigwedge \mathfrak{p})^{K}$

3.1. The decomposition of the spin module. Let $G / K$ be a compact symmetric space corresponding to an involution $\sigma$ of $G$ and let $\mathfrak{g}=\mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of the complexified Lie algebra $\mathfrak{g}$ of $G$ into eigenspaces of $\sigma$. In particular, $\mathfrak{k}$ is the complexified Lie algebra of $K$. We note that if $G_{\mathbb{R}} / K$ is the noncompact dual of $G / K$, then $\sigma$ corresponds to the Cartan involution of $G_{\mathbb{R}}$ ( $\sigma$ coincides with $\theta$ on $G_{\mathbb{R}}$ ).

We define the Clifford algebra $C(\mathfrak{p})$ using the nondegenerate invariant symmetric bilinear form $B$ on $\mathfrak{g}$ obtained by extending the Killing form over the center of $\mathfrak{g}$. Alternatively, for matrix groups from our table, we can replace $B$ by the trace form $\operatorname{tr} X Y$. Recall that $C(\mathfrak{p})$ is the associative unital algebra generated by $\mathfrak{p}$, with relations $X Y+Y X=2 B(X, Y), X, Y \in \mathfrak{p}$.

Let $S$ be a spin module for $C(\mathfrak{p})$. Recall that $S$ is constructed as follows. Let $\mathfrak{p}^{+}$and $\mathfrak{p}^{-}$be two maximal isotropic subspaces of $\mathfrak{p}$, dual under $B$. Let $S=\wedge \mathfrak{p}^{+}$, with elements of $\mathfrak{p}^{+} \subset C(\mathfrak{p})$ acting by wedging, and elements of $\mathfrak{p}^{-} \subset C(\mathfrak{p})$ acting by contracting. If $\operatorname{dim} \mathfrak{p}$ is even, $\mathfrak{p}=\mathfrak{p}^{+} \oplus \mathfrak{p}^{-}$ and hence this determines $S$ completely. Moreover, $S$ is the only irreducible $C(\mathfrak{p})$-module, and $C(\mathfrak{p})=\operatorname{End} S$. If $\operatorname{dim} \mathfrak{p}$ is odd, then $\mathfrak{p}=\mathfrak{p}^{+} \oplus \mathfrak{p}^{-} \oplus \mathbb{C} Z$ where $Z$ is an element of $\mathfrak{p}$ not contained in $\mathfrak{p}^{+} \oplus \mathfrak{p}^{-}$, such that $B(Z, Z)=1$. Now we can make $Z$ act on $\Lambda \mathfrak{p}^{+}$in two different ways; it can act by 1 on $\bigwedge^{\text {even }} \mathfrak{p}^{+}$and by -1 on $\bigwedge^{\text {odd }} \mathfrak{p}^{+}$, or by -1 on $\bigwedge^{\text {even }} \mathfrak{p}^{+}$and by 1 on $\bigwedge^{\text {odd }} \mathfrak{p}^{+}$. In this way we get two inequivalent $C(\mathfrak{p})$-modules $S_{1}$ and $S_{2}$. These are the only irreducible $C(\mathfrak{p})$-modules and $C(\mathfrak{p})=$ End $S_{1} \oplus$ End $S_{2}$. In the following, $S$ denotes either one of these two modules. For more details about Clifford algebras, spin modules, and also pin and spin groups, see HP06, Ch.2].

Since the pin group $\operatorname{Pin}(\mathfrak{p})$ is contained in $C(\mathfrak{p})$, the pin double cover $\widetilde{K}$ of $K$ acts on $S$. Recall that $\widetilde{K}$ is obtained from the following pullback diagram

where the map $K \rightarrow \mathrm{O}(\mathfrak{p})$ is given by the adjoint action of $K$ on $\mathfrak{p}$.
It now follows that

$$
C(\mathfrak{p})^{K}=C(\mathfrak{p})^{\widetilde{K}}= \begin{cases}\operatorname{End}_{\widetilde{K}} S, & \operatorname{dim} \mathfrak{p} \text { even }  \tag{3.1}\\ \operatorname{End}_{\widetilde{K}} S \operatorname{End}_{\widetilde{K}} S_{2}, & \operatorname{dim} \mathfrak{p} \text { odd. }\end{cases}
$$

Since the algebra $C(\mathfrak{p})^{K}$ and its graded version $(\bigwedge \mathfrak{p})^{K}$ are of our primary interest, we are led to study the $\widetilde{K}$-decomposition of $S$. We first study the decomposition of $S$ under the complexified Lie algebra $\mathfrak{k}$ of $\widetilde{K}$. This Lie algebra acts on $S$ through the map $\alpha: \mathfrak{k} \rightarrow C(\mathfrak{p})$, which is defined as the action map $\mathfrak{k} \rightarrow \mathfrak{s o}(\mathfrak{p})$ followed by the Chevalley map (i.e., the skew symmetrization) $\mathfrak{s o}(\mathfrak{p}) \cong$ $\bigwedge^{2} \mathfrak{p} \hookrightarrow C(\mathfrak{p})$. Explicitly, if $b_{i}$ is a basis of $\mathfrak{p}$ with dual basis $d_{i}$ with respect to the form $B$, then

$$
\begin{equation*}
\alpha(X)=\frac{1}{4} \sum_{i}\left[X, b_{i}\right] d_{i}, \quad X \in \mathfrak{k} \tag{3.2}
\end{equation*}
$$

(See HP06, §2.3.3]; the difference in sign comes from using different conventions to define the Clifford algebra.)

Let $\mathfrak{t}_{0}$ be a Cartan subalgebra of the (real) Lie algebra $\mathfrak{k}_{0}$ of $K$ and let $\mathfrak{t}=\left(\mathfrak{t}_{0}\right)_{\mathbb{C}}$. Let $\Delta^{+}(\mathfrak{g}, \mathfrak{t}) \supseteq$ $\Delta^{+}(\mathfrak{k}, \mathfrak{t})$ be compatible choices of positive roots for $(\mathfrak{g}, \mathfrak{t})$ respectively $(\mathfrak{k}, \mathfrak{t})$. Let $\rho$ respectively $\rho_{\mathfrak{k}}$ be the corresponding half sums of positive roots. Let $W_{\mathfrak{g}}$ respectively $W_{\mathfrak{k}}$ be the Weyl groups of $\Delta(\mathfrak{g}, \mathfrak{t})$ respectively $\Delta(\mathfrak{k}, \mathfrak{t})$.

Let $W_{\mathfrak{g}, \mathfrak{k}}^{1}$ be the set of minimal length representatives of $W_{\mathfrak{k}}$-cosets in $W_{\mathfrak{g}}$. Alternatively,

$$
W_{\mathfrak{g}, \mathfrak{k}}^{1}=\left\{\sigma \in W_{\mathfrak{g}} \mid \sigma \rho \text { is } \mathfrak{k} \text {-dominant }\right\} .
$$

It is well known ([Par72]; see also [HP06] ) that the decomposition of $S$ under the action of $\mathfrak{k}$ is given by

$$
\begin{equation*}
S=m \cdot \bigoplus_{\sigma \in W_{\mathfrak{g}, \mathfrak{k}}^{1}} E_{\sigma} \tag{3.3}
\end{equation*}
$$

where $E_{\sigma}$ denotes the irreducible finite-dimensional $\mathfrak{k}$-module with highest weight $\sigma \rho-\rho_{\mathfrak{k}}$, and the multiplicity $m$ is equal to $2^{\left[\frac{1}{2} \text { dima }\right]}$ where $\mathfrak{a}$ is the centralizer of $\mathfrak{t}$ in $\mathfrak{p}$ (so that $\mathfrak{h}=\mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ ).

Since $m$ is exactly the dimension of the spin module for the Clifford algebra $C(\mathfrak{a})$, (3.1) and (3.3) imply that

$$
\begin{equation*}
C(\mathfrak{p})^{\mathfrak{k}} \cong C(\mathfrak{a}) \otimes \operatorname{Pr}(S) \tag{3.4}
\end{equation*}
$$

where $\operatorname{Pr}(S)$ is the algebra spanned by the $\mathfrak{k}$-equivariant projections $\operatorname{pr}_{\sigma}: S \rightarrow m \cdot E_{\sigma}, \sigma \in W_{\mathfrak{g}, \mathfrak{k}}^{1}$.
If $K$ is connected, then the adjoint action map maps $K$ into $\mathrm{SO}(\mathfrak{p})$, so $\widetilde{K}$ is the spin double cover of $K$,


If the double covering map $\widetilde{K} \rightarrow K$ does not split, then $\widetilde{K}$ is connected and (3.3) gives a decomposition of $S$ with respect to $\widetilde{K}$. If the covering $\widetilde{K} \rightarrow K$ splits, then $\widetilde{K}=K \times \mathbb{Z}_{2}$, where the generator $z$ of $\mathbb{Z}_{2}$ maps to $1 \in K$ under the covering map. This implies that $z$ maps to the preimage in $\operatorname{Spin}(\mathfrak{p})$ of $1 \in \operatorname{SO}(\mathfrak{p})$, that is to $\pm 1 \in C(\mathfrak{p})$. Thus $z$ acts by the scalar 1 or -1 on $S$, in particular it preserves the decomposition (3.3), and hence this decomposition is also a decomposition of the $\widetilde{K}$-module $S$ into irreducibles. To conclude:
Proposition 3.5. If $K$ is connected, then the $\widetilde{K}$-decomposition of $S$ into irreducibles is the same as the $\mathfrak{k}$-decomposition (3.3).

In general, the $\mathfrak{k}$-decomposition is the same as the decomposition under $\widetilde{K}_{e}$, the connected component of the identity in $\widetilde{K}$, but the $\widetilde{K}$-action may combine several irreducible $\mathfrak{k}$-modules into one irreducible $\widetilde{K}$-module. More precisely, the component group $\widetilde{K} / \widetilde{K}_{e} \cong K / K_{e}$ acts by permuting the components $E_{\sigma}$ of $S$, and the $E_{\sigma}$ combining to produce an irreducible $\widetilde{K}$-module belong to the same orbit of $K / K_{e}$. (Recall that $K_{e}$ is a normal subgroup of $K$ and that $K / K_{e}$ is a finite group.) We will treat the case of disconnected $K$ in Subsections 3.7 and 3.8 below. Before that we describe the structure of $C(\mathfrak{p})^{\mathfrak{k}}$ more precisely.
3.2. Top degree element and Poincaré duality. We identify $C(\mathfrak{p})$ and $\wedge \mathfrak{p}$ using the Chevalley map, and think of them as one vector space with two different multiplications.

Proposition 3.6. Let $T$ be the unique (up to a scalar multiple) element of the top wedge of $\mathfrak{p}$; let $d=\operatorname{dimp}$ denote the degree of $T$.
(1) $T$ squares to a nonzero constant with respect to Clifford multiplication; consequently we can rescale $T$ and assume that $T^{2}=1$.
(2) If $d$ is odd, $T$ is in the center of $C(\mathfrak{p})$. If $d$ is even, $T$ commutes with $C(\mathfrak{p})_{\text {even }}$.
(3) $T$ is $\mathfrak{k}$-invariant (with respect to the adjoint action).
(4) Clifford multiplication by $T$ from the left is a linear isomorphism from $\bigwedge^{j} \mathfrak{p}$ to $\bigwedge^{d-j} \mathfrak{p}$, for any $j=0,1, \ldots, d$. This isomorphism, denoted by $*$, preserves the $\mathfrak{k}$-invariants and therefore gives an isomorphism from $\left(\bigwedge^{j} \mathfrak{p}\right)^{\mathfrak{k}}$ to $\left(\bigwedge^{d-j} \mathfrak{p}\right)^{\mathfrak{k}}$ for any $j$.
(5) The isomorphism * can up to sign be expressed as $x \mapsto \iota_{x} T$.
(6) For any $x, y \in \bigwedge^{j} \mathfrak{p}$,

$$
x \wedge * y=B(x, y) T
$$

In other words, $*$ is the usual Hodge star operator.
Proof. Let $Z_{1}, \ldots, Z_{d}$ be an orthonormal basis of $\mathfrak{p}$. Then, up to a nonzero scalar, $T=Z_{1} \cdots Z_{d}$. It follows that $T^{2}$ is a nonzero constant, since $Z_{1} \cdots Z_{d}$ squares to $\pm 1$. This proves (1). (2) is a straightforward computation: one checks that $Z_{1} \cdots Z_{d}$ commutes with all $Z_{j}$ if $d$ is odd, and with all $Z_{j} Z_{k}$ if $d$ is even.

To prove (3), we note that the adjoint action of $X \in \mathfrak{k}$ on $C(\mathfrak{p})$ is the same as the Clifford commutator with $\alpha(X)$. Since $\alpha$ maps $\mathfrak{k}$ into $C(\mathfrak{p})_{\text {even }}$, the claim follows from (2).

To prove (4), we note that if we set $Z_{I}=Z_{i_{1}} \ldots Z_{i_{a}}$ for $I=\left\{i_{1}, \ldots, i_{a}\right\} \subseteq\{1, \ldots, d\}$, then $T Z_{I}= \pm Z_{I^{c}}$, where $I^{c}=\{1, \ldots, d\} \backslash I$. This implies (4). (5) follows from the fact that Clifford multiplication by $y \in \mathfrak{p}$ equals $\iota_{y}+\varepsilon_{y}$ where $\varepsilon_{y}$ denotes wedging by $y$. Since $T$ is of top degree, it is annihilated by all $\varepsilon_{y}, y \in \mathfrak{p}$, and this implies the claim.

To prove (6), we note that both sides of the equation are bilinear in $x$ and $y$, so we can assume $x=Z_{I}, y=Z_{J}$ for some $I, J \subseteq\{1, \ldots, d\}$. If $I \neq J$, both sides of the equation are zero. Finally, if $I=J$, then we are to check that $Z_{I} \wedge * Z_{I}=T$, which is a straightforward computation.
Lemma 3.7. Let $x$ be any element of $C(\mathfrak{p})$ such that $x^{2}=1$ (with respect to Clifford multiplication). Then $B(x, x)=1$, where $B$ is the extended Killing form on $C(\mathfrak{p}) \cong \bigwedge \mathfrak{p}$. Consequently the elements 1 and $x$ span a subalgebra of $C(\mathfrak{p})$ isomorphic to the Clifford algebra on the one-dimensional space $\mathbb{C} x$.

In particular, if $T$ is the top element of $C(\mathfrak{p})$ as above, rescaled so that $T^{2}=1$, then $B(T, T)=1$ and $\operatorname{span}_{\mathbb{C}}(1, T)$ is a subalgebra of $C(\mathfrak{p})$ is isomorphic to the Clifford algebra $C(\mathbb{C} T)$.
Proof. This follows from the fact that the constant term of $x^{2}$ is $\iota_{x} x$, so

$$
1=B(1,1)=B\left(\iota_{x} x, 1\right)=B(x, x \wedge 1)=B(x, x)
$$

3.3. $C(\mathfrak{p})^{\mathfrak{k}}$ and $(\bigwedge \mathfrak{p})^{\mathfrak{k}}$ in the equal rank cases. The equal rank cases on our list are:

$$
\begin{array}{ll}
G / K=\mathrm{U}(p+q) / \mathrm{U}(p) \times \mathrm{U}(q) & \text { (Subsection 2.2); } \\
G / K=\mathrm{Sp}(p+q) / \mathrm{Sp}(p) \times \mathrm{Sp}(q) & \text { (Subsection 2.2); } \\
G / K=\mathrm{SO}(2 p+2 q) / \mathrm{S}(\mathrm{O}(2 p) \times \mathrm{O}(2 q)) & \text { (Subsection 2.2); } \\
G / K=\mathrm{SO}(2 p+2 q+1) / \mathrm{S}(\mathrm{O}(2 p) \times \mathrm{O}(2 q+1)) & \text { (Subsection 2.2); } \\
G / K=\mathrm{Sp}(n) / \mathrm{U}(n) & \text { (Subsection 2.3); } \\
G / K=\mathrm{SO}(2 n) / \mathrm{U}(n) & \text { (Subsection 2.4). }
\end{array}
$$

In each of these cases the situation is as in Kostant's email (see the introduction). In other words, the spin module $S$ is multiplicity free under $\mathfrak{k}$ and since $\operatorname{dim} \mathfrak{p}$ is even, $C(\mathfrak{p})=\operatorname{End} S$. Therefore Schur's lemma implies that $C(\mathfrak{p})^{\mathfrak{k}}=\operatorname{End}_{\mathfrak{k}} S=\operatorname{Pr}(S)$, the algebra spanned by the $\mathfrak{k}$-equivariant projections to the $\mathfrak{k}$-irreducible constituents of the spin module. The map $\alpha: \mathfrak{k} \rightarrow C(\mathfrak{p})$ from (3.2) extends to $U(\mathfrak{k})$ and its restriction to the center $Z(\mathfrak{k})$ of $U(\mathfrak{k})$ is the algebra homomorphism

$$
\alpha_{\mathfrak{k}}: Z(\mathfrak{k}) \rightarrow C(\mathfrak{p})^{\mathfrak{k}} .
$$

(The notation $\alpha_{\mathfrak{k}}$ is to distinguish this map from the analogous map $\alpha_{K}$ on the level of $K$-invariants; for connected $K$, there is no difference between these two maps.)

Since the $\mathfrak{k}$-infinitesimal character of $E_{\sigma}$ corresponds to $\sigma \rho$ under the Harish-Chandra isomorphism $Z(\mathfrak{k}) \cong \mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{\mathfrak{k}}}$, we can identify $\alpha_{\mathfrak{k}}$ with

$$
\begin{equation*}
\alpha_{\mathfrak{k}}: \mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{\mathfrak{e}}} \rightarrow C(\mathfrak{p})^{\mathfrak{k}}, \quad \alpha_{\mathfrak{k}}(P)=\sum_{\sigma \in W_{\mathfrak{g}}^{1}, \mathfrak{k}} P(\sigma \rho) \operatorname{pr}_{\sigma} \tag{3.8}
\end{equation*}
$$

where $\mathrm{pr}_{\sigma}$ denotes the $\mathfrak{k}$-equivariant projection $S \rightarrow E_{\sigma}$.
Proposition 3.9. The map $\alpha_{\mathfrak{k}}$ of (3.8) is a filtered algebra homomorphism, which doubles the degree. Here the filtration on the algebra $\mathbb{C}\left[t^{*}\right]^{W_{\mathrm{e}}}$ is induced by the grading, while the filtration on the algebra $C(\mathfrak{p})^{\mathfrak{k}}$ is inherited from $C(\mathfrak{p})$.

Proof. The claim follows from the fact that $\alpha_{\mathfrak{k}}$ is the restriction of $\alpha: U(\mathfrak{k}) \rightarrow C(\mathfrak{p})$ given by extending (3.2).

In the next subsection we consider a more general setting. We will in particular prove that the map (3.8) is onto; consequently, the map gr $\alpha_{\mathfrak{k}}: \mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{\mathfrak{e}}} \rightarrow(\bigwedge \mathfrak{p})^{\mathfrak{k}}$ is also onto. Moreover, we will give a description of $\operatorname{ker} \alpha$ and of $\operatorname{ker} \operatorname{gr} \alpha=\operatorname{gr} \operatorname{ker} \alpha$. It is clear from (3.8) that ker $\alpha_{\mathfrak{k}}$ consists of polynomials vanishing at all $\sigma \rho, \sigma \in W_{\mathfrak{g}, \mathfrak{e}}^{1}$. Thus ker $\alpha$ contains all $W_{\mathfrak{g}}$-invariant polynomials on $\mathfrak{t}^{*}$ that vanish at $\rho$ (and thus automatically on all $\sigma \rho$ ); we will prove that these polynomials in fact generate $\operatorname{ker} \alpha_{\mathfrak{k}}$. Likewise, we will see that $\operatorname{ker} \mathrm{gr} \alpha$ is generated by $W_{\mathfrak{g}}$-invariant polynomials on $\mathfrak{t}^{*}$ vanishing at 0 .
3.4. Relative coinvariant algebra and filtered deformations. Let $W$ be a finite group inside GL( $\mathfrak{t})$, with subgroup $H \subset W$. Let $\nu \in \mathfrak{t}^{*}$ be a point such that $\operatorname{Stab}_{W}(\nu)=\{\mathrm{id}\}$. Let $\mathbb{C}[W]$ denote the algebra of functions from $W$ to $\mathbb{C}$ with pointwise multiplication. Give $\mathbb{C}[W]$ basis $\left\{f_{w}: w \in W\right\}$, $f_{w}\left(w^{\prime}\right)=\delta_{w, w^{\prime}} 1$.

Definition 3.10. Define $\operatorname{Ev}_{\nu}: \mathbb{C}\left[t^{*}\right] \rightarrow \mathbb{C}[W]$ by

$$
\operatorname{Ev}_{\nu}(p)=\sum_{w \in W} p(w \nu) f_{w}
$$

Restricting $\operatorname{Ev}_{\nu}$ to $\mathbb{C}\left[\mathrm{t}^{*}\right]^{H}$ we define $\operatorname{Ev}_{\nu}^{H}: \mathbb{C}\left[\mathfrak{t}^{*}\right]^{H} \rightarrow \mathbb{C}[W]^{H}=\mathbb{C}[W / H]$.
Lemma 3.11. The map $\mathrm{Ev}_{\nu}$ is a surjective $W$-module and algebra homomorphism and $\mathrm{Ev}_{\nu}^{H}$ is a surjective algebra homomorphism.

Proof. Clearly $\mathrm{Ev}_{\nu}$ is a $W$ module homomorphism. Let $p_{\nu}$ be a linear polynomial that is zero on $\nu$ and non-zero on all $w \nu$, for all $w \neq 1$. The polynomial $\prod_{w \in W \backslash\{1\}} w p_{\nu}$ evaluated on $\nu$ is non-zero and is zero on every other element in the orbit of $\nu$. Suitably scaled, $\operatorname{Ev}_{\nu}\left(\prod_{w \in W \backslash\{1\}} w p_{\nu}\right)=f_{1}$. Since $f_{1}$ is a cyclic generator for the module $\mathbb{C}[W]$ and is in the image of $\mathrm{Ev}_{\nu}$ then the homomorphism is surjective. Since $f_{w} f_{w^{\prime}}=\delta_{w w^{\prime}} f_{w}$, then $\operatorname{Ev}_{\nu}(p) \operatorname{Ev}_{\nu}(q)=\sum_{w \in W} p(w \nu) f_{w} \sum_{w^{\prime} \in W} q\left(w^{\prime} \nu\right) f_{w^{\prime}}$ is equal to $\sum_{w \in W} p q(w \nu) f_{w}=\operatorname{Ev}_{\nu}(p q)$. Taking $H$ invariants on both sides proves that $\operatorname{Ev}_{\nu}^{H}$ is a surjective algebra homomorphism.

We define two ideals of $\mathbb{C}\left[\mathfrak{t}^{*}\right], I_{W,+}$ is the two sided graded ideal generated by

$$
\left\{p \in \mathbb{C}\left[\mathrm{t}^{*}\right]^{W}: \operatorname{deg} p>0\right\}=\left\{p \in \mathbb{C}\left[\mathfrak{t}^{*}\right]^{W}: p(0)=0\right\}
$$

and $I_{W, \nu}$ is the filtered ideal generated by $\left\{p \in \mathbb{C}\left[\mathrm{t}^{*}\right]^{W}: p(\nu)=0\right\}$.
Lemma 3.12. The ideal $I_{W,+}$ is equal to $\operatorname{gr} I_{W, \nu}$.
Proof. $I_{W,+}$ is generated by $\mathbb{C}\left[\mathfrak{t}^{*}\right]_{+}^{W}=\left\{p \in \mathbb{C}\left[\mathrm{t}^{*}\right]^{W}: p(0)=0\right\}$ and $I_{W, \nu}$ is generated by $\mathbb{C}\left[\mathfrak{t}^{*}\right]_{\nu}^{W}=$ $\left\{p \in \mathbb{C}\left[\mathfrak{t}^{*}\right]^{W}: p(\nu)=0\right\}$. Both of which are codimension 1 in $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{W}$. Furthermore, $\operatorname{gr}\left(\mathbb{C}\left[\mathfrak{t}^{*}\right]_{\nu}^{W}\right)$ is a codimension one graded ideal of $\mathbb{C}\left[t^{*}\right]^{W}$. Since $\mathbb{C}\left[\mathfrak{t}^{*}\right]_{+}^{W}$ is the only graded ideal of codimension one, then $\operatorname{gr} \mathbb{C}\left[\mathrm{t}^{*}\right]_{\nu}^{W}=\mathbb{C}\left[\mathrm{t}^{*}\right]_{+}^{W}$.

Lemma 3.13. The kernel of $\operatorname{Ev}_{\nu}$ is equal to $I_{W, \nu}$ and the $\operatorname{ker}^{\operatorname{Ev}}{ }_{\nu}^{H}=I_{W, \nu}^{H}$.
Proof. The kernel of $\mathrm{Ev}_{\nu}$ is precisely polynomials that evaluate to zero on the full $W$-orbit of $\nu$. The kernel $I_{W, \nu}$ is generated by $W$-invariant polynomials that evaluate to zero on $\nu$. Since they are $W$-invariant they also evaluate to zero on the full $W$-orbit. Hence ker $\operatorname{Ev}_{\nu} \supset I_{W, \nu}$. The quotient of $\mathbb{C}\left[\mathfrak{t}^{*}\right]$ by $I_{W,+}$ is the coinvariant algebra, which, in particular, is of dimension $|W|$. Lemma 3.12 then shows that the codimension of $I_{W, \nu}$ is $|W|$. Since $\mathrm{Ev}_{\nu}$ is surjective onto $\mathbb{C}[W]$ then ker $\mathrm{Ev}_{\nu}$ is also of codimension $|W|$ and hence $\operatorname{ker} \mathrm{Ev}_{\nu}=I_{W, \nu}$. The second statement follows by taking $H$ invariants of both sides.

The polynomials $\mathbb{C}\left[\mathfrak{t}^{*}\right]$ (resp. $\mathbb{C}\left[t^{*}\right]^{H}$ ) are graded, therefore the map $\operatorname{Ev}_{\nu}$ gives $\mathbb{C}[W]$ (resp. $\mathbb{C}[W / H])$ a filtration.

Theorem 3.14. Let $W \subset \mathrm{GL}(\mathfrak{t})$ be such that $\mathbb{C}\left[\mathfrak{t}^{*}\right]$ is a free $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{W}$ module of rank $|W|$ and let $H$ be any subgroup of $W$. With the filtration on $\mathbb{C}[W]$ endowed by $\mathrm{Ev}_{\nu}$, the associated graded algebra of $\mathbb{C}[W]$ is the coinvariant algebra $\mathbb{C}\left[\mathfrak{t}^{*}\right] /\left\langle\mathbb{C}\left[\mathrm{t}^{*}\right]_{+}^{W}\right\rangle$, similarly $\operatorname{gr}(\mathbb{C}[W / H]) \cong \mathbb{C}\left[\mathfrak{t}^{*}\right]^{H} /\left\langle\mathbb{C}\left[\mathrm{t}^{*}\right]_{+}^{W}\right\rangle_{\mathbb{C}\left[\mathrm{t}^{*}\right]^{H} .}$
Proof. Lemmas 3.12 and 3.13 prove that gr $\operatorname{ker} \operatorname{Ev}_{\nu}=I_{W,+}$ and $\operatorname{gr}\left(\operatorname{ker} \operatorname{Ev}_{\nu}^{H}\right)=I_{W,+}^{H}$. Since $\mathbb{C}[W]=$ $\mathbb{C}\left[t^{*}\right] / \operatorname{ker}\left(\operatorname{Ev}_{\nu}\right)$ then $\operatorname{gr} \mathbb{C}[W]=\mathbb{C}\left[t^{*}\right] / \operatorname{gr}\left(\operatorname{ker}\left(\operatorname{Ev}_{\nu}\right)\right)=\mathbb{C}\left[\mathrm{t}^{*}\right] / I_{W,+}$ which is the definition of the coinvariant algebra of $W$ acting on $\mathbb{C}\left[\epsilon^{*}\right]$. An identical statement holds for $\mathrm{Ev}_{\nu}{ }^{H}$.

Hence for any finite group $W$ acting by complex reflections on $\mathfrak{t}$ with any subgroup $H$ we can define a filtration on $\mathbb{C}[W / H]$ such that the associated graded algebra is isomorphic to the relative coinvariant algebra of $W$ and $H$.

Corollary 3.15. With notation of Subsections 3.1 and 3.3, the map $\alpha_{\mathfrak{k}}$ is surjective onto $C(\mathfrak{p})^{\mathfrak{k}}$ and the kernel of $\alpha_{\mathfrak{E}}$ is generated by the $W_{\mathfrak{g}}$ invariant polynomials in $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{\mathfrak{e}}}$ which evaluate to zero at $\rho,\left(I_{W, \rho}\right)^{W_{\mathfrak{k}}}$. Furthermore, the map gr $\alpha_{\mathfrak{k}}$ is surjective onto $(\bigwedge \mathfrak{p})^{\mathfrak{k}}$ and the kernel of gr $\alpha_{\mathfrak{k}}$ is $I_{W_{\mathfrak{g}},+}^{W_{\mathfrak{k}}}$.
Proof. The map $\alpha_{\mathfrak{k}}$ defined by (3.8) is given by evaluation of polynomials in $\mathbb{C}\left[\mathrm{t}^{*}\right]^{W_{\mathfrak{k}}}$ at $\sigma \rho$ for $\sigma \in W_{\mathfrak{g}, \mathfrak{k}}^{1}$, defining an isomorphism between $C(\mathfrak{p})^{\mathfrak{k}}=\operatorname{Pr}(S)$ and $\mathbb{C}\left[W_{G} / W_{K}\right]$
where $\operatorname{pr}_{\sigma}$ maps to $\sum_{w \in \sigma W_{\mathfrak{t}}} f_{w}$. So the map $\alpha_{\mathfrak{k}}$ is equal to the restriction of $\operatorname{Ev}_{\rho}: \mathbb{C}\left[\mathfrak{t}^{*}\right] \rightarrow \mathbb{C}\left[W_{\mathfrak{g}}\right]$ to $W_{\mathfrak{k}}$ invariants

$$
\operatorname{Ev}_{\rho}^{W_{\mathfrak{t}}}: \mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{\mathfrak{k}}} \rightarrow \mathbb{C}\left[W_{\mathfrak{g}} / W_{\mathfrak{k}}\right]=\mathbb{C}\left[W_{\mathfrak{g}, \mathfrak{k}}^{1}\right]
$$

Lemma 3.11 then states that $\alpha_{\mathfrak{k}}$ is surjective onto $C(\mathfrak{p})^{\mathfrak{k}}=\operatorname{Pr}(S)$ and Lemma 3.13 describes the kernel. Theorem 3.14 provides the statement for $\operatorname{gr} \alpha_{\mathfrak{k}}$.
3.5. $C(\mathfrak{p})^{\mathfrak{k}}$ and $(\bigwedge \mathfrak{p})^{\mathfrak{k}}$ in the almost equal rank case: $G / K=\mathrm{SO}(2 p+2 q+2) / S(\mathrm{O}(2 p+1) \times$ $\mathrm{O}(2 q+1))$. We call this case "almost equal rank", because $\operatorname{dim} \mathfrak{a}=1$ for all $p$ and $q$. To see that indeed $\operatorname{dim} \mathfrak{a}=1$, and also for later purposes, we first describe a Cartan subalgebra $\mathfrak{h}=\mathfrak{t} \oplus \mathfrak{a}$ of $\mathfrak{g}$.

For the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ we choose block diagonal matrices with diagonal blocks

$$
\begin{equation*}
t_{1} J, \ldots, t_{p} J, 0, t_{p+1} J, \ldots, t_{p+q} J, 0 \tag{3.16}
\end{equation*}
$$

where $J=J_{1}=\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$, and $t_{1}, \ldots, t_{p+q}$ are (complex) scalars. The centralizer $\mathfrak{a}$ of $\mathfrak{t}$ in $\mathfrak{p}$ is onedimensional, spanned by $E_{k k+m}-E_{k+m k}$. We identify $\mathfrak{t}$ with $\mathbb{R}^{p+q} \times 0 \subset \mathbb{R}^{p+q+1}$ by sending the matrix (3.16) to $\left(t_{1}, \ldots, t_{p+q}, 0\right)$, and we identify $\mathfrak{a}$ with $0 \times \mathbb{R} \subset \mathbb{R}^{p+q+1}$ by sending $E_{k k+m}-E_{k+m k}$ to $(0, \ldots, 0,1)$.

Since $\operatorname{dim} \mathfrak{p}$ is odd, $C(\mathfrak{p})=$ End $S_{1} \oplus \operatorname{End} S_{2}$, where $S_{1}$ and $S_{2}$ are the two spin modules. These spin modules are not isomorphic as $C(\mathfrak{p})$-modules, but they are isomorphic as modules over $C(\mathfrak{p})_{\text {even }}$, in particular they are isomorphic as $\mathfrak{k}$-modules. Moreover, the $\mathfrak{k}$-module $S=S_{1}=S_{2}$ is multiplicity free (since the multiplicity $m=2^{\left[\frac{1}{2} \text { dima }\right]}=1$ ).

To understand the decomposition $C(\mathfrak{p})=$ End $S_{1} \oplus$ End $S_{2}$ more explicitly, we first note that the top element $T$ of $C(\mathfrak{p})$, which is central in $C(\mathfrak{p})$ since $\operatorname{dim} \mathfrak{p}$ is odd, acts as 1 on $S_{1}$ and as -1 on $S_{2}$. Therefore the central idempotents

$$
\mathrm{pr}_{1}=\frac{1}{2}(1+T), \quad \mathrm{pr}_{2}=\frac{1}{2}(1-T)
$$

satisfy the following: $\mathrm{pr}_{1}$ is 1 on $S_{1}$ and 0 on $S_{2}$, while $\mathrm{pr}_{2}$ is 0 on $S_{1}$ and 1 on $S_{2}$. It follows that

$$
\text { End } S_{1}=C(\mathfrak{p}) \operatorname{pr}_{1} \quad \text { and } \quad \text { End } S_{2}=C(\mathfrak{p}) \operatorname{pr}_{2}
$$

By Proposition 3.6, multiplication by $T$ is an isomorphism between $C(\mathfrak{p})_{\text {even }}$ and $C(\mathfrak{p})_{\text {odd }}$, and moreover

$$
\begin{equation*}
C(\mathfrak{p}) \cong C(\mathbb{C} T) \otimes C(\mathfrak{p})_{\mathrm{even}} \tag{3.17}
\end{equation*}
$$

with the isomorphism implemented by the multiplication. It follows that

$$
\text { End } S_{1}=C(\mathfrak{p})_{\text {even }} \operatorname{pr}_{1} \quad \text { and } \quad \text { End } S_{2}=C(\mathfrak{p})_{\text {even }} \operatorname{pr}_{2}
$$

Namely, End $S_{i}$ corresponds to $\operatorname{pr}_{i} \otimes C(\mathfrak{p})_{\text {even }}$ under the decomposition (3.17).

Since the $\mathfrak{k}$-action on $C(\mathbb{C} T)$ is trivial, and since End $S_{1}=\operatorname{End} S_{2}=\operatorname{End} S$ as $\mathfrak{k}$-modules, we see that for any $c \in C(\mathbb{C} T), c \otimes C(\mathfrak{p})_{\text {even }}$ is a copy of End $S$. In particular, $1 \otimes C(\mathfrak{p})_{\text {even }}=C(\mathfrak{p})_{\text {even }}$ is isomorphic to End $S$, and in the following when we write End $S$ we mean this particular copy. It follows that $\operatorname{End}_{\mathfrak{k}} S=C(\mathfrak{p})_{\text {even }}^{\mathfrak{k}}$; this is also the image of the map $\alpha_{\mathfrak{k}}$ of (3.8), which now sends $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{\mathfrak{k}}}$ onto $\operatorname{Pr}(S)=\operatorname{End}_{\mathfrak{k}} S \subset C(\mathfrak{p})^{\mathfrak{k}}$. ( $\operatorname{End}_{\mathfrak{k}} S$ is equal to the algebra $\operatorname{Pr}(S)$ of projections onto isotypic components since $S$ is multiplicity free.) Furthermore, an analogue of Proposition 3.9 holds, with $C(\mathfrak{p})^{\mathfrak{k}}$ replaced by $\operatorname{Pr}(S)$. Finally, the above discussion shows that

$$
C(\mathfrak{p})^{\mathfrak{k}}=C(\mathbb{C} T) \otimes \operatorname{Pr}(S)
$$

We now go back to our Cartan subalgebra $\mathfrak{h}=\mathfrak{t} \oplus \mathfrak{a}$. Since $(\mathfrak{g}, \mathfrak{t})$-roots are the restrictions of $(\mathfrak{g}, \mathfrak{h})$-roots to $\mathfrak{t}$, we see that $\Delta(\mathfrak{g}, \mathfrak{t})$ is of type $B_{p+q}$, while $\Delta(\mathfrak{k}, \mathfrak{t})$ is of type $B_{p} \times B_{q}$. (On the other hand, $\Delta(\mathfrak{g}, \mathfrak{h})$ is of type $D_{p+q+1}$.)

Lemma 3.18. The filtered algebra $\operatorname{Pr}(S)$ is isomorphic to the filtered algebra $C\left(\mathfrak{p}^{\prime}\right)^{\mathfrak{k}^{\prime}}=\operatorname{Pr}\left(S^{\prime}\right)$ for the equal rank symmetric space

$$
G^{\prime} / K^{\prime}=\operatorname{Sp}(p+q) / \operatorname{Sp}(p) \times \operatorname{Sp}(q)
$$

This algebra is isomorphic to the algebra $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{\mathfrak{e}}^{\prime}}$ modulo the ideal generated by $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{\mathfrak{g}}^{\prime}}$. (We identify the isomorphic spaces $\mathfrak{t}$ and $\mathfrak{t}^{\prime}$.) It can be identified with the space $\mathbb{C}\left[r_{1}, \ldots, r_{p}\right]_{\leq q}$, spanned by monomials of degree at most $q$ in the elementary symmetric functions $r_{1}, \ldots, r_{p}$ of the squares of the variables $x_{1}, \ldots, x_{p}$, as in Subsection 4.3 below. The degrees of these monomials as functions of the $x_{i}$ range from 0 to $4 p q$ and are divisible by 4.
Proof. It will be shown in Subsection 4.3] that $C\left(\mathfrak{p}^{\prime}\right)^{\mathfrak{k}^{\prime}}$ for $G^{\prime} / K^{\prime}$ is $\mathbb{C}\left[r_{1}, \ldots, r_{p}\right]_{\leq q}$ as in the statement of the lemma. Since types $B$ and $C$ have the same Weyl group, $W_{\mathfrak{g}}$ and $W_{\mathfrak{k}}$ are the same for $G / K$ and $G^{\prime} / K^{\prime}$. Moreover, $\rho=\rho^{\prime}=(p+q, p+q-1, \ldots, 1) \in \mathfrak{t}^{*}$. Since the algebra $\operatorname{Pr}(S)$ is isomorphic to the algebra $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{\mathfrak{t}}}$ modulo the ideal generated by $\mathbb{C}\left[\mathfrak{t}^{*}\right]_{\rho}^{W_{\mathfrak{g}}}$, this implies the lemma.

The degree of the top element $T$ of $C(\mathfrak{p})$ is $d=\operatorname{dim} \mathfrak{p}=4 p q+2 p+2 q+1$. The algebra $\operatorname{Pr}(S)$ contains a unique element $t$ of degree $4 p q$. Let $e=T t$ be the corresponding odd element. Then $e$ is the unique element of lowest odd degree; this degree is

$$
\operatorname{deg} e=d-4 p q=2 p+2 q+1
$$

Lemma 3.19. The elements $t$ and e square to nonzero constants in $C(\mathfrak{p})$. Therefore, we can rescale these two elements and assume that $t^{2}=e^{2}=1$.
Proof. By Lemma 3.18, the filtered algebra $\operatorname{Pr}(S)$ is isomorphic to the filtered algebra $C\left(\mathfrak{p}^{\prime}\right)^{\mathfrak{k}^{\prime}}$ for $\left(\mathfrak{g}^{\prime}, \mathfrak{k}^{\prime}\right)=(\mathfrak{s p}(p, q), \mathfrak{s p}(p) \times \mathfrak{s p}(q))$. By Proposition 3.6, the top degree element of $C\left(\mathfrak{p}^{\prime}\right)^{\mathfrak{e}^{\prime}}$ squares to a nonzero constant. So $t^{2}$ is a nonzero constant.

Since $T^{2}=1$, it follows that $e=T t$ squares to the same constant.
Corollary 3.20. The element $e \in C(\mathfrak{p})^{\mathfrak{k}}$ satisfies $B(e, e)=1$ (here $B$ is the extended Killing form on $C(\mathfrak{p}) \cong \wedge \mathfrak{p})$. Consequently the elements 1 and e span a subalgebra of $C(\mathfrak{p})^{\mathfrak{k}}$ isomorphic to the Clifford algebra on the one-dimensional space $\mathbb{C}$. The same is true if we replace e by $t$.

Proof. This follows from Lemma 3.19 and Lemma 3.7
Theorem 3.21. There are tensor product decompositions

$$
\begin{aligned}
& C(\mathfrak{p})^{\mathfrak{k}} \cong C(\mathbb{C} e) \otimes \operatorname{Pr}(S) \\
& (\bigwedge \mathfrak{p})^{\mathfrak{k}} \cong \bigwedge \mathbb{C} e \otimes \operatorname{gr} \operatorname{Pr}(S)
\end{aligned}
$$

with the isomorphisms implemented by the multiplication.
Proof. To show that $C(\mathfrak{p})^{\mathfrak{k}} \cong C(\mathbb{C} e) \otimes \operatorname{Pr}(S)$, we first note that by (3.17), $\operatorname{Pr}(S)=C(\mathfrak{p})_{\text {even }}$ is a subalgebra of $C(\mathfrak{p})^{\mathfrak{k}}$ of half the dimension. Moreover, by Corollary 3.20, span $\{1, e\}$ is a subalgebra of $C(\mathfrak{p})^{\mathfrak{k}}$ isomorphic to the Clifford algebra $C(\mathbb{C e})$ of the space $\mathbb{C} e$. It is thus enough to show that (Clifford) multiplication by $e$ is injective on $\operatorname{Pr}(S)$. This follows immediately from $e^{2}=1$. Since $\operatorname{Pr}(S)$ commutes with $C(\mathfrak{p})^{\mathfrak{k}}$, this concludes the proof of $C(\mathfrak{p})^{\mathfrak{k}} \cong C(\mathbb{C} e) \otimes \operatorname{Pr}(S)$.

To prove $(\bigwedge \mathfrak{p})^{\mathfrak{k}} \cong \bigwedge \mathbb{C} e \otimes \operatorname{gr} \operatorname{Pr}(S)$, we again start from the fact that $\operatorname{gr} \operatorname{Pr}(S)$ is a subalgebra of $(\bigwedge \mathfrak{p})^{\mathfrak{k}}$ of half the dimension. Moreover, $e \wedge e=0$, since the degree of $e \wedge e$ is $4 p+4 q+2$, which is even but not divisible by 4 . It thus suffices to see that wedging by $e$ is injective on $\operatorname{gr} \operatorname{Pr}(S)$.

We first note that $e \wedge t=T$ up to (nonzero) scalar. Indeed, since the degrees match, it is enough to see that $e \wedge t \neq 0$. But

$$
B(e \wedge t, T)=-B\left(e, \iota_{t} T\right)=-B(e, e) \neq 0
$$

(We have already seen that $B(e, e)=1$. Alternatively, since $e$ is the only $\mathfrak{k}$-invariant in its degree, and since $B$ is nondegenerate on $\mathfrak{k}$-invariants, $B(e, e) \neq 0$.)

Assume now that $p \in \operatorname{gr} \operatorname{Pr}(S)$ is nonzero; we want to show that $e \wedge p \neq 0$. We can assume $p$ is homogeneous. We will be done if we can show that there is $p^{\prime} \in \operatorname{gr} \operatorname{Pr}(S)$ such that $p \wedge p^{\prime}=t$; then

$$
(e \wedge p) \wedge p^{\prime}=e \wedge t \neq 0
$$

so also $e \wedge p \neq 0$.
By Lemma 3.18, the algebra $\operatorname{Pr}(S)$ is isomorphic (as a filtered algebra) to the algebra $C\left(\mathfrak{p}^{\prime}\right)^{\mathfrak{k}^{\prime}}$, where $\left(\mathfrak{g}^{\prime}, \mathfrak{k}^{\prime}\right)=(\mathfrak{s p}(p+q), \mathfrak{s p}(p) \times \mathfrak{s p}(q))$. The corresponding graded algebras are thus also isomorphic. The claim now follows from Proposition 3.6. (6).
3.6. $(\bigwedge \mathfrak{p})^{K}$ in the primary case and almost primary case. In this section we cite results from GHV76] that describe the structure of $(\bigwedge \mathfrak{p})^{\mathfrak{k}}$, extend this description to $(\bigwedge \mathfrak{p})^{K}$ and explicitly give a generating subspace when $G / K$ is primary or almost primary.
Definition 3.22. Let $\lambda_{\mathfrak{k}}=\operatorname{gr} \alpha_{\mathfrak{k}}: U(\mathfrak{k})^{\mathfrak{k}} \rightarrow(\bigwedge \mathfrak{p})^{\mathfrak{k}}$ and let $\lambda_{K}$ be the restriction of $\lambda_{\mathfrak{k}}$ to $U(\mathfrak{k})^{K}$.
Theorem 3.23. GHV76, X. 4 Th VII] Let ( $\mathfrak{g}, \mathfrak{k}$ ) be a symmetric pair of Lie algebras, with Cartan involution $\theta$. Let $\mathcal{P}_{\mathfrak{g}}$ be a graded subspace that generates $\bigwedge(\mathfrak{g})^{\mathfrak{g}}$ and define the Samelson subspace $\mathcal{P}_{\mathfrak{a}}=\mathcal{P}_{\mathfrak{g}}^{-\theta}$ then there is an isomorphism of graded algebras

$$
(\bigwedge \mathfrak{p})^{\mathfrak{k}} \cong \bigwedge \mathcal{P}_{\mathfrak{a}} \otimes \operatorname{im} \lambda_{\mathfrak{k}}
$$

We extend the graded algebra description of $(\bigwedge \mathfrak{p})^{\mathfrak{k}}$ from GHV76] to $(\bigwedge \mathfrak{p})^{K}$ in the below proposition.

Proposition 3.24. Let $G / K$ be a symmetric space with $G$ connected ( $K$ may be disconnected). Then, with notation as in Theorem 3.23, there is an isomorphism of graded algebras

$$
(\bigwedge \mathfrak{p})^{K} \cong \bigwedge \mathcal{P}_{\mathfrak{a}} \otimes \operatorname{im} \lambda_{K}
$$

Proof. When one identifies the de Rham cohomology $H\left(G / K_{e}\right)$ of the space $G / K_{e}$ with $\wedge \mathfrak{p}^{\mathfrak{k}}$ and the de Rham cohomology $H(G)$ of $G$ with $\bigwedge(\mathfrak{g})^{\mathfrak{g}}$, then the map on cohomology from $H\left(G / K_{e}\right)$ to $H(G)$ is given by the inclusion of $(\bigwedge \mathfrak{p})^{\mathfrak{k}}$ into $\bigwedge \mathfrak{g}$ followed by the projection of $\bigwedge \mathfrak{g}$ onto $(\bigwedge \mathfrak{g})^{\mathfrak{g}}$ along $\mathfrak{g} \cdot \wedge \mathfrak{g}$ (see GHV76, X.4] for extra details). Both of these maps are $K$-module homomophisms and the image of the composition is $\bigwedge \mathcal{P}_{\mathfrak{a}} \subset \bigwedge \mathcal{P}_{\mathfrak{g}}$ [GHV76, X. 4 Th VII (2)]. Since $(\bigwedge \mathfrak{g})^{\mathfrak{g}}$ is $G$-fixed
(and hence $K$-fixed) and the above map is a $K$-module homomorphism, we can conclude that the subspace of $(\bigwedge \mathfrak{p})^{\mathfrak{k}}$ congruent to $\bigwedge \mathcal{P}_{\mathfrak{a}}$ is $K$-fixed. The space $\bigwedge(\mathfrak{p})^{K}$ is equal to

$$
\left(\bigwedge(\mathfrak{p})^{\mathfrak{k}}\right)^{K} \cong\left(\bigwedge\left(\mathcal{P}_{\mathfrak{a}}\right) \otimes \operatorname{im} \lambda_{\mathfrak{k}}\right)^{K}=\bigwedge\left(\mathcal{P}_{\mathfrak{a}}\right) \otimes\left(\operatorname{im} \lambda_{\mathfrak{k}}\right)^{K},
$$

the second equality following from the fact that the first tensorand is entirely $K$-fixed. To finish, note that $\left(\operatorname{im} \lambda_{\mathfrak{E}}\right)^{K}=\operatorname{im} \lambda_{K}$, hence

$$
(\bigwedge \mathfrak{p})^{K} \cong \bigwedge \mathcal{P}_{\mathfrak{a}} \otimes \operatorname{im} \lambda_{K}
$$

Definition 3.25. The symmetric space $G / K$ is primary if $W_{\mathfrak{g}, \mathfrak{t}}=W_{\mathfrak{k}}$ and almost primary if $W_{\mathfrak{g}, \mathfrak{t}}=W_{K} \neq W_{\mathfrak{e}}$.
Definition 3.26. Define $\mathcal{P}_{\wedge}(\mathfrak{p})$ to be the subspace of $(\bigwedge \mathfrak{p})^{K}$ orthogonal to square of the augmentation ideal $\left((\bigwedge \mathfrak{p})_{+}^{K}\right)^{2}$.
Proposition 3.27. Oni94, proposition 4 p 105 ] Suppose that the algebra $A$ is isomorphic to an exterior algebra, and let $S$ be a subset of $A$. Then the following are equivalent:
(a) the algebra $A$ is generated by $S$ and 1;
(b) the augmentation ideal $A_{+}$is equal to the $A$-submodule generated by $S$;
(c) the augmentation ideal $A_{+}$is equal to $\operatorname{span}_{\mathbb{C}}(S) \oplus A_{+}^{2}$.

In particular, Proposition 3.27 shows that modifying a generating set $S$ by any elements from $A_{+}^{2}$ retains the property of generating $A$; we will use this to show that $P_{\wedge}(\mathfrak{p})$ generates $(\bigwedge \mathfrak{p})^{K}$ in the (almost) primary case.
Corollary 3.28. Suppose that $A$ is a graded algebra with non-degenerate bilinear form such that $A$ is isomorphic to an exterior algebra, and different graded components are orthogonal to each other. Let $R$ be the subspace of $A_{+}$orthogonal to $A_{+}^{2}$ then $A$ is generated by $R$.
Proof. Since $A$ is isomorphic to an exterior algebra, suppose $P$ is any graded subspace of $A$ such that $A=\bigwedge P$. The proof follows by performing a Gram-Schmidt algorithm on $P$ modifying by elements in $A_{+}^{2}$ until the generating subspace is orthogonal to $A_{+}^{2}$. By Proposition 3.27, at each step of the Gram-Schmidt process the new subspace still generates $A$ and is graded since the different graded components are orthogonal. The end result is a graded subspace $P^{\prime}$ orthogonal to $A_{+}^{2}$ that generates $A$. Hence we have a direct sum of orthogonal components

$$
A=\mathbb{C} \oplus P^{\prime} \oplus A_{+}^{2}
$$

and $P^{\prime}$ is contained in $R$. Any element in $R \backslash P^{\prime}$ would be orthogonal to $P^{\prime}, \mathbb{C}$ and $A_{+}^{2}$ thus contradicting the fact that the form on $A$ is non-degenerate, hence $R=P^{\prime}$ and $R$ generates $A$.
Theorem 3.29. Let $G / K$ be primary or almost primary, then the inclusion $\mathcal{P}_{\wedge}(\mathfrak{p}) \hookrightarrow(\bigwedge \mathfrak{p})^{K}$ extends to an isomorphism of graded algebra

$$
(\bigwedge \mathfrak{p})^{K}=\bigwedge \mathcal{P}_{\wedge}(\mathfrak{p})
$$

Proof. If $G / K$ is primary then $\operatorname{im} \lambda_{\mathfrak{k}}$ is $\mathbb{C}$ and $(\bigwedge \mathfrak{p})^{\mathfrak{k}} \cong \bigwedge_{\mathfrak{a}}$, if $G / K$ is almost primary then im $\lambda_{K}$ is $\mathbb{C}$ and $(\bigwedge \mathfrak{p})^{K} \cong \bigwedge \mathcal{P}_{\mathfrak{a}}$. Hence, in both cases $(\bigwedge \mathfrak{p})^{K}$ is isomorphic to an exterior algebra, denote this isomorphism by $f: \bigwedge \mathcal{P}_{\mathfrak{a}} \cong(\bigwedge \mathfrak{p})^{K}$. Then $(\bigwedge \mathfrak{p})^{K}$ is generated by the graded subspace $f\left(\mathcal{P}_{\mathfrak{a}}\right)$ and the form on $(\bigwedge \mathfrak{p})^{K}$ induced by the Killing form on $\mathfrak{p}$ is non degenerate on $(\bigwedge \mathfrak{p})^{K}$ with differing graded components orthogonal. Hence Corollary 3.28 proves that $P_{\wedge}(\mathfrak{p})$ (Definition 3.26) generates $(\bigwedge \mathfrak{p})^{K} ;(\bigwedge \mathfrak{p})^{K}=\bigwedge \mathcal{P}_{\wedge}(\mathfrak{p})$.

The degrees of $\mathcal{P}_{\wedge}(\mathfrak{p})$ are the same as the degrees of $\mathcal{P}_{\mathfrak{a}}=\mathcal{P}_{\mathfrak{g}}^{-\theta}$ which are given in GHV76, Table I,II, III p. 492-496] and repeated below for reference. There is paper in preparation CGKP that will prove a transgression theorem when $G / K$ is primary or almost primary and will directly give the degrees of $\mathcal{P}_{\wedge}(\mathfrak{p})$.

TABLE 2. Degrees of $\mathcal{P}_{\wedge}(\mathfrak{p})$

| $G$ | $K$ | degrees of $\mathcal{P}_{\wedge}(\mathfrak{p})$ |
| :--- | :--- | :--- |
| Group |  |  |
| $\mathrm{U}_{2 n+1}(\mathbb{R})^{2}$ | $\mathrm{U}_{2 n+1}(\mathbb{R})$ | $4 p-1: 1 \leq p \leq n$ |
| $\mathrm{U}_{2 n}(\mathbb{R})^{2}$ | $\mathrm{U}_{2 n}(\mathbb{R})$ | $4 p-1: 1 \leq p \leq n-1$ and $2 n-1$ |
| $\mathrm{U}_{n}(\mathbb{C})^{2}$ | $\mathrm{U}_{n}(\mathbb{C})$ | $2 p-1: 1 \leq p \leq n$ |
| $\mathrm{U}_{n}(\mathbb{H})^{2}$ | $\mathrm{U}_{n}(\mathbb{H})$ | $4 p-1: 1 \leq p \leq n$ |
| Primary |  |  |
| $\mathrm{U}_{2 n+1}(\mathbb{C})$ | $\mathrm{U}_{2 n+1}(\mathbb{R})$ | $4 p-3: 1 \leq p \leq n+1$ |
| $\mathrm{U}_{2 n}(\mathbb{C})$ | $\mathrm{U}_{n}(\mathbb{H})$ | $4 p-3: 1 \leq p \leq n$ |
| Almost Primary |  |  |
| $\mathrm{U}_{2 n}(\mathbb{C})$ | $\mathrm{U}_{2 n}(\mathbb{R})$ | $4 p-3: 1 \leq p \leq n$ |
| $\quad\left(\right.$ Recall $\mathrm{U}_{n}(\mathbb{R})=$ | $\mathrm{O}(n), \mathrm{U}_{n}(\mathbb{C})=\mathrm{U}(n)$, and $\left.\mathrm{U}_{n}(\mathbb{H})=\operatorname{Sp}(n).\right)$ |  |

3.7. $C(\mathfrak{p})^{K}$ and $(\bigwedge \mathfrak{p})^{K}$ for disconnected $K$ : the case $G / K=\mathrm{SO}(k+m) / S(\mathrm{O}(k) \times \mathrm{O}(m))$. To understand what happens with the decomposition (3.3) when $K$ is disconnected, we consider the group theoretic Weyl group $W_{K}$ defined as

$$
\begin{equation*}
W_{K}=N_{K}(\mathfrak{t}) / Z_{K}(\mathfrak{t}) \tag{3.30}
\end{equation*}
$$

where $N_{K}(\mathfrak{t})$ denotes the normalizer in $K$ of the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$, while $Z_{K}(\mathfrak{t})$ denotes the centralizer of $\mathfrak{t}$ in $K$. It is well known (see Kna96, Theorem 4.54]) that for connected $K, W_{K}=W_{\mathfrak{k}}$, the Weyl group of the root system of $(\mathfrak{k}, \mathfrak{t})$. Analogously, we define $W_{G}$, which is equal to $W_{\mathfrak{g}}=W_{\mathfrak{g}, \mathfrak{t}}$ since $G$ is assumed connected.

We note that it looks like we should consider the groups $W_{\widetilde{K}}$ and $W_{\widetilde{K}_{e}}$, but these are in fact the same as $W_{K}$ respectively $W_{K_{e}}$. This follows from the fact that the adjoint action of $k \in \widetilde{K}$ is the same as the adjoint action of $\pi(k)$ where $\pi$ denotes the covering map from $\widetilde{K}$ to $K$.

Let now $G / K=\mathrm{SO}(k+m) / S(\mathrm{O}(k) \times \mathrm{O}(m))$. In these cases $K$ is disconnected and the group theoretic Weyl group $W_{K}$ may be different from $W_{\mathfrak{k}}$. In the following we describe $W_{K}$ explicitly.

The group $K=S(\mathrm{O}(k) \times \mathrm{O}(m))$ has two connected components, and we choose the following explicit representative $s$ of the disconnected component:

$$
\begin{align*}
& s=\operatorname{diag}(\underbrace{1, \ldots, 1,-1}_{k}, \underbrace{1, \ldots, 1,-1}_{m}) \quad \text { if } k \text { and } m \text { are both even; }  \tag{3.31}\\
& s=\operatorname{diag}(\underbrace{1, \ldots, 1,-1}_{k}, \underbrace{-1, \ldots,-1}_{m}) \quad \text { if } k \text { is even and } m \text { is odd; } \\
& s=\operatorname{diag}(\underbrace{-1, \ldots,-1}_{k}, \underbrace{-1, \ldots,-1}_{m}) \text { if } k \text { and } m \text { are both odd. }
\end{align*}
$$

For the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ we choose block diagonal matrices with diagonal blocks

$$
\begin{array}{cl}
t_{1} J, \ldots, t_{p} J, t_{p+1} J, \ldots, t_{p+q} J & \text { if }(k, m)=(2 p, 2 q) \\
t_{1} J, \ldots, t_{p} J, t_{p+1} J, \ldots, t_{p+q} J, 0 & \text { if }(k, m)=(2 p, 2 q+1) \\
t_{1} J, \ldots, t_{p} J, 0, t_{p+1} J, \ldots, t_{p+q} J, 0 & \text { if }(k, m)=(2 p+1,2 q+1) ;
\end{array}
$$

where $J=J_{1}=\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$, and $t_{1}, \ldots, t_{p+q}$ are (complex) scalars.
In the equal rank cases $(k, m)=(2 p, 2 q)$ and $(k, m)=(2 p, 2 q+1), \mathfrak{t}$ is also a Cartan subalgebra of $\mathfrak{g}$, while for $(k, m)=(2 p+1,2 q+1)$, as noted in Subsection 3.5, a Cartan subalgebra for $\mathfrak{g}$ is $\mathfrak{h}=\mathfrak{t} \oplus \mathfrak{a}$ where $\mathfrak{a}$ is one-dimensional, spanned by $E_{k k+m}-E_{k+m k}$. For $(k, m)=(2 p, 2 q)$ and $(k, m)=(2 p, 2 q+1)$, we identify $\mathfrak{t}$ with $\mathbb{R}^{p+q}$ by sending the above described matrix to $\left(t_{1}, \ldots, t_{p+q}\right)$. For $(k, m)=(2 p+1,2 q+1)$ we identify $\mathfrak{t}$ with $\mathbb{R}^{p+q} \times 0 \subset \mathbb{R}^{p+q+1}$ by sending the above described matrix to $\left(t_{1}, \ldots, t_{p+q}, 0\right)$, and we identify $\mathfrak{a}$ with $0 \times \mathbb{R} \subset \mathbb{R}^{p+q+1}$ by sending $E_{k k+m}-E_{k+m k}$ to $(0, \ldots, 0,1)$. In this last case we see that $\Delta(\mathfrak{g}, \mathfrak{t})$ is of type $B_{p+q}$, while $\Delta(\mathfrak{k}, \mathfrak{t})$ is of type $B_{p} \times B_{q}$. (On the other hand, $\Delta(\mathfrak{g}, \mathfrak{h})$ is of type $D_{p+q+1}$.)

Since

$$
\left(\begin{array}{cc}
1 & 0  \tag{3.32}\\
0 & -1
\end{array}\right)\left(\begin{array}{cc}
0 & t \\
-t & 0
\end{array}\right)\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)=\left(\begin{array}{cc}
0 & -t \\
t & 0
\end{array}\right)
$$

we see that in each of the cases $s$ given in (3.31) normalizes $t$ and acts on it as follows:
Lemma 3.33. For $(k, m)=(2 p, 2 q)$,

$$
s\left(t_{1}, \ldots, t_{p-1}, t_{p}, t_{p+1}, \ldots, t_{p+q-1}, t_{p+q}\right)=\left(t_{1}, \ldots, t_{p-1},-t_{p}, t_{p+1}, \ldots, t_{p+q-1},-t_{p+q}\right)
$$

For $(k, m)=(2 p, 2 q+1)$,

$$
s\left(t_{1}, \ldots, t_{p-1}, t_{p}, t_{p+1}, \ldots, t_{p+q}\right)=\left(t_{1}, \ldots, t_{p-1},-t_{p}, t_{p+1}, \ldots, t_{p+q}\right)
$$

For $(k, m)=(2 p+1,2 q+1)$, s centralizes $\mathfrak{t}$ (i.e., s acts trivially on $\mathfrak{t})$.
In particular, s normalizes $\mathfrak{t}$ in all cases.
Since in the cases $(k, m)=(2 p, 2 q)$ and $(k, m)=(2 p, 2 q+1)$ we have $K=K_{e} \rtimes\{1, s\}$, and we know by Lemma 3.33 that $s$ normalizes $\mathfrak{t}$, we conclude that

$$
W_{K}=W_{K_{e}} \rtimes\{1, s\}=W_{\mathfrak{k}} \rtimes\{1, s\}
$$

For $(k, m)=(2 p+1,2 q+1), s$ centralizes $\mathfrak{t}$ and thus $W_{K}=W_{\mathfrak{k}}$. To conclude:
Proposition 3.34. For $G / K=\mathrm{SO}(k+m) / S(\mathrm{O}(k) \times \mathrm{O}(m))$,

$$
W_{K}=\left\{\begin{array}{cc}
S\left(B_{p} \times B_{q}\right) & \text { if }(k, m)=(2 p, 2 q) ; \\
B_{p} \times B_{q} & \text { if }(k, m)=(2 p, 2 q+1) ; \\
B_{p} \times B_{q} & \text { if }(k, m)=(2 p+1,2 q+1) .
\end{array}\right.
$$

Here $B_{p} \times B_{q}$ is the group consisting of permutations and sign changes of the first $p$ and the last $q$ coordinates, while $S\left(B_{p} \times B_{q}\right)$ is the group consisting of permutations and sign changes of the first $p$ and the last $q$ coordinates, so that the total number of sign changes is even.

The group $W_{G}$ is equal to $D_{p+q}$ if $(k, m)=(2 p, 2 q)$, and to $B_{p+q}$ if $(k, m)=(2 p, 2 q+1)$ or if $(k, m)=(2 p+1,2 q+1)$.

In Lemma 3.33 we have described the adjoint action of $s$ (and hence also of $\tilde{s}$ ) on $\mathfrak{t}$. It follows from that and from passing to the dual $\mathfrak{t}^{*}$ that

Lemma 3.35. Let $s \in K$ be defined by (3.31), and let $\tilde{s}$ be a lift of $s$ in $\widetilde{K}$. Then the coadjoint action of $s$ and $\tilde{s}$ permutes the positive roots of $(\mathfrak{k}, \mathfrak{t})$.

Proof. As already noted, $\operatorname{Ad}(\tilde{s})=\operatorname{Ad}(s)$ so the coadjoint actions of $s$ and $\tilde{s}$ are also the same. To compute the action of $s$, we note that the formulas in Lemma 3.33 imply:
(1) For $G / K=S O(2 p+2 q) / S(O(2 p) \times O(2 q)), s$ interchanges the roots $\varepsilon_{i}-\varepsilon_{p}$ and $\varepsilon_{i}+\varepsilon_{p}$ $(1 \leq i<p)$, and the roots $\varepsilon_{p+j}-\varepsilon_{p+q}$ and $\varepsilon_{p+j}+\varepsilon_{p+q}(1 \leq j<q)$, while fixing all the other positive ( $\mathfrak{k}, \mathfrak{t}$ )-roots.
(2) For $G / K=S O(2 p+2 q+1) / S(O(2 p) \times O(2 q+1))$, $s$ interchanges the roots $\varepsilon_{i}-\varepsilon_{p}$ and $\varepsilon_{i}+\varepsilon_{p}(1 \leq i<p)$, while fixing all the other positive $(\mathfrak{k}, \mathfrak{t})$-roots.
(3) For $G / K=S O(2 p+2 q+2) / S(O(2 p+1) \times O(2 q+1)), s$ fixes all the positive $(\mathfrak{k}, \mathfrak{t})$-roots.

The formulas in Lemma 3.33 also describe the action of $s$ (and hence of $\tilde{s}$ ) on weights $\lambda \in \mathfrak{t}^{*}$. In coordinates, the action is exactly the same as on coordinates of elements of $\mathfrak{t}$ :
(1) For $G / K=S O(2 p+2 q) / S(O(2 p) \times O(2 q))$, $s$ acts by changing the sign of the $p$-th and the $(p+q)$-th coordinate;
(2) For $G / K=S O(2 p+2 q+1) / S(O(2 p) \times O(2 q+1))$, $s$ acts by changing the sign of the $p$-th coordinate;
(3) For $G / K=S O(2 p+2 q+2) / S(O(2 p+1) \times O(2 q+1))$, $s$ acts trivially.

We are now ready to describe the $\widetilde{K}$-decomposition of the spin module $S$ in each of the cases. Recall that the $\mathfrak{k}$-decomposition of $S$, or equivalently the $\widetilde{K}_{e}$-decomposition where $\widetilde{K}_{e}$ is the spin double cover of the connected component $K_{e}$ of the identity in $K$, is multiplicity free (since dim $\mathfrak{a} \leq$ 1), and given by
(1) For $G / K=S O(2 p+2 q) / S(O(2 p) \times O(2 q))$, the infinitesimal characters of the irreducible $\mathfrak{k}$-submodules of $S$ are the $\mathfrak{k}$-dominant $W_{G}$-conjugates of $\rho=(p+q-1, p+q-2, \ldots, 1,0)$. These are the $(p, q)$-shuffles of $\rho$, and the $(p, q)$-shuffles of $\rho$ with the sign of the $p$-th and the $(p+q)$-th coordinates changed to negative (note that one of these coordinates is zero, so the sign change does not affect it). The highest weights are obtained from these infinitesimal characters by subtracting $\rho_{\mathfrak{k}}$.
(2) For $G / K=S O(2 p+2 q+1) / S(O(2 p) \times O(2 q+1))$, the infinitesimal characters of the irreducible $\mathfrak{k}$-submodules of $S$ are the $\mathfrak{k}$-dominant $W_{G}$-conjugates of $\rho=\left(p+q-\frac{1}{2}, p+q-\frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2}\right)$. These are the $(p, q)$-shuffles of $\rho$, and the $(p, q)$-shuffles of $\rho$ with the sign of the $p$-th coordinate changed to negative. The highest weights are obtained from these infinitesimal characters by subtracting $\rho_{\mathfrak{k}}$.
(3) For $G / K=S O(2 p+2 q+2) / S(O(2 p+1) \times O(2 q+1))$, the infinitesimal characters of the irreducible $\mathfrak{k}$-submodules of $S$ are the $\mathfrak{k}$-dominant $W_{G}$-conjugates of $\rho=\rho_{(\mathfrak{g}, \mathfrak{t})}=\left.\rho_{(\mathfrak{g}, \mathfrak{h})}\right|_{\mathfrak{t}}=$ $(p+q, p+q-1, \ldots, 2,1)$. These are the $(p, q)$-shuffles of $\rho$. The highest weights are obtained from these infinitesimal characters by subtracting $\rho_{\mathfrak{k}}$.

It is now clear that in Cases (1) and (2) the action of $s$ (or $\tilde{s}$ ) interchanges the infinitesimal characters that differ only by the sign of one of the coordinates. Since in each of the cases $s$ permutes $\Delta^{+}(\mathfrak{k}, \mathfrak{t})$, it fixes $\rho_{\mathfrak{k}}$ and thus also interchanges the highest weights corresponding to the above infinitesimal characters.

In Case (3), $s$ (and $\tilde{s}$ ) fix all the $\mathfrak{k}$-infinitesimal characters and hence also all the highest weights in $S$.

Proposition 3.36. For $G / K=\mathrm{SO}(2 p+2 q) / S(\mathrm{O}(2 p) \times \mathrm{O}(2 q))$ or $G / K=\mathrm{SO}(2 p+2 q+1) / S(\mathrm{O}(2 p) \times$ $\mathrm{O}(2 q+1)$ ), each irreducible $\widetilde{K}$-module in $S$, when viewed as a $\mathfrak{k}$-module, decomposes into two
irreducible $\mathfrak{k}$-modules. The highest weights of these two modules are interchanged by $\tilde{s}$ and $s$, and differ from each other by one sign change.

For $G / K=\mathrm{SO}(2 p+2 q+2) / S(\mathrm{O}(2 p+1) \times \mathrm{O}(2 q+1))$, the $\widetilde{K}$-decomposition of the spin module $S$ is the same as the $\mathfrak{k}$-decomposition.

Proof. Let $v$ be any highest weight vector for $\mathfrak{k}$ in $S$, and let its weight be $\lambda$ (it is one of the weights described above). We claim that $\tilde{s} v$ is a highest weight vector of weight $s \lambda$.

To see this, we first note that $\widetilde{K}$-equivariance of the $\mathfrak{k}$-action on $S$ implies that for any $X \in \mathfrak{k}$,

$$
\begin{equation*}
X(\tilde{s} v)=\tilde{s}\left(\operatorname{Ad}(\tilde{s})^{-1} X\right) v=\tilde{s}\left(\operatorname{Ad}(s)^{-1} X\right) v=\tilde{s}(\operatorname{Ad}(s) X) v \tag{3.37}
\end{equation*}
$$

By Lemma 3.35, if $X$ is a positive root vector, then $\operatorname{Ad}(s) X$ is also a positive root vector. It follows that

$$
X(\tilde{s} v)=\tilde{s}(\operatorname{Ad}(s) X) v=0
$$

so $\tilde{s} v$ is a highest weight vector. To see the weight of $\tilde{s} v$, we apply (3.37) for $X \in \mathfrak{t}$. Recall from Lemma 3.33 that then also $\operatorname{Ad}(s) X \in \mathfrak{t}$, so we have

$$
X(\tilde{s} v)=\tilde{s}(\operatorname{Ad}(s) X) v=\lambda(\operatorname{Ad}(s) X) \tilde{s} v=s \lambda(X) \tilde{s} v
$$

so $\tilde{s} v$ is of weight $s \lambda$.
Let now $\widetilde{K}_{e}$ be the spin double cover of the identity component $K_{e}$ of $K$. Then by Proposition 3.5 the $\widetilde{K}_{e}$-decomposition of $S$ is the same as the $\mathfrak{k}$-decomposition. Since $\widetilde{K}_{e}$ and $\tilde{s}$ generate $\widetilde{K}$, and since we have described the action of $\tilde{s}$, the proposition follows.

We now describe a version of the Harish-Chandra isomorphism for the disconnected case. Let

$$
\gamma_{0}: Z(\mathfrak{k})=U(\mathfrak{k})^{\mathfrak{k}}=U(\mathfrak{k})^{K_{e}} \rightarrow S(\mathfrak{t})^{W_{\mathfrak{t}}}=S(\mathfrak{t})^{W_{K_{e}}}
$$

be the usual Harish-Chandra isomorphism.
Proposition 3.38. For $K=S(\mathrm{O}(k) \times \mathrm{O}(m))$, the Harish-Chandra map $\gamma_{0}: U(\mathfrak{k})^{K_{e}} \rightarrow S(\mathfrak{t})^{W_{K_{e}}}$ restricts to an isomorphism

$$
\gamma: U(\mathfrak{k})^{K} \rightarrow S(\mathfrak{t})^{W_{K}}
$$

Proof. We have seen that $K=K_{e} \rtimes\{1, s\}$, where $s$ is defined as above (the product is direct if $k, m$ are both odd). Since $s$ normalizes $K_{e}, \mathfrak{k}, \mathfrak{t}$ and $W_{K_{e}}$, it acts on $U(\mathfrak{k})^{K_{e}}$ and on $S(\mathfrak{t})^{W_{K_{e}}}$, and the map $\gamma_{0}$ intertwines these actions. The claim now follows by taking $s$-invariants.

Remark 3.39. It is possible to generalize the above proposition to the case when $K$ is an arbitrary compact Lie group.

We now define

$$
\alpha_{K}: U(\mathfrak{k})^{K} \cong \mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{K}} \rightarrow \operatorname{Pr}(S) \subseteq C(\mathfrak{p})^{K},
$$

as the restriction of the map $\alpha_{\mathfrak{k}}$ of (3.8) to the $K$-invariants. Here $\operatorname{Pr}(S)$ denotes the algebra of $\widetilde{K}$-equivariant projections of the $K$-module $S$ to its isotypic components $E_{\sigma}$, where $\sigma \in W_{G, K}^{1}$, the set of minimal length representatives of $W_{K}$-cosets in $W_{G}$. Since the $E_{\sigma}$ are of multiplicity 1 , $\operatorname{Pr}(S)=\operatorname{End}_{\tilde{K}} S$. The map $\alpha_{K}$ is given by the analogue of (3.8), i.e.,

$$
\alpha_{K}(P)=\sum_{\sigma \in W_{G, K}^{1}} P(\sigma \rho) \operatorname{pr}_{\sigma}, \quad P \in \mathbb{C}\left[\mathrm{t}^{*}\right]^{W_{K}}
$$

where $\mathrm{pr}_{\sigma}: S \rightarrow E_{\sigma}$ is the $K$-equivariant projection. From the above considerations we conclude

Corollary 3.40. (1) For $G / K=\mathrm{SO}(2 p+2 q) / S(\mathrm{O}(2 p) \times \mathrm{O}(2 q))$, the algebra $C(\mathfrak{p})^{K}$ is isomorphic to $\operatorname{Pr}(S)$, which is isomorphic to $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{S\left(B_{p} \times B_{q}\right)}$ modulo the ideal generated by $D_{p+q}$-invariants in $\mathbb{C}\left[\mathfrak{t}^{*}\right]$ evaluating to 0 at $\rho=(p+q-1, \ldots, 1,0)$. The algebra $(\bigwedge \mathfrak{p})^{K}$ is isomorphic to $\operatorname{gr} \operatorname{Pr}(S)$, which is isomorphic to $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{S\left(B_{p} \times B_{q}\right)}$ modulo the ideal generated by $D_{p+q}$-invariants in $\mathbb{C}\left[t^{*}\right]$ evaluating to 0 at 0 .
(2) For $G / K=\mathrm{SO}(2 p+2 q+1) / S(\mathrm{O}(2 p) \times \mathrm{O}(2 q+1))$, the algebra $C(\mathfrak{p})^{K}$ is isomorphic to $\operatorname{Pr}(S)$, which is isomorphic to $\mathbb{C}\left[\mathrm{t}^{*}\right]^{B_{p} \times B_{q}}$ modulo the ideal generated by $B_{p+q}$-invariants in $\mathbb{C}\left[\mathrm{t}^{*}\right]$ evaluating to 0 at $\rho=\left(p+q-\frac{1}{2}, \ldots, \frac{3}{2}, \frac{1}{2}\right)$. The algebra $(\bigwedge \mathfrak{p})^{K}$ is isomorphic to $\operatorname{gr} \operatorname{Pr}(S)$, which is isomorphic to $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{B_{p} \times B_{q}}$ modulo the ideal generated by $B_{p+q}$-invariants in $\mathbb{C}\left[\mathfrak{t}^{*}\right]$ evaluating to 0 at 0.
(3) For $G / K=S O(2 p+2 q+2) / S(\mathrm{O}(2 p+1) \times \mathrm{O}(2 q+1))$, the algebra $C(\mathfrak{p})^{K}$ is isomorphic to $C(\mathbb{C e}) \otimes \operatorname{Pr}(S)$, where $e$ is a generator of degree $2 p+2 q+1$ squaring to 1 , and $\operatorname{Pr}(S)$ is isomorphic to $\mathbb{C}\left[\epsilon^{*}\right]^{B_{p} \times B_{q}}$ modulo the ideal generated by $B_{p+q}$-invariants in $\mathbb{C}\left[\mathrm{t}^{*}\right]$ evaluating to 0 at $\rho=(p+$ $q, \ldots, 2,1)$. The algebra $(\bigwedge \mathfrak{p})^{K}$ is isomorphic to $\bigwedge \mathbb{C e} \otimes \operatorname{gr} \operatorname{Pr}(S)$, where e is a generator of degree $2 p+2 q+1$ squaring to 0 , and where $\operatorname{gr} \operatorname{Pr}(S)$ is isomorphic to $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{B_{p} \times B_{q}}$ modulo the ideal generated by $B_{p+q}$-invariants in $\mathbb{C}\left[\mathrm{t}^{*}\right]$ evaluating to 0 at 0 .
3.8. $(\bigwedge \mathfrak{p})^{K}$ for disconnected $K$ : the case $G / K=\mathrm{U}(n) / \mathrm{O}(n)$. We use the standard matrix realizations of $G=\mathrm{U}(n)$ and $K=\mathrm{O}(n)$. Since $K$ is disconnected, the group $W_{K}$ may be different from $W_{\mathfrak{k}}$ and we want to describe it explicitly. We prove that $G / K=\mathrm{U}(n) / \mathrm{O}(n)$ is primary (resp. almost primary) when $n$ is even (resp. odd), we then apply the results of Section 3.6.

The group $K=\mathrm{O}(n)$ has two connected components and we choose the following representative for the disconnected component:

$$
\begin{array}{ll}
s=\operatorname{diag}(1, \ldots, 1,-1) & \text { if } n=2 k \\
s=\operatorname{diag}(-1, \ldots,-1) & \text { if } n=2 k+1
\end{array}
$$

For the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ we take the space of block diagonal matrices with diagonal blocks

$$
\begin{aligned}
t_{1} J, \ldots, t_{k} J & \text { if } n=2 k \\
t_{1} J, \ldots, t_{k} J, 0 & \text { if } n=2 k+1
\end{aligned}
$$

where as before $J=J_{1}=\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$ and $t_{1}, \ldots, t_{k}$ are complex scalars. We extend $\mathfrak{t}$ to a Cartan subalgebra $\mathfrak{h}=\mathfrak{t} \oplus \mathfrak{a}$ of $\mathfrak{g}$, where $\mathfrak{a}$ is the space of block diagonal matrices with diagonal blocks

$$
\begin{aligned}
a_{1} I, \ldots, a_{k} I & \text { if } n=2 k \\
a_{1} I, \ldots, a_{k} I, a_{k+1} & \text { if } n=2 k+1,
\end{aligned}
$$

where $I=I_{2}=\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ and $a_{1}, \ldots, a_{k+1}$ are complex scalars.
We can identify $\mathfrak{t} \subset \mathfrak{h} \cong \mathbb{C}^{n}$ with

$$
\begin{aligned}
\left\{\left(t_{1}, \ldots, t_{k},-t_{k}, \ldots,-t_{1}\right) \mid t_{1}, \ldots, t_{k} \in \mathbb{C}\right\} & \text { if } n=2 k \\
\left\{\left(t_{1}, \ldots, t_{k}, 0,-t_{k}, \ldots,-t_{1}\right) \mid t_{1}, \ldots, t_{k} \in \mathbb{C}\right\} & \text { if } n=2 k+1
\end{aligned}
$$

and $\mathfrak{a} \subset \mathfrak{h} \cong \mathbb{C}^{n}$ with

$$
\begin{aligned}
\left\{\left(a_{1}, \ldots, a_{k}, a_{k}, \ldots, a_{1}\right) \mid a_{1}, \ldots, a_{k} \in \mathbb{C}\right\} & \text { if } n=2 k \\
\left\{\left(a_{1}, \ldots, a_{k}, a_{k+1}, a_{k}, \ldots, a_{1}\right) \mid a_{1}, \ldots, a_{k+1} \in \mathbb{C}\right\} & \text { if } n=2 k+1
\end{aligned}
$$

This corresponds to the action of the involution $\sigma$ on $\mathfrak{h}$ being

$$
\sigma\left(h_{1}, \ldots, h_{n}\right)=\left(-h_{n}, \ldots,-h_{1}\right)
$$

Here $\sigma$ is the restriction to $G=\mathrm{U}(n)$ of the involution in Subsection 2.3. in particular, $\mathrm{U}(n)^{\sigma}=$ $\mathrm{O}(n)$.

The above identification enables us to identify the $(\mathfrak{g}, \mathfrak{t})$ roots easily: $\Delta(\mathfrak{g}, \mathfrak{t})$ is of type $C_{k}$ if $n=2 k$ and of type $B C_{k}$ if $n=2 k+1$. Thus

$$
W_{G}=W_{\mathfrak{g}}=B_{k}
$$

recall that the Weyl group $B_{k}$, which is the same as the Weyl group of type $C_{k}$ or $B C_{k}$, consists of permutations and sign changes of coordinates of $\mathfrak{t} \cong \mathbb{C}^{k}$.

Of course, $W_{\mathfrak{k}}$ is $D_{k}$ if $n=2 k$ and $B_{k}$ if $n=2 k+1$; here $D_{k}$ is the Weyl group of type $D_{k}$, consisting of permutations of the coordinates and sign changes of an even number of coordinates of $\mathfrak{t} \cong \mathbb{C}^{k}$. We however want to identify the group theoretic Weyl group $W_{K}$.

To do this, we identify $\mathfrak{t}$ with $\mathbb{C}^{k}$ by sending $\left(t_{1}, \ldots, t_{k},-t_{k}, \ldots,-t_{1}\right)$ to $\left(t_{1}, \ldots, t_{k}\right)$. Using (3.32) we see that for $n=2 k$ the element $s$ normalizes $\mathfrak{t}$ and sends $\left(t_{1}, \ldots, t_{k-1}, t_{k}\right) \in \mathfrak{t}$ to $\left(t_{1}, \ldots, t_{k-1},-t_{k}\right)$. Thus $W_{K}=W_{\mathfrak{k}} \rtimes\{1, s\}=B_{k}$. For $n=2 k+1, s$ centralizes $\mathfrak{t}$, hence $W_{K}=W_{\mathfrak{k}}=B_{k}$.

We have proved that $G / K=U(n) / O(n)$ is primary for odd $n$ and almost primary for even $n$, therefore Theorem 3.29 gives the graded algebra structure of $(\bigwedge \mathfrak{p})^{K}$.

We conclude
Corollary 3.41. For $G / K=\mathrm{U}(n) / \mathrm{O}(n)$, either $G / K$ is primary or almost primary. Hence, by Theorem 3.29,

$$
(\bigwedge \mathfrak{p})^{K} \cong \bigwedge\left(\mathcal{P}_{\wedge}(\mathfrak{p})\right)
$$

where $\mathcal{P}_{\wedge}(\mathfrak{p})$ is the subspace defined in Definition 3.26 and the degrees are given in Table 2.

## 4. Cohomology Rings of compact symmetric spaces

4.1. Some general facts. Let $G / K$ be a compact symmetric space, with $G$ a compact connected Lie group and $K$ a closed symmetric subgroup. Then the de Rham cohomology (with real coefficients) of $G / K$ can be identified, as an algebra, with $\left(\wedge \mathfrak{p}_{0}^{*}\right)^{K}$, where $\mathfrak{p}_{0}$ stands for the tangent space $\mathfrak{g}_{0} / \mathfrak{k}_{0}$ to $G / K$ at $e K\left(\mathfrak{g}_{0}\right.$ and $\mathfrak{k}_{0}$ are the Lie algebras of $G$ respectively $\left.K\right)$. In the examples we are interested in, $\mathfrak{p}_{0}$ can be $K$-equivariantly embedded into $\mathfrak{g}_{0}$, and also $\mathfrak{p}_{0}^{*} \cong \mathfrak{p}_{0}$.

As mentioned in the introduction, this fact is well known, but it is difficult to find an appropriate reference. We present here a proof we learned from Sebastian Goette Goe23.

Any $g \in G$ acts on $G / K$ by a map that is homotopic to the identity. Indeed, since $G$ is connected, there is a smooth path $g(t), t \in[0,1]$, from $g$ to the unit element $e \in G$. Then $H: G / K \times[0,1] \rightarrow G / K, H(x, t)=g(t) x$, is a smooth homotopy from $g: G / K \rightarrow G / K$ to the identity map on $G / K$.

It now follows that if $\omega$ is a closed form, then it represents the same cohomology class as $g^{*} \omega$, for any $g \in G$. Namely, Lee03, Proposition 15.5] says that $g^{*}$ and $e^{*}=\mathrm{id}$ induce the same map on cohomology, so the class of $\omega$ is the same as the class of $g^{*} \omega$. Since $G$ is compact, we can average over $g$ and get a $G$-invariant differential form that represents the same cohomology class as $\omega$.

Next, assume that $\omega$ is $G$-invariant and $\omega=d \mu$. Even if $\mu$ is not $G$-invariant itself, we know that $d \mu=g^{*} d \mu=d g^{*} \mu$. So we can average over $g \in G$ again to get a $G$-invariant differential form $\bar{\mu}$ such that $\omega=d \bar{\mu}$. So we see that the de Rham cohomology is captured by the subcomplex of $G$-invariant differential forms.

Now we recall that the differential forms on $G / K$ are sections of the homogeneous vector bundle

$$
G \times_{K} \wedge \mathfrak{p}_{0}^{*} \rightarrow G / K
$$

The bundle $G \times_{K} \bigwedge \mathfrak{p}_{0}^{*}$ is defined as $\left(G \times \bigwedge \mathfrak{p}_{0}^{*}\right) / \sim$, where $\sim$ is the equivalence relation defined by

$$
(g k, \nu) \sim\left(g, \operatorname{Ad}^{*}(k) \nu\right), \quad g \in G, k \in K, \nu \in \bigwedge \mathfrak{p}_{0}^{*}
$$

The differential forms, or sections of the above bundle, are maps

$$
\omega: G \rightarrow \bigwedge \mathfrak{p}_{0}^{*} \quad \text { such that } \quad \omega(g k)=\mathrm{Ad}^{*}\left(k^{-1}\right) \omega(g), \quad g \in G, k \in K
$$

The group $G$ acts on such $\omega$ by left translation, i.e.,

$$
(g \omega)\left(g^{\prime}\right)=\omega\left(g^{-1} g^{\prime}\right), \quad g, g^{\prime} \in G
$$

Thus $\omega$ is $G$-invariant if and only if it is constant as a function on $G$, i.e., $\omega(g)=\omega(e)$ for any $g \in G$.

For such an invariant form $\omega$, set $\bar{\omega}=\omega(e) \in \bigwedge \mathfrak{p}_{0}^{*}$. We claim that $\bar{\omega} \in\left(\bigwedge \mathfrak{p}_{0}^{*}\right)^{K}$. Indeed, for any $k \in K$ we have

$$
\begin{aligned}
& \operatorname{Ad}^{*}(k) \bar{\omega}=\operatorname{Ad}^{*}(k) \omega(e)=(\text { since } \omega \text { is a section })= \\
& \qquad \omega\left(e k^{-1}\right)=\omega\left(k^{-1}\right)=(\text { since } \omega \text { is } G \text {-invariant })=\omega(e)=\bar{\omega} .
\end{aligned}
$$

Conversely, if $\bar{\omega} \in\left(\bigwedge \mathfrak{p}_{0}^{*}\right)^{K}$, then $\omega(g)=\bar{\omega}$ defines a $G$-invariant form $\omega$ on $G / K$.
So we see that for any compact homogeneous space $G / K$, with $G$ a compact connected Lie group and $K$ a closed subgroup of $G$, the de Rham cohomology $H(G / K)$ is the cohomology of the complex $\left(\left(\bigwedge \mathfrak{p}_{0}^{*}\right)^{K}, d\right)$, where the differential $d$ is induced by the de Rham differential. Let us describe the differential $d$ more explicitly. We first recall a coordinate free formula for the de Rham differential $d$ on a manifold $M$. Any differential $q$-form is determined if we know how to evaluate it on any $q$-tuple of (smooth) vector fields. In this interpretation, the de Rham differential of a $q$-form $\omega$ is the $(q+1)$-form given by

$$
\begin{gather*}
d \omega\left(X_{1} \wedge \ldots \wedge X_{q+1}\right)=\sum_{i}(-1)^{i-1} X_{i}\left(\omega\left(X_{1} \wedge \cdots \wedge \widehat{X}_{i} \wedge \cdots \wedge X_{q+1}\right)\right)+  \tag{4.1}\\
\sum_{i<j}(-1)^{i+j} \omega\left(\left[X_{i}, X_{j}\right] \wedge X_{1} \wedge \cdots \wedge \widehat{X}_{i} \wedge \cdots \wedge \widehat{X}_{j} \wedge \cdots \wedge X_{q+1}\right)
\end{gather*}
$$

where $X_{1}, \ldots, X_{q+1}$ are vector fields on $M$, the bracket denotes the Lie bracket of vector fields, and the hat over a variable means this variable is omitted. See e.g. [Lee03, Proposition 12.19].

If $M=G / K$ and if the form $\omega$ is $G$-invariant (as we saw we may assume), then we know $\omega$ at any point $g K$ if we know it at the base point $e K$. More precisely, if $Y_{1}, \ldots, Y_{q}$ is any $q$-tuple of tangent vectors at a point $g K$ in $G / K$, then

$$
\omega(g K)\left(Y_{1}, \ldots, Y_{q}\right)=\omega(e K)\left(g_{*}^{-1} Y_{1}, \ldots, g_{*}^{-1} Y_{q}\right)
$$

It follows that it is enough to know the value of $\omega$ at $q$-tuples of $G$-invariant vector fields, which correspond to the tangent space $\mathfrak{g}_{0} / \mathfrak{k}_{0} \cong \mathfrak{p}_{0}$ to $G / K$ at $e K$. The $G$-invariant vector fields can in turn be obtained as push-forwards of left invariant vector fields on $G$ under the projection $\Pi: G \rightarrow G / K$. Since $\Pi_{*}$ is compatible with Lie brackets, and since the left invariant vector fields on $G$ can be identified with the Lie algebra $\mathfrak{g}_{0}$ of $G$, we see that if $\tilde{X}, \tilde{Y}$ are invariant vector fields on $G / K$ corresponding to tangent vectors $X, Y \in \mathfrak{p}_{0}$, then the vector field $[\tilde{X}, \tilde{Y}]$ corresponds to the tangent vector $\pi([X, Y])$, where the bracket $[X, Y]$ is taken in $\mathfrak{g}_{0}$ and $\pi=d \Pi: \mathfrak{g}_{0} \rightarrow \mathfrak{p}_{0}$ is the canonical projection. Thus the formula (4.1) can be rewritten with $X_{1}, \ldots, X_{q+1}$ in $\mathfrak{p}_{0}$ and with
[ $\left.X_{i}, X_{j}\right]$ replaced by $\pi\left(\left[X_{i}, X_{j}\right]\right)$. Moreover, the first sum in the formula vanishes, since the action of a vector field involves the differentiation, and $\omega$ is constant. So the de Rham differential becomes

$$
\begin{equation*}
d \omega\left(X_{1} \wedge \ldots \wedge X_{q+1}\right)=\sum_{i<j}(-1)^{i+j} \omega\left(\pi\left(\left[X_{i}, X_{j}\right]\right) \wedge X_{1} \wedge \cdots \wedge \widehat{X}_{i} \wedge \cdots \wedge \widehat{X}_{j} \wedge \cdots \wedge X_{q+1}\right) \tag{4.2}
\end{equation*}
$$

with $X_{i}$ in $\mathfrak{p}_{0}$.
Now let us assume in addition that $K$ is a symmetric subgroup of $G$, i.e., that ( $\mathfrak{g}_{0}, \mathfrak{k}_{0}$ ) is a symmetric pair. Then $\left[\mathfrak{p}_{0}, \mathfrak{p}_{0}\right] \subseteq \mathfrak{k}_{0}$, so $\pi\left(\left[X_{i}, X_{j}\right]\right)=0$ for any $X_{i}, X_{j} \in \mathfrak{p}_{0}$, and we see that $d \omega=0$. Hence

$$
\begin{equation*}
H(G / K)=\left(\bigwedge \mathfrak{p}_{0}^{*}\right)^{K} \cong\left(\bigwedge \mathfrak{p}_{0}\right)^{K} \tag{4.3}
\end{equation*}
$$

In the following we will pass to the cohomology of $G / K$ with complex coefficients:

$$
H(G / K ; \mathbb{C})=H(G / K) \otimes_{\mathbb{R}} \mathbb{C}=\left(\bigwedge \mathfrak{p}_{0}\right)^{K} \otimes_{\mathbb{R}} \mathbb{C}=(\bigwedge \mathfrak{p})^{K}
$$

where $\mathfrak{p}$ stands for the complexification of $\mathfrak{p}_{0}$. We will also denote by $\mathfrak{g}$ respectively $\mathfrak{k}$ the complexifications of $\mathfrak{g}_{0}$ respectively $\mathfrak{k}_{0}$.

Recall from Section 3, the paragraph above Corollary 3.40 that there is a surjection $\alpha=\alpha_{K}$ : $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{K}} \rightarrow \operatorname{Pr}(S)$, given by

$$
\alpha_{K}(P)=\sum_{\tilde{\sigma} \in W_{G, K}^{1}} P(\tilde{\sigma} \rho) \operatorname{pr}_{\tilde{\sigma}}
$$

with kernel equal to the ideal $\left\langle\mathbb{C}\left[\mathfrak{t}^{*}\right]_{\rho}^{W_{G}}\right\rangle$ in $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{K}}$ generated by the $W_{G}$-invariants that vanish at $\rho$. Likewise, the kernel of the surjection $\operatorname{gr} \alpha: \mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{K}} \rightarrow \operatorname{gr} \operatorname{Pr}(S) \subseteq(\bigwedge \mathfrak{p})^{K}$ is the ideal $\left\langle\mathbb{C}\left[\mathfrak{t}^{*}\right]_{+}^{W_{G}}\right\rangle$ generated by the $W_{G}$-invariants that vanish at 0 . In this section we will only use the obvious inclusions

$$
\begin{equation*}
\left\langle\mathbb{C}\left[\mathrm{t}^{*}\right]_{\rho}^{W_{G}}\right\rangle \subseteq \operatorname{ker} \alpha \quad \text { and } \quad\left\langle\mathbb{C}\left[\mathrm{t}^{*}\right]_{+}^{W_{G}}\right\rangle \subseteq \operatorname{ker} \operatorname{gr} \alpha \tag{4.4}
\end{equation*}
$$

The opposite inclusions will get reproved as a byproduct of our analysis. Namely, in each of the cases we will have a candidate set for a basis of the quotient $\mathbb{C}\left[\mathrm{t}^{*}\right]^{W_{K}} / \operatorname{ker} \alpha$ respectively $\mathbb{C}\left[\mathrm{t}^{*}\right]^{W_{K}} / \operatorname{ker} \operatorname{gr} \alpha$ consisting of certain monomials, of the cardinality equal to

$$
\operatorname{dim} \operatorname{Pr}(S)=\operatorname{dim} \operatorname{gr} \operatorname{Pr}(S)=\left|W_{G, K}^{1}\right|
$$

We will use the relations coming from $\mathbb{C}\left[\mathrm{t}^{*}\right]_{\rho}^{W_{G}}$ respectively $\mathbb{C}\left[\mathrm{t}^{*}\right]_{+}^{W_{G}}$ to show that the images of these candidate monomials span the respective quotients. It will follow that they form a basis, and also that there can be no additional relations outside of $\left\langle\mathbb{C}\left[\mathrm{t}^{*}\right]_{\rho}^{W_{G}}\right\rangle$ respectively $\left\langle\mathbb{C}\left[\mathrm{t}^{*}\right]_{+}^{W_{G}}\right\rangle$.
4.2. The case $G / K=\mathrm{U}(p+q) / \mathrm{U}(p) \times \mathrm{U}(q)$. This is an equal rank case, so $\operatorname{Pr}(S)=C(\mathfrak{p})^{K}$ and $\operatorname{gr} \operatorname{Pr}(S)=(\bigwedge \mathfrak{p})^{K}$. Since $G$ and $K$ are both connected, the Weyl groups are $W_{G}=W_{\mathfrak{g}}=S_{p+q}$ and $W_{K}=W_{\mathfrak{k}}=S_{p} \times S_{q}$. Let $x_{1}, \ldots, x_{p+q}$ be coordinates for the Cartan subalgebra $\mathfrak{t}$. Let

- $r_{1}, \ldots, r_{p}$ be the elementary symmetric functions in variables $x_{1}, \ldots, x_{p}$;
- $s_{1}, \ldots, s_{q}$ be the elementary symmetric functions in variables $x_{p+1}, \ldots, x_{p+q}$;
- $t_{1}, \ldots, t_{p+q}$ be the elementary symmetric functions in variables $x_{1}, \ldots, x_{p+q}$.

Then the space of $W_{K}$-invariants in $S(\mathfrak{t})$ is generated by $r_{1}, \ldots, r_{p}$ and $s_{1}, \ldots, s_{q}$.
Theorem 4.5. For $G / K=\mathrm{U}(p+q) / \mathrm{U}(p) \times \mathrm{U}(q), 1 \leq p \leq q$, the algebra $C(\mathfrak{p})^{K}$ is isomorphic to the algebra $\mathfrak{H}(p, q ; c)$ of Definition 1.5, where $c=\left(t_{1}(\rho), \ldots, t_{p+q}(\rho)\right)$. The algebra $(\bigwedge \mathfrak{p})^{K}$ is isomorphic to $\mathfrak{H}(p, q ; 0)$.

In other words, the algebras $C(\mathfrak{p})^{K}$ and $(\bigwedge \mathfrak{p})^{K}$ are both generated by $r_{1}, \ldots, r_{p}$ (or more precisely, their images in the respective quotients), with relations generated by

$$
\sum_{i, j \geq 0 ; i+j=k} r_{i} s_{j}=t_{k}=\left\{\begin{array}{cl}
t_{k}(\rho) & \text { for the case of } C(\mathfrak{p})^{K}  \tag{4.6}\\
0 & \text { for the case of }(\bigwedge \mathfrak{p})^{K}
\end{array}\right.
$$

for $k=1, \ldots, p+q$, where we set $r_{0}=s_{0}=1$ and $r_{i}=0$ if $i>p, s_{j}=0$ if $j>q$.
To summarize,

$$
\begin{aligned}
C(\mathfrak{p})^{K} & =\mathfrak{H}(p, q ; c)=\frac{\mathbf{C}\left[r_{1}, \ldots, r_{p} ; s_{1}, \ldots, s_{q}\right]}{\left(\sum_{i, j \geq 0 ; i+j=k} r_{i} s_{j}=t_{k}(\rho)\right)} \\
(\bigwedge \mathfrak{p})^{K} & =\mathfrak{H}(p, q ; 0)=\frac{\mathbf{C}\left[r_{1}, \ldots, r_{p} ; s_{1}, \ldots, s_{q}\right]}{\left(\sum_{i, j \geq 0 ; i+j=k} r_{i} s_{j}=0\right)}
\end{aligned}
$$

The latter algebra is isomorphic to $H^{*}\left(\operatorname{Gr}_{p}\left(\mathbb{C}^{p+q}\right), \mathbb{C}\right)$.
A basis for each of these algebras is given by the monomials $r^{\alpha}=r_{1}^{\alpha_{1}} \ldots r_{p}^{\alpha_{p}}$ of degree $|\alpha| \leq q$, so that our algebras can be identified with the space

$$
\mathbb{C}\left[r_{1}, \ldots, r_{p}\right]_{\leq q}
$$

In particular, each monomial in $r_{1}, \ldots, r_{p}$ of degree $q+1$ can be expressed as a linear combination of lower degree monomials in $r_{1}, \ldots, r_{p}$. Such expressions follow from (4.9) and (4.14) below, and they provide another set of defining relations for each of our algebras.

The filtration degree of $C(\mathfrak{p})^{K}$ inherited from $C(\mathfrak{p})$, and the gradation degree of $(\bigwedge \mathfrak{p})^{K}$ inherited from $\wedge \mathfrak{p}$, are obtained by setting $\operatorname{deg} r_{i}=2 i$ for $i=1, \ldots, p$.
Proof. Let us first note that it is clear that the algebras $C(\mathfrak{p})^{K}$ and $(\bigwedge \mathfrak{p})^{K}$ are generated by the $r_{i}$ and the $s_{j}$. Also, the inclusions (4.4) imply that the relations for these algebras include the relations (4.6). We are going to see that these relations in fact generate all the relations.

Using the first $q$ of the relations (4.6), we can express all $s_{j}$ as polynomials in the $r_{i}$. Indeed, the first relation is

$$
r_{1}+s_{1}=t_{1}
$$

and we see that $s_{1}=t_{1}-r_{1}$. Now the second relation is

$$
r_{2}+r_{1} s_{1}+s_{2}=t_{2}
$$

so $s_{2}$ can be expressed as a polynomial in the $r_{i}$ since we have already expressed $s_{1}$. We continue inductively. So each of our algebras is generated by the (images of the) polynomials in the $r_{i}$.

We now prove that every monomial in the $r_{i}$ of degree $q+1$ can be expressed as a linear combination of lower degree monomials. This will finish the proof. Namely, this will show that the monomials in the $r_{i}$ of degree at most $q$ span each of our algebras, and their number is $\binom{p+q}{p}$, the same as the dimension of both algebras. So they have to form a basis. Since we only use the relations (4.6), it follows that these relations generate all the relations, otherwise the dimension would be lower which is impossible.

We order the monomials in the $r_{i}$ first by degree, and then inside each degree by the reverse lexicographical order. We will show by induction on this order that all degree $q+1$ monomials can be expressed by lower degree monomials.

We start with the first monomial, $r_{1}^{q+1}$. We express the $s_{j}$ in terms of the $r_{i}$ from relations $2,3, \ldots, q+1$. These relations are linear in the $s_{j}$, with coefficients that are either constant or the
$r_{i}$. In matrix form, this system of equations is

$$
\left(\begin{array}{ccccccc}
r_{1} & 1 & 0 & 0 & 0 & \ldots & 0  \tag{4.7}\\
r_{2} & r_{1} & 1 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
r_{p-1} & \ldots & r_{1} & 1 & 0 & \ldots & 0 \\
r_{p} & r_{p-1} & \ldots & r_{1} & 1 & 0 & \ldots \\
0 & r_{p} & r_{p-1} & \ldots & r_{1} & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & r_{p} & \cdots & r_{2} & r_{1}
\end{array}\right)\left(\begin{array}{c}
s_{1} \\
s_{2} \\
\vdots \\
s_{q}
\end{array}\right)=\left(\begin{array}{c}
t_{2}-r_{2} \\
t_{3}-r_{3} \\
\vdots \\
t_{p}-r_{p} \\
t_{p+1} \\
\vdots \\
t_{q+1}
\end{array}\right)
$$

The determinant $D$ of this system contains a unique monomial $r_{1}^{q}$ of degree $q$, since upon expanding successively along the first row we can always pick either $r_{1}$, or 1 which leads to lower degree. In particular $D \neq 0$, so we can solve the system by Cramer's rule and obtain

$$
\begin{equation*}
D s_{i}=D_{i}, \quad i=1, \ldots, q \tag{4.8}
\end{equation*}
$$

where $D_{i}$ is obtained from $D$ by replacing the $i$-th column by the right hand side of (4.7).
Now we multiply the first equation, $r_{1}+s_{1}=t_{1}$, by $D$, and use (4.8) to get

$$
\begin{equation*}
r_{1} D+D_{1}=t_{1} D \tag{4.9}
\end{equation*}
$$

Since all monomials in $D_{1}$ and in $t_{1} D$ are of degree $\leq q$, and since all the monomials of $r_{1} D$ are of degree $\leq q$ except for the monomial $r_{1}^{q+1}$, we have expressed $r_{1}^{q+1}$ as a linear combination of lower degree monomials.

We now do the induction step. Let $1 \leq i_{1}<i_{2}<\cdots<i_{a} \leq p$ be integers and let

$$
\begin{equation*}
r_{i_{1}}^{m_{1}} r_{i_{2}}^{m_{2}} \ldots r_{i_{a}}^{m_{a}} \tag{4.10}
\end{equation*}
$$

be a monomial of degree $q+1$ different from $r_{1}^{q+1}$. Suppose that we have already expressed all the degree $q+1$ monomials that are before (4.10) in reverse lexicographical order.

Let us consider the degree $q$ monomial

$$
\begin{equation*}
r_{i_{1}}^{m_{1}-1} r_{i_{2}}^{m_{2}} \ldots r_{i_{a}}^{m_{a}} \tag{4.11}
\end{equation*}
$$

Notice that for each $i$ and $j, r_{i} s_{j}$ appears in exactly one of the equations (the $(i+j)$-th one). We first assume that $m_{1}>1$ and pick the equations that contain respectively

$$
\begin{gather*}
r_{i_{1}} s_{1}, r_{i_{1}} s_{2}, \ldots r_{i_{1}} s_{m_{1}-1} ; r_{i_{2}} s_{m_{1}}, \ldots, r_{i_{2}} s_{m_{1}+m_{2}-1} ; \ldots  \tag{4.12}\\
\ldots ; r_{i_{a}} s_{m_{1}+\cdots+m_{a-1}}, \ldots, r_{i_{a}} s_{m_{1}+\cdots+m_{a}-1}\left(=r_{i_{a}} s_{q}\right)
\end{gather*}
$$

We view these equations as a linear system for $s_{1}, \ldots, s_{q}$ and note that the diagonal coefficients are exactly the coefficients of the terms (4.12), i.e.,

$$
r_{i_{1}}, \ldots, r_{i_{1}}, r_{i_{2}}, \ldots, r_{i_{2}}, \ldots, r_{i_{a}}, \ldots, r_{i_{a}}
$$

with $r_{i_{1}}$ repeating $m_{1}-1$ times, $r_{i_{2}}$ repeating $m_{2}$ times, $\ldots, r_{i_{a}}$ repeating $m_{a}$ times. Thus the determinant of the system, denoted again by $D$, contains the monomial (4.11), and we claim this is the leading term of the expanded determinant $D$. (We warn the reader not to confuse the present $D$ with the one in (4.8).)

The first of the picked equations is

$$
r_{i_{1}} s_{1}+r_{i_{1}-1} s_{2}+\cdots+r_{1} s_{i_{1}}+s_{i_{1}+1}=t_{i_{1}+1}-r_{i_{1}+1}
$$

(This covers all the cases since we defined $r_{i}=0$ for $i>p$ and $s_{j}=0$ for $j>q$.) The first row of $D$ is thus

$$
r_{i_{1}} r_{i_{1}-1} \ldots r_{1} 10 \ldots 0
$$

with 1 and/or zeros possibly missing. When we expand $D$ along the first row and then write out all the lower order determinants as combinations of monomials, the terms containing the first row elements $r_{i_{1}-1}, r_{i_{1}-2}, \ldots$ are all either of lower degree or before the term (4.11) in our ordering. To get the leading term we thus have to pick $r_{i_{1}}$ and cross the first row and column. The remaining determinant (if $m_{1}>2$ ) has the first row equal to

$$
r_{i_{1}} r_{i_{1}-1} r_{i_{1}-2} \ldots
$$

and we use the same argument to conclude that we should pick $r_{i_{1}}$ to obtain the leading term.
After we go over all the rows containing $r_{i_{1}}$, we continue with the next row

$$
r_{i_{2}} r_{i_{2}-1} r_{i_{2}-2} \ldots
$$

We again see that to obtain the leading term we have to pick $r_{i_{2}}$ from this row.
The conclusion is that the leading term of $D$ is indeed the monomial (4.11); in particular, $D \neq 0$. We now again write the Cramer's rule

$$
\begin{equation*}
D s_{i}=D_{i}, \quad i=1, \ldots, q \tag{4.13}
\end{equation*}
$$

We multiply the equation containing $r_{i_{1}}$ with coefficient 1 , i.e., the equation

$$
r_{i_{1}}+r_{i_{1}-1} s_{1}+\cdots+r_{1} s_{i_{1}-1}+s_{i_{1}}=t_{i_{1}}
$$

by $D$, and use (4.13) to get

$$
\begin{equation*}
r_{i_{1}} D+r_{i_{1}-1} D_{1}+\cdots+r_{1} D_{i_{1}-1}+D_{i_{1}}=t_{i_{1}} D \tag{4.14}
\end{equation*}
$$

The leading term of $r_{i_{1}} D$ is (4.10), and the other terms in the above equation are either of lower degree, or of the same degree but of lower order with respect to the reverse lexicographical order. Expressing these last terms by lower degree terms using the inductive assumption, we see that we have expressed (4.10) as a linear combination of lower degree terms.

This finishes the proof if $m_{1}>1$. If $m_{1}=1$, we proceed analogously, starting by picking the equation containing $r_{i_{2}} s_{1}$. The argument is entirely similar.

The statement about degrees follows from the fact that $r_{i}$ is of degree $i$ as a polynomial in the variables $x_{j}$, and that the map $\alpha: U(\mathfrak{k}) \rightarrow C(\mathfrak{p})$ doubles the degree.

Remark 4.15. In the course of the proof of Theorem 4.5 we have obtained explicit relations (4.9) and (4.14) for the generators $r_{1}, \ldots, r_{p}$. It is clear that $C(\mathfrak{p})^{K}$ is the algebra generated by the $r_{i}$ with these relations if we set $t_{i}=t_{i}(\rho)$ and $(\bigwedge \mathfrak{p})^{K}$ is the algebra generated by the $r_{i}$ with the same relations if we set $t_{i}=0$.

Remark 4.16. The monomials in Theorem4.5span the same space as the Schur polynomials $s_{\lambda}$ for $\lambda$ in the $p \times q$ box. In particular, these Schur polynomials also form a basis of our algebra(s), since their number is equal to the dimension of each of the two algebras.

To pursue this relationship in more detail, we first recall the well known Jacobi-Trudi formulas that express Schur polynomials as polynomials in the elementary symmetric functions: if $\lambda$ is a
partition with Young diagram inside the $p \times q$ box, let $\lambda^{t}=\left(\lambda_{1}^{t}, \ldots, \lambda_{l}^{t}\right)$ be the transpose of $\lambda(l \leq q$ is the length of $\left.\lambda^{t}\right)$. Then

$$
s_{\lambda}=\operatorname{det}\left(\begin{array}{cccc}
r_{\lambda_{1}^{t}} & r_{\lambda_{1}^{t}+1} & \ldots & r_{\lambda_{1}^{t}+l-1}  \tag{4.17}\\
r_{\lambda_{2}^{t}-1} & r_{\lambda_{2}^{t}} & \ldots & r_{\lambda_{2}^{t}+l-2} \\
\vdots & \vdots & \vdots & \vdots \\
r_{\lambda_{l}^{t}-l+1} & r_{\lambda_{l}^{t}-l+2} & \cdots & r_{\lambda_{l}^{t}}
\end{array}\right)
$$

Here $r_{j}$ is the $j$ th elementary symmetric function on $x_{1}, \ldots, x_{p}$ if $1 \leq j \leq p, r_{0}=1$, and $r_{j}=0$ if $j<0$ or $j>p$.

This is an expression of $s_{\lambda}$ as a linear combination of monomials in $r_{1}, \ldots, r_{p}$ of degree at most $q$. We claim that in this way we obtain a triangular change of basis between the $s_{\lambda}$ and the monomials in $r_{1}, \ldots, r_{p}$ of degree at most $q$. To see this, we order the monomials first be degree (in the $r_{j}$ ), and then by reverse lexicographical order inside each degree. We claim that upon expanding the determinant (4.17) the leading term is the diagonal monomial $r_{\lambda_{1}^{t}} r_{\lambda_{2}^{t}} \ldots r_{\lambda_{l}^{t}}$. Indeed, let us expand the determinant along the first row. Since $\lambda_{1}^{t} \geq \lambda_{2}^{t} \geq \cdots \geq \lambda_{l}^{t}$, the diagonal monomial has no $r_{j}$ with $j>\lambda_{1}^{t}$, but if we pick any element of the first row other than $r_{\lambda_{1}^{t}}$, all monomials in the corresponding piece of the expansion will contain $r_{j}$ with $j>\lambda_{1}^{t}$. So to obtain the leading term we must pick $r_{\lambda_{1}^{t}}$. We now repeat this argument inductively, always expanding along the first row.

The main advantage of the Schur polynomials is the fact that their multiplication table is well understood, using Littlewood-Richardson coefficients. While the computation of the LR coefficients is only algorithmic, computer programs for computing them are widely known and available; for example, there is an online calculator available from the web page of Joel Gibson Gib. Our approach using monomials in the elementary functions and the relations between them that we obtained can also lead to a multiplication table, as illustrated by the following example. In this way, we get an alternative way of computing the LR coefficients.

Example 4.18. Let $p=2$ and $q=3$. The expressions of the Schur polynomials for $\lambda$ in the $2 \times 3$ box in terms of monomials in the elementary symmetric functions $r_{1}=x_{1}+x_{2}, r_{2}=x_{1} x_{2}$ are

$$
\begin{array}{lll}
s_{(0,0)}=1 ; & s_{(1,0)}=r_{1} ; & s_{(2,0)}=r_{1}^{2}-r_{2} ;
\end{array} l \begin{aligned}
& s_{(3,0)}=r_{1}^{3}-2 r_{1} r_{2} ; \\
& s_{(1,1)}=r_{2} ;
\end{aligned} \begin{array}{lll}
s_{(2,1)}=r_{1} r_{2} ; & s_{(3,1)}=r_{1}^{2} r_{2}-r_{2}^{2} ; \\
s_{(2,2)}=r_{2}^{2} ; & s_{(3,2)}=r_{1} r_{2}^{2} ; & s_{(3,3)}=r_{2}^{3}
\end{array}
$$

Our relations expressing monomials of degree four in terms of monomials of degree at most three are

$$
r_{1}^{4}=3 r_{1}^{2} r_{2}-r_{2}^{2} ; \quad r_{1}^{3} r_{2}=2 r_{1} r_{2}^{2} ; \quad r_{1}^{2} r_{2}^{2}=r_{2}^{3} ; \quad r_{1} r_{2}^{3}=0 ; \quad r_{2}^{4}=0
$$

The multiplication table for the monomials is

|  | $r_{1}$ | $r_{2}$ | $r_{1}^{2}$ | $r_{1} r_{2}$ | $r_{2}^{2}$ | $r_{1}^{3}$ | $r_{1}^{2} r_{2}$ | $r_{1} r_{2}^{2}$ | $r_{2}^{3}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $r_{1}$ | $r_{1}^{2}$ | $r_{1} r_{2}$ | $r_{1}^{3}$ | $r_{1}^{2} r_{2}$ | $r_{1} r_{2}^{2}$ | $3 r_{1}^{2} r_{2}-r_{2}^{2}$ | $2 r_{1} r_{2}^{2}$ | $r_{2}^{3}$ | 0 |
| $r_{2}$ |  | $r_{2}^{2}$ | $r_{1}^{2} r_{2}$ | $r_{1} r_{2}^{2}$ | $r_{2}^{3}$ | $2 r_{1} r_{2}^{2}$ | 0 | 0 | 0 |
| $r_{1}^{2}$ |  |  | $3 r_{1}^{2} r_{2}-r_{2}^{2}$ | $2 r_{1} r_{2}^{2}$ | $r_{2}^{3}$ | $5 r_{1} r_{2}^{2}$ | $2 r_{2}^{3}$ | 0 | 0 |
| $r_{1} r_{2}$ |  |  |  | $r_{2}^{3}$ | 0 | $2 r_{2}^{3}$ | 0 | 0 | 0 |
| $r_{2}^{2}$ |  |  |  |  | 0 | 0 | 0 | 0 | 0 |
| $r_{1}^{3}$ |  |  |  |  |  | $5 r_{2}^{3}$ | 0 | 0 | 0 |
| $r_{1}^{2} r_{2}$ |  |  |  |  |  |  | 0 | 0 | 0 |
| $r_{1} r_{2}^{2}$ |  |  |  |  |  |  |  | 0 | 0 |
| $r_{2}^{3}$ |  |  |  |  |  |  |  |  | 0 |

The reader is invited to compare this with the multiplication table for the Schur polynomials obtained from [Gib]; to use the online calculator one has to remember that the Schur polynomials for $\lambda$ outside of the $2 \times 3$ box have to be replaced by zeros.
4.3. The cases $G / K=\operatorname{Sp}(p+q) / \operatorname{Sp}(p) \times \operatorname{Sp}(q)$. Since $G$ and $K$ have equal rank, $\operatorname{Pr}(S)=C(\mathfrak{p})^{K}$ and $\operatorname{gr} \operatorname{Pr}(S)=(\bigwedge \mathfrak{p})^{K}$. Since $G$ and $K$ are both connected, the Weyl group $W_{G}$ is equal to $W_{\mathfrak{g}}$ which is isomorphic to $B_{p+q}$, while the Weyl group $W_{K}=W_{\mathfrak{k}}$ is $B_{p} \times B_{q}$. (Recall that type $B$ and type $C$ have the same Weyl group. It consists of permutations and sign changes of the variables.)

As in type A, the set $W_{G, K}^{1}$ consists of $(p, q)$-shuffles. In particular,

$$
\left|W_{G, K}^{1}\right|=\operatorname{dim} C(\mathfrak{p})^{K}=\operatorname{dim}(\bigwedge \mathfrak{p})^{K}=\binom{p+q}{p}
$$

It is well known (see Hum90, p.67]) that the algebra of $B_{k}$-invariants is a polynomial algebra generated by symmetric functions of the squares of the variables. Thus $S(\mathfrak{t})^{W_{K}}$ is generated by

$$
\begin{gathered}
r_{1}=x_{1}^{2}+\cdots+x_{p}^{2}, \quad r_{2}=x_{1}^{2} x_{2}^{2}+\cdots+x_{p-1}^{2} x_{p}^{2}, \quad \ldots, \quad r_{p}=x_{1}^{2} x_{2}^{2} \ldots x_{p}^{2} \\
s_{1}=x_{p+1}^{2}+\cdots+x_{p+q}^{2}, s_{2}=x_{p+1}^{2} x_{p+2}^{2}+\cdots+x_{p+q-1}^{2} x_{p+q}^{2}, \ldots, s_{q}=x_{p+1}^{2} \ldots x_{p+q}^{2}
\end{gathered}
$$

and $S(\mathfrak{t})^{W_{G}}$ is generated by

$$
t_{1}=x_{1}^{2}+\cdots+x_{p+q}^{2}, \quad t_{2}=x_{1}^{2} x_{2}^{2}+\cdots+x_{p+q-1}^{2} x_{p+q}^{2}, \quad \ldots, \quad t_{p+q}=x_{1}^{2} x_{2}^{2} \ldots x_{p+q}^{2}
$$

As in the type A case, the relations for $C(\mathfrak{p})^{K}$ respectively $(\bigwedge \mathfrak{p})^{K}$ include the relations (4.6). Moreover, we have
Theorem 4.19. For $G / K=\operatorname{Sp}(p+q) / \operatorname{Sp}(p) \times \operatorname{Sp}(q), 1 \leq p \leq q$, the algebra $C(\mathfrak{p})^{K}$ is isomorphic to the algebra $\mathfrak{H}(p, q ; c)$ of Definition 1.5, with generators the above $r_{i}$, and with $c=\left(t_{1}(\rho), \ldots, t_{p+q}(\rho)\right)$. The algebra $(\bigwedge \mathfrak{p})^{K}$ is isomorphic to $\mathfrak{H}(p, q ; 0)$.

In other words,

$$
\begin{aligned}
C(\mathfrak{p})^{K} & =\mathfrak{H}(p, q ; c)
\end{aligned}=\frac{\mathbb{C}\left[r_{1}, \ldots, r_{p} ; s_{1}, \ldots, s_{q}\right]}{\left(\sum_{i, j \geq 0 ; i+j=k} r_{i} s_{j}=t_{k}(\rho)\right)}, ~(\bigwedge \mathfrak{p})^{K}=\mathfrak{H}(p, q ; 0)=\frac{\mathbb{C}\left[r_{1}, \ldots, r_{p} ; s_{1}, \ldots, s_{q}\right]}{\left(\sum_{i, j \geq 0 ; i+j=k} r_{i} s_{j}=0\right)}
$$

The latter algebra is isomorphic to $H^{*}\left(\operatorname{Gr}_{p}\left(\mathbb{H}^{p+q}\right), \mathbb{C}\right)$.
Both algebras can be identified with the space

$$
\mathbb{C}\left[r_{1}, \ldots, r_{p}\right]_{\leq q}
$$

The filtration degree of $C(\mathfrak{p})^{K}$ inherited from $C(\mathfrak{p})$, and the gradation degree of $(\bigwedge \mathfrak{p})^{K}$ inherited from $\wedge \mathfrak{p}$, are obtained by setting $\operatorname{deg} r_{i}=4 i$ for $i=1, \ldots, p$.
Proof. The same as the proof of Theorem 4.5. (The statement about degrees follows from the fact that $r_{i}$ is of degree $2 i$ as a polynomial in the variables $x_{j}$, and from the fact that the map $\alpha: U(\mathfrak{k}) \rightarrow C(\mathfrak{p})$ doubles the degree.)

Remark 4.20. As in Remark 4.16, we can replace the monomials in the $r_{i}$ by the Schur polynomials $s_{\lambda}$ for $\lambda$ in the $p \times q$ box. This allows us to write the multiplication table in the usual way.
4.4. The cases $G / K=\mathrm{SO}(k+m) / S(\mathrm{O}(k) \times \mathrm{O}(m))$. If $(k, m)=(2 p, 2 q)$ or $(k, m)=(2 p, 2 q+1)$ then $G$ and $K$ have equal rank, so $C(\mathfrak{p})^{K}=\operatorname{Pr}(S)$ and $(\bigwedge \mathfrak{p})^{K}=\operatorname{gr} \operatorname{Pr}(S)$.

If $(k, m)=(2 p, 2 q+1)$, then by Proposition 3.34 $W_{G}=B_{p+q}$ and $W_{K}=B_{p} \times B_{q}$, so $C(\mathfrak{p})^{K}$ and $(\bigwedge \mathfrak{p})^{K}$ are described by Theorem 4.19,

If $(k, m)=(2 p, 2 q)$, then by Proposition 3.34 $W_{G}=D_{p+q}$ and $W_{K}=S\left(B_{p} \times B_{q}\right)$. The invariants in $S(\mathfrak{t})$ under $B_{p} \times B_{q} \supset W_{K}$ are generated by the symmetric functions $r_{1}, \ldots, r_{p}$ of $x_{1}^{2}, \ldots, x_{p}^{2}$ and the symmetric functions $s_{1}, \ldots, s_{q}$ of $x_{p+1}^{2}, \ldots, x_{p+q}^{2}$, while the invariants under $D_{p} \times D_{q} \subset W_{K}$ are generated by $r_{1}, \ldots, r_{p-1}, s_{1}, \ldots, s_{q-1}$, and the Pfaffians $\bar{r}_{p}=x_{1} \ldots x_{p}, \bar{s}_{q}=x_{p+1} \ldots x_{p+q}$ (see [Hum90, p.68]). It follows that the invariants under $W_{K}$ are generated by

$$
r_{1}, \ldots, r_{p} ; s_{1}, \ldots, s_{q} ; \bar{r}_{p} \bar{s}_{q}
$$

Of course, these generators are not independent, as $\left(\bar{r}_{p} \bar{s}_{q}\right)^{2}=r_{p} s_{q}$.
Since our algebras $C(\mathfrak{p})^{K}$ and $(\bigwedge \mathfrak{p})^{K}$ are quotients of $\mathbb{C}\left[\mathfrak{t}^{*}\right]^{W_{K}}$ by the ideal generated by $\mathbb{C}\left[\mathfrak{t}^{*}\right]_{\rho}^{W_{G}}$ respectively $\mathbb{C}\left[\mathrm{t}^{*}\right]_{+}^{W_{G}}$, and since $\bar{r}_{p} \bar{s}_{q}=x_{1} \ldots x_{p+q}$ is $W_{G}$-invariant, we can remove $\bar{r}_{p} \bar{s}_{q}$ from the list of generators. (Note that the value of $\bar{r}_{p} \bar{s}_{q}$ at $\rho$ is 0 , since 0 is a coordinate of $\rho$.) It follows that the algebras $C(\mathfrak{p})^{K}$ and $(\bigwedge \mathfrak{p})^{K}$ are again described by Theorem 4.19,

Finally, suppose that $(k, m)=(2 p+1,2 q+1)$. This is an unequal (but almost equal) rank case and as we saw in Subsection 3.5, the fundamental Cartan subalgebra is $\mathfrak{h}=\mathfrak{t} \oplus \mathfrak{a} \cong \mathbb{C}^{p+q+1}$ with

$$
\mathfrak{t}=\left\{\left(t_{1}, \ldots t_{p+q}, 0\right) \mid t_{i} \in \mathbb{C}\right\} ; \quad \mathfrak{a}=\{(0, \ldots, 0, a) \mid a \in \mathbb{C}\}
$$

The root system $\Delta(\mathfrak{g}, \mathfrak{t})$ is $B_{p+q}$ while $\Delta(\mathfrak{k}, \mathfrak{t})=B_{p} \times B_{q}$. (Note that $\Delta(\mathfrak{g}, \mathfrak{h})$ is $D_{p+q+1}$. )
By Corollary $3.40(3), C(\mathfrak{p})^{K}=C\left(\mathcal{P}_{\mathfrak{a}}\right) \otimes \operatorname{Pr}(S)$; since $\mathfrak{a}$ is one-dimensional, $C\left(\mathcal{P}_{\mathfrak{a}}\right)$ is twodimensional, spanned by 1 and a generator $e$ squaring to 1 . Likewise, $(\bigwedge \mathfrak{p})^{K}=\Lambda \mathcal{P}_{\mathfrak{a}} \otimes \operatorname{Pr}(S)$, with $\bigwedge \mathcal{P}_{\mathfrak{a}}$ spanned by 1 and by a generator $e$ squaring to 0 .

We know from Proposition 3.34 that $W_{G}=B_{p+q}$ and $W_{K}=B_{p} \times B_{q}$. It follows that the algebras $\operatorname{Pr}(S)$ and $\operatorname{gr} \operatorname{Pr}(S)$ are described by Theorem4.19, with notation given by that theorem and the text above it.

Theorem 4.21. Let $G / K=\mathrm{SO}(k+m) / S(\mathrm{O}(k) \times \mathrm{O}(m))$.
(a) If $(k, m)=(2 p, 2 q)$ or $(k, m)=(2 p, 2 q+1)$, then the algebra $C(\mathfrak{p})^{K}$ is isomorphic to the algebra $\mathfrak{H}(p, q ; c)$ of Definition 1.5, with generators $r_{1}, \ldots, r_{p}$ as above, and with $c=\left(t_{1}(\rho), \ldots, t_{p+q}(\rho)\right)$. The algebra $(\bigwedge \mathfrak{p})^{K}$ is isomorphic to $\mathfrak{H}(p, q ; 0)$. In other words,

$$
\begin{aligned}
& C(\mathfrak{p})^{K}=\mathfrak{H}(p, q ; c)=\frac{\mathbb{C}\left[r_{1}, \ldots, r_{p} ; s_{1}, \ldots, s_{q}\right]}{\left(\sum_{i, j \geq 0 ; i+j=k} r_{i} s_{j}=t_{k}(\rho)\right)} \\
&(\bigwedge \mathfrak{p})^{K}=\mathfrak{H}(p, q ; 0)=\frac{\mathbb{C}\left[r_{1}, \ldots, r_{p} ; s_{1}, \ldots, s_{q}\right]}{\left(\sum_{i, j \geq 0 ; i+j=k} r_{i} s_{j}=0\right)}
\end{aligned}
$$

The latter algebra is isomorphic to $H^{*}\left(\operatorname{Gr}_{k}\left(\mathbb{R}^{k+m}\right), \mathbb{C}\right)$.
Both algebras can be identified with the space

$$
\mathbb{C}\left[r_{1}, \ldots, r_{p}\right]_{\leq q}
$$

The filtration degree of $C(\mathfrak{p})^{K}$ inherited from $C(\mathfrak{p})$, and the gradation degree of $(\bigwedge \mathfrak{p})^{K}$ inherited from $\wedge \mathfrak{p}$, are obtained by setting $\operatorname{deg} r_{i}=4 i$ for $i=1, \ldots, p$.
(b) If $(k, m)=(2 p+1,2 q+1)$, then the algebra $C(\mathfrak{p})^{K}$ contains the algebra $\mathfrak{H}(p, q ; c)$ as in (a), and an additional generator e squaring to 1 . The algebra $(\bigwedge \mathfrak{p})^{K}$ contains the algebra $\mathfrak{H}(p, q ; 0)$ as in (a), and an additional generator e squaring to 0 .

$$
\begin{aligned}
C(\mathfrak{p})^{K} & =\frac{\mathbb{C}\left[r_{1}, \ldots, r_{p} ; s_{1}, \ldots, s_{q} ; e\right]}{\left(\sum_{i, j \geq 0 ; i+j=k} r_{i} s_{j}=t_{k}(\rho), e^{2}=1\right)} \\
(\bigwedge \mathfrak{p})^{K} & =\frac{\mathbb{C}\left[r_{1}, \ldots, r_{p} ; s_{1}, \ldots, s_{q} ; e\right]}{\left(\sum_{i, j \geq 0 ; i+j=k} r_{i} s_{j}=0, e^{2}=0\right)}
\end{aligned}
$$

The latter algebra is isomorphic to $\left.H^{*}\left(\operatorname{Gr}_{k}\left(\mathbb{R}^{k+m}\right), \mathbb{C}\right)\right)$.
Each of the algebras can be identified with

$$
\mathbb{C}\left[r_{1}, \ldots, r_{p}\right]_{\leq q} \oplus \mathbb{C}\left[r_{1}, \ldots, r_{p}\right]_{\leq q} e
$$

The filtration degree of $C(\mathfrak{p})^{K}$ inherited from $C(\mathfrak{p})$, and the gradation degree of $(\bigwedge \mathfrak{p})^{K}$ inherited from $\wedge \mathfrak{p}$, are obtained by setting $\operatorname{deg} r_{i}=4 i$ for $i=1, \ldots, p$ and $\operatorname{deg} e=2 p+2 q+1$.
Proof. This follows from the discussion above and from Corollary 3.40,
Remark 4.22. As in Remark 4.16, we can replace the monomials in the $r_{i}$ by the Schur polynomials $s_{\lambda}$ for $\lambda$ in the $p \times q$ box. This allows us to write the multiplication table in the usual way.
4.5. The case $G / K=\operatorname{Sp}(n) / \mathrm{U}(n)$. Since $G$ and $K$ have equal rank, $\operatorname{Pr}(S)=C(\mathfrak{p})^{K}$ and $\operatorname{gr} \operatorname{Pr}(S)=(\bigwedge \mathfrak{p})^{K}$.

Since $G$ and $K$ are both connected, the Weyl groups are $W_{G}=W_{\mathfrak{g}}=B_{n}$ and $W_{K}=W_{\mathfrak{k}}=A_{n-1}$. In other words, $W_{K}$ consists of the permutations of the variables $x_{1}, \ldots, x_{n}$, while $W_{G}$ consists of permutations and sign changes of $x_{1}, \ldots, x_{n}$. The set $W_{G, K}^{1}$ has $2^{n}$ elements and can be identified with the sign changes. The algebra $S(\mathfrak{t})^{W_{K}}$ is generated by

$$
r_{1}=x_{1}+\cdots+x_{n}, \quad r_{2}=x_{1} x_{2}+x_{1} x_{3}+\cdots+x_{n-1} x_{n}, \ldots, \quad r_{n}=x_{1} x_{2} \ldots x_{n}
$$

and $S(\mathfrak{t})^{W_{G}}$ is generated by

$$
t_{1}=x_{1}^{2}+\cdots+x_{n}^{2}, \quad t_{2}=x_{1}^{2} x_{2}^{2}+\cdots+x_{n-1}^{2} x_{n}^{2}, \ldots, t_{n}=x_{1}^{2} x_{2}^{2} \ldots x_{n}^{2}
$$

To write down the relations coming from $S(\mathfrak{t})^{W_{G}}$, let $z$ be a formal variable and note that

$$
\begin{aligned}
\sum_{k=0}^{n}(-1)^{k} t_{k} z^{2 k}=\prod_{k=1}^{n}\left(1-x_{k}^{2} z^{2}\right) & =\prod_{i=1}^{n}\left(1-x_{i} z\right) \prod_{i=1}^{n}\left(1+x_{i} z\right) \\
& =\left(\sum_{i=0}^{n}(-1)^{i} r_{i} z^{i}\right)\left(\sum_{j=0}^{n} r_{j} z^{j}\right)=\sum_{k=0}^{n}\left(\sum_{i+j=2 k}(-1)^{i} r_{i} r_{j}\right) z^{2 k}
\end{aligned}
$$

It follows that the relations are

$$
\begin{equation*}
\sum_{i+j=2 k}(-1)^{i} r_{i} r_{j}=(-1)^{k} t_{k}, \quad k=1, \ldots, n \tag{4.23}
\end{equation*}
$$

Equivalently,

$$
\begin{equation*}
r_{k}^{2}=t_{k}+2 r_{k-1} r_{k+1}-2 r_{k-2} r_{k+2}+\ldots, \quad k=1, \ldots, n, \tag{4.24}
\end{equation*}
$$

where as usual we set $r_{0}=1$ and $r_{i}=0$ for $i>n$.
Theorem 4.25. For $G / K=\operatorname{Sp}(n) / \mathrm{U}(n)$, the algebras $C(\mathfrak{p})^{K}$ and $(\bigwedge \mathfrak{p})^{K}$ are both generated by $r_{1}, \ldots, r_{p}$ (or more precisely, their images in the respective quotients), with relations generated by (4.24) with $t_{k}$ replaced by $t_{k}(\rho)$ for $C(\mathfrak{p})^{K}$, and by 0 for $(\wedge \mathfrak{p})^{K}$.

In other words,

$$
\begin{aligned}
C(\mathfrak{p})^{K} & =\frac{\mathbb{C}\left[r_{1}, \ldots, r_{n}\right]}{\left(\sum_{i+j=2 k}(-1)^{i} r_{i} r_{j}=(-1)^{k} t_{k}(\rho)(1 \leq k \leq n)\right)} \\
(\bigwedge \mathfrak{p})^{K} & =\frac{\mathbb{C}\left[r_{1}, \ldots, r_{n}\right]}{\left(\sum_{i+j=2 k}(-1)^{i} r_{i} r_{j}=0(1 \leq k \leq n)\right)}
\end{aligned}
$$

The latter algebra is isomorphic to $H^{*}\left(\operatorname{LGr}\left(\mathbb{C}^{2 n}\right), \mathbb{C}\right)$.
A basis for each of the algebras is represented by the monomials

$$
\begin{equation*}
r_{1}^{\varepsilon_{1}} r_{2}^{\varepsilon_{2}} \ldots r_{n}^{\varepsilon_{n}}, \quad \varepsilon_{i} \in\{0,1\} . \tag{4.26}
\end{equation*}
$$

The filtration degree of $C(\mathfrak{p})^{K}$ inherited from $C(\mathfrak{p})$, and the gradation degree of $(\wedge \mathfrak{p})^{K}$ inherited from $\wedge \mathfrak{p}$, are obtained by setting deg $r_{i}=2 i$ for $i=1, \ldots, n$.
Proof. Note that the cardinality of the set (4.26) is correct, $2^{n}$, so it is enough to show that every monomial can be written as a linear combination of the monomials in (4.26).

We proceed by induction on degree. If the degree is 0 , the only possible monomial is 1 , and it is on the list (4.26). If an arbitrary monomial contains either $r_{1}$ or $r_{n}$ with degree $\geq 2$, then we can use the relations (4.24) for $k=1\left(r_{1}^{2}=t_{1}+r_{2}\right)$ or for $k=n\left(r_{n}^{2}=t_{n}\right)$ to write this monomial as a combination of smaller degree monomials.

Assume now that a monomial

$$
\begin{equation*}
x^{d}=x_{1}^{d_{1}} \ldots x_{n}^{d_{n}}, \quad d_{i} \in \mathbb{Z}_{+} \tag{4.27}
\end{equation*}
$$

has $d_{k} \geq 2$, for some $1<k<n$. We identify monomials (4.27) with the strings of exponents $\left(d_{1}, \ldots, d_{n}\right) \in \mathbb{Z}_{+}^{n}$. Let $f:[1, n] \rightarrow \mathbb{R}^{+}$be a concave function taking integer values on $[1, n] \cap \mathbb{Z}$; for example, we can take $f(x)=x(n+1-x)$. We define $F: \mathbb{Z}_{+}^{n} \rightarrow \mathbb{Z}_{+}$by

$$
F\left(d_{1}, \ldots, d_{n}\right)=\sum_{k=1}^{n} f(k) d_{k} .
$$

By relations (4.24), the monomial $x^{d}$ is, up to lower degree monomials, equal to a linear combinations of monomials with exponents of the form

$$
\left(\ldots, d_{k-i}+1, \ldots d_{k}-2, \ldots, d_{k+i}+1, \ldots\right)=d+e_{k-i}-2 e_{k}+e_{k+i},
$$

with $i$ a positive integer such that $k-i \geq 1$ and $k+i \leq n$. Here $e_{1}, \ldots, e_{n}$ is the usual standard basis of $\mathbb{R}^{n}$.

We now have

$$
\begin{aligned}
& F\left(d+e_{k-i}-2 e_{k}+e_{k+i}\right)-F(d)= \\
& \quad f(k-i)\left[\left(d_{k-i}+1\right)-d_{k-i}\right]+f(k)\left[\left(d_{k}-2\right)-d_{k}\right]+f(k+i)\left[\left(d_{k+i)}+1\right)-d_{k+i}\right]= \\
& \\
& f(k-i)-2 f(k)+f(k+i),
\end{aligned}
$$

which is negative since $f$ is concave. So all the monomials in the expression for $d$ using the relations have values of $F$ lower than $F(d)$.

We can repeat this procedure as long as we have some $d_{k} \geq 2,1<k<n$ (recall that we already handled the cases $d_{1} \geq 2$ and $d_{n} \geq 2$ ). Since the value of $F$ gets strictly smaller each time, and since these values are positive integers, the process has to stop, meaning that there are no $d_{k} \geq 2$, hence we have arrived at a monomial of the form (4.26).

The statement about degrees follows from the fact that $r_{i}$ is of degree $i$ as a polynomial in the variables $x_{j}$, and that the map $\alpha: U(\mathfrak{k}) \rightarrow C(\mathfrak{p})$ doubles the degree.
4.6. The case $G / K=\mathrm{SO}(2 n) / \mathrm{U}(n)$. Since $G$ and $K$ have equal rank, $\operatorname{Pr}(S)=C(\mathfrak{p})^{K}$ and $\operatorname{gr} \operatorname{Pr}(S)=(\bigwedge \mathfrak{p})^{K}$.

Since $G$ and $K$ are both connected, the Weyl groups are $W_{G}=W_{\mathfrak{g}}=D_{n}$ and $W_{K}=W_{\mathfrak{e}}=A_{n-1}$, i.e., $W_{K}$ consists of the permutations of the variables $x_{1}, \ldots, x_{n}$, while $W_{G}$ consists of permutations and sign changes of an even number of variables $x_{1}, \ldots, x_{n}$. The set $W_{G, K}^{1}$ has $2^{n-1}$ elements and can be identified with the sign changes of an even number of variables. The algebra $S(\mathfrak{t})^{W_{K}}$ is generated by

$$
r_{1}=x_{1}+\cdots+x_{n}, \quad r_{2}=x_{1} x_{2}+x_{1} x_{3}+\cdots+x_{n-1} x_{n}, \ldots, \quad r_{n}=x_{1} x_{2} \ldots x_{n}
$$

and $S(\mathfrak{t})^{W_{G}}$ is generated by

$$
t_{1}=x_{1}^{2}+\cdots+x_{n}^{2}, \ldots, \quad t_{n-1}=x_{1}^{2} \ldots x_{n-1}^{2}+\cdots+x_{2}^{2} \ldots x_{n}^{2}, \quad \bar{t}_{n}=x_{1} x_{2} \ldots x_{n}
$$

We set $t_{n}=\bar{t}_{n}^{2}$. The relations are the same as (4.23) or (4.24) (where as before, $t_{k}$ is replaced by $t_{k}(\rho)$ for $C(\mathfrak{p})^{K}$ and by 0 for $\left.(\bigwedge \mathfrak{p})^{K}\right)$,
except that for $k=n$ we have $r_{n}=\bar{t}_{n}$ instead of $r_{n}^{2}=t_{n}$. This last equation enables us to eliminate $r_{n}$ from the list of generators. Thus we have

Theorem 4.28. For $G / K=\operatorname{SO}(2 n) / \mathrm{U}(n)$, the algebras $C(\mathfrak{p})^{K}$ and $(\bigwedge \mathfrak{p})^{K}$ are both generated by $r_{1}, \ldots, r_{n-1}$ (or more precisely, their images in the respective quotients). The relations are generated by (4.24) with $t_{k}$ replaced by $t_{k}(\rho)$ for $C(\mathfrak{p})^{K}$ and by 0 for $(\bigwedge \mathfrak{p})^{K}$, and with the last relation replaced by $r_{n}=\bar{t}_{n}$.

In other words,

$$
\begin{aligned}
C(\mathfrak{p})^{K} & =\frac{\mathbb{C}\left[r_{1}, \ldots, r_{n}\right]}{\left(\sum_{i+j=2 k}(-1)^{i} r_{i} r_{j}=(-1)^{k} t_{k}(\rho)(1 \leq k \leq n), r_{n}=\bar{t}_{n}(\rho)\right)} \\
(\bigwedge \mathfrak{p})^{K} & =\frac{\mathbb{C}\left[r_{1}, \ldots, r_{n}\right]}{\left(\sum_{i+j=2 k}(-1)^{i} r_{i} r_{j}=0(1 \leq k \leq n), r_{n}=0\right)}
\end{aligned}
$$

The latter algebra is isomorphic to $H^{*}\left(\mathrm{OLGr}^{+}\left(\mathbb{C}^{2 n}\right), \mathbb{C}\right)$.
A basis for each of the algebras is represented by the monomials

$$
\begin{equation*}
r_{1}^{\varepsilon_{1}} r_{2}^{\varepsilon_{2}} \ldots r_{n-1}^{\varepsilon_{n-1}}, \quad \varepsilon_{i} \in\{0,1\} \tag{4.29}
\end{equation*}
$$

The filtration degree of $C(\mathfrak{p})^{K}$ inherited from $C(\mathfrak{p})$, and the gradation degree of $(\bigwedge \mathfrak{p})^{K}$ inherited from $\wedge \mathfrak{p}$, are obtained by setting $\operatorname{deg} r_{i}=2 i$ for $i=1, \ldots, n-1$.

Proof. The same as the proof of Theorem4.25.
4.7. The group cases. In this subsection we consider the group cases $G \times G / \Delta G \cong G$, where $G$ is $\mathrm{SO}(n)$, or $\mathrm{U}(n)$, or $\mathrm{Sp}(n)$.

In each of the three cases, the complexified Lie algebra of $G \times G$ is $\mathfrak{g} \oplus \mathfrak{g}$ where $\mathfrak{g}$ is the complexified Lie algebra of $G, \mathfrak{k}$ is the diagonal subalgebra $\Delta \mathfrak{g} \cong \mathfrak{g}$ of $\mathfrak{g} \oplus \mathfrak{g}$, and $\mathfrak{p}$ is the antidiagonal subspace of $\mathfrak{g} \oplus \mathfrak{g}$, which is isomorphic to $\mathfrak{g}$ as a $\mathfrak{g}$-module.

Thus we are looking for the description of $C(\mathfrak{g})^{G}=C(\mathfrak{g})^{\mathfrak{g}}$ and of $(\bigwedge \mathfrak{g})^{G}=(\bigwedge \mathfrak{g})^{\mathfrak{g}}$. Thus we can use the results of Kos97 in the Clifford case and the well known Hopf-Koszul-Samelson Theorem in the exterior case [Car51, Sam41, Kos50] see [GHV76, p.568] for full bibliographic details.
to conclude
Theorem 4.30. Kos97, Sam41 The algebras $C(\mathfrak{p})^{K}=C(\mathfrak{g})^{\mathfrak{g}}$ and $(\bigwedge \mathfrak{p})^{K}=(\bigwedge \mathfrak{g})^{\mathfrak{g}}$ are isomorphic to $C\left(\mathcal{P}_{\wedge}(\mathfrak{p})\right)$, respectively $\wedge \mathcal{P}_{\wedge}(\mathfrak{p})$, where $\mathcal{P}_{\wedge}(\mathfrak{p}) \cong \mathfrak{h}$ denotes the graded subspace defined in Definition 3.26. The degrees are given in Table 2.
4.8. The cases $G / K=\mathrm{U}(2 n) / \operatorname{Sp}(n)$. In these cases $K$ is connected, so $(\bigwedge \mathfrak{p})^{K}=(\bigwedge \mathfrak{p})^{\mathfrak{k}}$. Since these are unequal rank cases, we first describe the fundamental Cartan subalgebra $\mathfrak{h}=\mathfrak{t} \oplus \mathfrak{a}$. The situation is similar to the case $G / K=\mathrm{U}(2 n) / \mathrm{O}(2 n)$.

The noncompact symmetric space corresponding to $\mathrm{U}(2 n) / \operatorname{Sp}(n)$ is $\mathrm{GL}(2 n, \mathbb{H}) / \operatorname{Sp}(n)=$ $\mathrm{U}^{*}(2 n) / \operatorname{Sp}(n)$. The following can be read off from the information about the classification of real forms in Kna96.

The fundamental Cartan subalgebra $\mathfrak{h}$ can be identified with $\mathbb{C}^{2 n}$, with the Cartan involution acting by $\theta\left(h_{1}, \ldots, h_{2 n}\right)=\left(-h_{2 n}, \ldots,-h_{1}\right)$. Hence

$$
\begin{aligned}
\mathfrak{t} & =\left\{\left(h_{1}, \ldots, h_{n},-h_{n}, \ldots,-h_{1}\right) \mid h_{1}, \ldots, h_{n} \in \mathbb{C}\right\} \cong \mathbb{C}^{n} \\
\mathfrak{a} & =\left\{\left(h_{1}, \ldots, h_{n}, h_{n}, \ldots, h_{1}\right) \mid h_{1}, \ldots, h_{n} \in \mathbb{C}\right\} \cong \mathbb{C}^{n}
\end{aligned}
$$

If we now restrict the roots $\pm\left(\varepsilon_{i}-\varepsilon_{j}\right), 1 \leq i<j \leq 2 n$, to $\mathfrak{t}$, we get

$$
\pm\left(\varepsilon_{i} \pm \varepsilon_{j}\right), 1 \leq i<j \leq n ; \quad 2 \varepsilon_{i}, 1 \leq i \leq n
$$

In other words, we have obtained the root system $C_{n}$. Since $\Delta(\mathfrak{k}, \mathfrak{t})$ is also of type $C_{n}$ (but with smaller multiplicities), $W_{\mathfrak{g}, \mathfrak{e}}^{1}$ consists only of the identity. (This means that the spin module $S$ is primary, as already observed in Han06.)

So the algebras $\operatorname{Pr}(S)$ and $\operatorname{gr} \operatorname{Pr}(S)$ are both equal to $\mathbb{C} \cdot 1$, and using Theorem 3.29 we get
Theorem 4.31. The algebra $(\bigwedge \mathfrak{p})^{K}$ is isomorphic to $\bigwedge\left(\mathcal{P}_{\wedge}(\mathfrak{p})\right)$, where $\mathcal{P}_{\wedge}(\mathfrak{p}) \cong \mathfrak{a}$ is defined in Definition 3.26 and the degrees are given in Table 2.

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