Tight Bounds on the Chromatic Edge Stability Index of Graphs

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Abstract

The chromatic edge stability index $\mathrm{es}_{\chi'}(G)$ of a graph G is the minimum number of edges whose removal results in a graph with smaller chromatic index. We give best-possible upper bounds on $\mathrm{es}_{\chi'}(G)$ in terms of the number of vertices of degree $\Delta(G)$ (if G is Class 2), and the numbers of vertices of degree $\Delta(G)$ and $\Delta(G) - 1$ (if G is Class 1). If G is bipartite we give an exact expression for $\mathrm{es}_{\chi'}(G)$ involving the maximum size of a matching in the subgraph induced by the vertices of degree $\Delta(G)$. Finally, we consider whether a minimum mitigating set, that is a set of size $\mathrm{es}_{\chi'}(G)$ whose removal reduces the chromatic index, has the property that every edge meets a vertex of degree at least $\Delta(G) - 1$; we prove that this is true for some minimum mitigating set of G, but not necessarily for every minimum mitigating set of G.

1 Introduction

Throughout this paper all graphs have at least one edge and are simple, that is, with no loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The
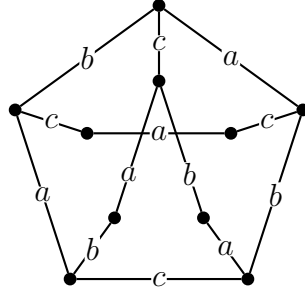
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Figure 1. A 3-edge-colorable graph obtained by removing two edges from P.

degree of a vertex v is denoted by $d_G(v)$. The *core* of G, denoted by G_Δ, is the subgraph of G induced by the vertices of degree $\Delta(G)$, where $\Delta(G)$ is the maximum degree of G. We denote the number of vertices of degree i by t_i. The graph G is called *acyclic* if G does not contain any cycles. If $S \subseteq V(G)$, then the subgraph of G induced by S is denoted by $G[S]$. Similarly, if $S' \subseteq E(G)$, then the subgraph of G induced by S' is denoted by $G[S']$.

A *proper edge coloring* (resp. vertex coloring) assigns colors to the edges (resp. vertices) of a graph so that no two incident edges (resp. adjacent vertices) have the same color. The *chromatic index* of G, $\chi'(G)$ is the minimum number of colors required for a proper edge coloring of G. The *chromatic number* of a graph G, $\chi(G)$ is similarly defined as the minimum number of colors required for a proper vertex coloring of G. A *matching M* in G is a set of pairwise non-incident edges. The *matching number $\alpha'(G)$* of a graph G is the size of a maximum matching.

The *chromatic edge stability number* $\mathrm{es}_\chi(G)$ of a graph G is the minimum number of edges whose removal results in a graph H with $\chi(H) = \chi(G) - 1$. This concept was first introduced in the 1980's by Staton [7], and some inequalities and sharp bounds on es_χ were proved in [3]. Similarly, the *chromatic vertex stability number* $vs_\chi(G)$ of a graph G, defined as the minimum number of vertices whose removal results in a graph H with $\chi(H) = \chi(G) - 1$, was first studied in [2].

A *mitigating set* is a set of edges F in G such that $\chi'(G \setminus F) < \chi'(G)$.

In this paper, we focus on the *chromatic edge stability index* of G denoted by $\mathrm{es}_{\chi'}(G)$, defined as the minimum number of edges whose removal results in a graph H with $\chi'(H) = \chi'(G) - 1$. This concept was first studied in [1].

As an example, consider the *Petersen graph*, P, in which $\chi'(P) = 4$, see [8, p. 7]. As was proved in [8, p. 16], removing one vertex of P does not reduce the chromatic index. Hence, the removal of one edge cannot reduce the chromatic index. However, one can reduce the chromatic index by removing two edges, as illustrated in Figure 1. Hence, $\mathrm{es}_{\chi'}(P) = 2$.

We now recall Vizing's Theorem [9].

Theorem 1. *[9] For a graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

A graph G is *Class 1* if $\chi'(G) = \Delta(G)$ and it is *Class 2* if $\chi'(G) = \Delta(G) + 1$. The

following theorem was first stated by Fournier [5].

Theorem 2. *[5] Let G be a graph. If G_Δ is acyclic, then G is Class 1.*

Theorem 3. *[4] Let G be a connected graph with $|G_\Delta| = 3$. Then G is Class 2 if and only if for some integer n, G is obtained from K_{2n+1} by removing $n-1$ independent edges.*

2 Main Results

We now state our first theorem regarding the chromatic edge stability index.

Theorem 4. *Let G be a graph of Class 2. Then, $\mathrm{es}_{\chi'}(G) \leqslant \frac{t_\Delta - 1}{2}$. Furthermore, if G is connected and t_Δ is odd, then equality holds if and only if G is obtained from K_{2n+1} by removing s independent edges for some integers n and s, $0 \leq s \leq n-1$.*

Proof. Note that it is sufficient to remove a set of edges F such that $|F| \leq \frac{t_\Delta - 1}{2}$ and $G \setminus F$ is Class 1, since then we have $\chi'(G \setminus F) = \Delta(G \setminus F) \leq \Delta(G) < \chi'(G)$.

By Theorem 2, since G is Class 2 it contains a cycle of vertices of degree $\Delta(G)$; removing an edge from that cycle reduces the number of vertices of degree $\Delta(G)$ by 2. Continue in this manner until either the remaining graph is Class 1 or $\lfloor \frac{t_\Delta - 1}{2} \rfloor$ edges have been removed. In the latter case the number of vertices of degree $\Delta(G)$ remaining is at most 2, so there is no cycle of such vertices and Theorem 2 implies the remaining graph is Class 1. Thus $\mathrm{es}_{\chi'}(G) \leqslant \frac{t_\Delta - 1}{2}$.

Now, we prove the second part of the theorem. Let A be the set of vertices of maximum degree. First, assume that $\mathrm{es}_{\chi'}(G) = \frac{t_\Delta - 1}{2}$. Consider a maximum matching M in G_Δ and denote by V_M the set of vertices saturated by M. Remove the edges of M from G and call the new graph G'. Since t_Δ is odd, $A \neq V_M$. Note that $G'_\Delta = G[A \setminus V_M]$ has no edges and by Theorem 2 G' is Class 1. Hence, $\mathrm{es}_{\chi'}(G) \leqslant \alpha'(G_\Delta)$. Notice that we have $\alpha'(G_\Delta) \leq \frac{t_\Delta - 1}{2}$. By combining these two inequalities and our initial assumption, we obtain $\alpha'(G_\Delta) = \frac{t_\Delta - 1}{2}$. Hence, $|A \setminus V_M| = 1$. Suppose that $A \setminus V_M = \{v\}$. We claim that v is adjacent to all vertices of V_M. By contradiction, assume $uv \notin E(G_\Delta)$ for some $u \in V_M$. Suppose $uw \in M$. Remove all edges in M except uw from G and call the new graph G''. Note that $V(G''_\Delta) = \{u, v, w\}$ and G''_Δ is a forest so by Theorem 2, G'' is Class 1. Hence, $\chi'(G'') = \Delta(G'') = \Delta(G) < \chi'(G)$ and G'' is obtained by removing $\mathrm{es}_{\chi'}(G) - 1$ edges from G, a contradiction. Similarly, for every vertex $u \in V_M$, if $uw \in M$, by considering the matching $M \setminus \{uw\} \cup \{vw\}$, it can be proved that u is adjacent to all vertices in A. Therefore, $G_\Delta \cong K_{t_\Delta}$. Now, remove $\alpha'(G_\Delta) - 1$ edges of M from G and call the remaining graph H. Since $\mathrm{es}_{\chi'}(G) - 1$ edges are removed from G, $\chi'(H) = \chi'(G)$. Note that $\Delta(H) = \Delta(G)$. Hence, we have $\chi'(H) = \chi'(G) = \Delta(G) + 1 = \Delta(H) + 1$ and H is Class 2. Since $H_\Delta \cong K_3$ and H is connected, by Theorem 3, H is obtained

from K_{2n+1} by removing $n-1$ independent edges, for some integer n. It is straight-forward to see that G is obtained from K_{2n+1} by removing s independent edges, where $s = (n-1) - (\alpha'(G_\Delta) - 1) = n - \alpha'(G_\Delta) = n - \mathrm{es}_{\chi'}(G) \le n - 1$.

Now, assume that G is obtained from K_{2n+1} by removing a matching M, where $|M| = s$ and $0 \le s \le n-1$. It is not hard to see that G is Class 2. Note that $\frac{t_\Delta - 1}{2} = \frac{2n+1-2s-1}{2} = n - s$. Therefore, by the first part of the theorem, we have $\mathrm{es}_{\chi'}(G) \le n - s$. Now, we prove that $\mathrm{es}_{\chi'}(G) \ge n - s$. Suppose $\chi'(G)$ is reduced by removing at most $n - s - 1$ edges. Then there is a subgraph G' with $\chi'(G') = 2n$ and $|E(G)| \le |E(G')| + n - s - 1$. Consider an edge-coloring of G' with $2n$ colors. Each color class is a matching, so contains at most n edges. Thus $E(G') \le 2n^2$, giving

$$|E(G)| \le 2n^2 + n - s - 1 = \binom{2n+1}{2} - s - 1 = |E(G)| - 1,$$

a contradiction. Hence, $\mathrm{es}_{\chi'}(G) = n - s = \frac{t_\Delta - 1}{2}$ and the proof is completed. □

Remark 1. For every value of $t_\Delta \ge 3$, the upper bound $\lfloor \frac{t_\Delta - 1}{2} \rfloor$ can be attained for $\mathrm{es}_{\chi'}(G)$. If t_Δ is odd, this follows from the equality case of Theorem 4. If $t_\Delta = 2k$ is even, construct G as follows: start from K_{2k+1} and remove an edge xy, then add a new vertex z and the edge xz. Note that $\Delta(G) = 2k$ and there are $2k$ vertices of degree $2k$ (all except y and z). In a proper edge-coloring of a subgraph of G, there are at most $k+1$ edges of the color used on xz, and at most k edges of any other color. Thus any $H \subset G$ with $\chi'(H) = 2k$ satisfies $|E(H)| \le 2k^2 + 1 = |E(G)| - (k-1)$, and so $\mathrm{es}_{\chi'}(G) \ge k - 1$.

Remark 2. In contrast to Class 2, for every function $f(t_\Delta)$ we can find a connected graph G of Class 1 such that $\mathrm{es}_{\chi'}(G) > f(t_\Delta)$. Consider the Petersen graph, remove one of its edges, and call the remaining graph Q, as shown in Figure 2.
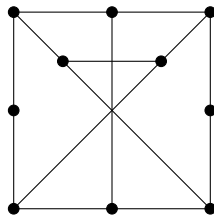
Figure 2. The graph $Q \cong P \setminus \{e\}$.

The graph G illustrated in Figure 3 is a chain of copies of Q joined to a copy of $K_{1,3}$. Note that $\chi'(G) = 4$, $t_\Delta = 1$ and $\Delta(G) = 4$. As was previously stated, one must remove at least one more edge from each copy of Q in order to reduce the chromatic index. Therefore, one can arbitrarily increase $\mathrm{es}_{\chi'}(G)$ by adding more copies of Q to the chain.

In the next result, we provide an upper bound for the chromatic edge stability index in terms of $t_{\Delta-1}$ and t_Δ.
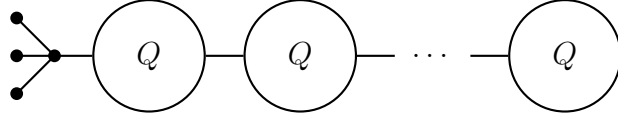
Figure 3. A graph G with $t_\Delta = 1$ and $es_{\chi'}(G)$ arbitrarily large.

Theorem 5. *For every graph G, $es_{\chi'}(G) \leq t_\Delta + \frac{t_{\Delta-1}}{2}$. Furthermore, equality holds if and only if $t_{\Delta-1} = 0$ and G_Δ has no edges.*

Proof. We prove the following slightly stronger statement by induction. Let H be a graph with maximum degree at most d. Let t be the number of vertices of degree d, and let s be the number of vertices of degree $d - 1$ that have a neighbor of degree at least $d - 1$. Then we may remove at most k edges from H to leave a graph of chromatic index at most $d - 1$, where $k = t$ if $s = 0$ and the vertices of degree d form an independent set, and $k = t + \lfloor \frac{s-1}{2} \rfloor$ otherwise.

We prove this by induction on t. If $t = 0$ and $s = 0$, the statement follows immediately from Theorem 1. If $t = 0$ and $s \geq 1$ then H has maximum degree $d - 1$ and it suffices to remove $\lfloor \frac{s-1}{2} \rfloor$ edges to leave a graph of Class 1. We do this as in the proof of Theorem 4: remove edges between vertices of degree $d - 1$ until at most one such edge remains, leaving a graph of Class 1 by Theorem 2. All such edges lie within the set A of vertices of degree $d - 1$ meeting another vertex of degree $d - 1$, initially $|A| = s$, and each edge removed reduces $|A|$ by 2, so after at most $\lfloor \frac{s-1}{2} \rfloor$ edges are removed we have $|A| \leq 2$ and so $H[A]$ has at most one edge, as required.

Now suppose $t > 0$. If there is an edge uv between two vertices of degree d, remove that edge. If not, let w be an arbitrary vertex of degree d. If w has a neighbor x of degree $d - 1$, remove the edge wx. Otherwise, choose an arbitrary neighbor y and remove the edge wy. In each case we have removed one edge to leave a graph H'; let t' be the number of vertices of H' having degree d, and s' be the number of vertices of degree $d - 1$ that are adjacent to some other vertex of degree at least $d - 1$. We now apply the induction hypothesis to H'.

In the first case, we have $t' = t - 2$ and $s' \leq s + 2$. We may therefore remove at most $t - 2 + \lfloor \frac{s+1}{2} \rfloor$ edges from H' to leave a graph H'' with $\chi'(H'') \leq d - 1$; together with the edge uv, we have removed at most $t + \lfloor \frac{s-1}{2} \rfloor$ edges from H to obtain H''.

In the second case, we have $t' = t - 1$ and $s' \leq s$, since the set of vertices of degree $d - 1$ with a neighbor of degree at least $d - 1$ may gain w, but must lose x. We may therefore remove at most k' edges from H' to leave a graph H'' with $\chi'(H'') \leq d - 1$, where $k' = t' = t - 1$ if $s' = 0$ and $k' = t' + \lfloor \frac{s'-1}{2} \rfloor$ if $s' \geq 1$. It follows that $k' \leq k - 1$ and so we have removed at most k edges from H to obtain H''.

In the final case, we have $t' = t - 1$ and $s' = s$, since $d_{H'}(w) = d - 1$ but w has no neighbor of degree at least $d - 1$, so is not counted in s'. Thus we may remove at most $k' = k - 1$ edges from H' to leave a graph H'' with $\chi'(H'') \leq d - 1$, as required.

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This completes the proof of the stronger statement. The required upper bounds follow by taking $H = G$ and $d = \Delta$, since $\chi'(G) \geq \Delta$. Note that we obtain the stronger bound of $t_\Delta + \frac{t_{\Delta-1}-1}{2}$ unless $t_{\Delta-1} = s = 0$ and G_Δ has no edges.

It remains to show that equality holds if $t_{\Delta-1} = 0$ and G_Δ has no edges. In this case, by Theorem 2, we have $\chi'(G) = \Delta$. In order to reduce the chromatic index to $\Delta - 1$, we must remove at least one edge meeting every vertex of degree Δ. Since no edge meets more than one of these vertices, we have $\mathrm{es}_{\chi'}(G) \geq t_\Delta$, as required. $\qquad\square$

Remark 3. Theorem 5 gives $\mathrm{es}_{\chi'}(G) \leq t_\Delta + \lfloor \frac{t_{\Delta-1}-1}{2} \rfloor$ whenever $t_{\Delta-1} \geq 1$. In fact this is sharp for any values of $t = t_\Delta \geq 1$ and $s = t_{\Delta-1} \geq 1$. To see this (for $s \geq 3$), let H_1 be a Class 2 graph having s vertices of maximum degree, such that $\mathrm{es}_{\chi'}(H_1) = \lfloor \frac{s-1}{2} \rfloor$ (see Remark 1). Let H_2 be a graph with t vertices of degree $\Delta(H_2) = \Delta(H_1) + 1$, forming an independent set, and none of degree $\Delta(H_1)$ (for example, a disjoint union of stars). Then the disjoint union $G = H_1 \cup H_2$ has $\chi'(G) = \Delta(G)$, $t_\Delta = t$, $t_{\Delta-1} = s$, and at least t edges from H_2 and $\lfloor \frac{s-1}{2} \rfloor$ edges from H_2 must be removed to reduce its chromatic index. If $s \leq 2$ then the bound reduces to t_Δ, so the graph H_1 should be omitted (and $\Delta(H_2)$ may be chosen arbitrarily).

Now, we provide a formula for the chromatic edge stability index of bipartite graphs in terms of t_Δ and the matching number of G_Δ.

Theorem 6. *For every bipartite graph G, $\mathrm{es}_{\chi'}(G) = t_\Delta - \alpha'(G_\Delta)$.*

Proof. First, we prove that $\mathrm{es}_{\chi'}(G) \geq t_\Delta - \alpha'(G_\Delta)$. Assume that S' is the smallest mitigating set of G. Note that since G is bipartite, it is Class 1 [6]. Hence, in order to reduce $\chi'(G)$, it is necessary and sufficient that there are no vertices of degree $\Delta(G)$ left in the remaining graph. We now state two claims:

Claim 1. $G[S']$ is acyclic. By contradiction, assume that $G[S']$ contains a cycle C. Consider an edge $e = uv$ in C and let G' be the graph obtained by removing the edges of the path $C - e$ from G. Note that $d_{G'}(u)$ and $d_{G'}(v)$ do not exceed $\Delta(G) - 1$. Hence, by removing $S' - e$ from G, $\chi'(G)$ is reduced, a contradiction.

Claim 2. $G[S']$ contains no path of length 3. Similarly to the previous claim, if $G[S']$ contains a path of length 3, by removing the middle edge of such a path from S', we get a subset of the edges of size $|S'| - 1$ whose removal reduces $\chi'(G)$, a contradiction.

By the claims above, one can conclude that each component of $G[S']$ is the graph $K_{1,r}$ for some $r \geq 1$.

Assume that the components of $G[S']$ are $S_1, \ldots, S_k$. If S_i is a component with at least two edges and l is a leaf of S_i, then $d_G(l) = \Delta(G)$, because otherwise by removing the edge incident with l from S', we get a subset of edges of size $|S'| - 1$ whose removal reduces $\chi'(G)$, a contradiction. By a similar argument, if S_i is a single edge, at least one of its endpoints has degree $\Delta(G)$ in G. Now, assume that in exactly r components of $G[S']$, all

vertices have degree $\Delta(G)$ in G. Hence, if we take one edge of each of these components, we get a matching of size r in G_Δ. Therefore, $\alpha'(G_\Delta) \geq r$. Note that $G[S']$ contains all vertices of degree $\Delta(G)$ in G. Since $t_\Delta = \sum_{i=1}^{k} |E(S_i)| + r$, we have the following:

$$\mathrm{es}_{\chi'}(G) = |S'| = \sum_{i=1}^{k} |E(S_i)| = t_\Delta - r \geq t_\Delta - \alpha'(G_\Delta).$$

Now, we prove that $\mathrm{es}_{\chi'}(G) \leq t_\Delta - \alpha'(G_\Delta)$. Consider a maximum matching M in G_Δ. There are $t_\Delta - 2|M|$ vertices of degree $\Delta(G)$ in G which are not saturated by M. Let G' be the graph obtained by removing the edges of M and one incident edge of each of these vertices from G. Notice that $\Delta(G') \leq \Delta(G) - 1$ and since G' is Class 1, $\chi'(G') = \Delta(G') < \Delta(G) = \chi'(G)$. Hence, we get the following:

$$\mathrm{es}_{\chi'}(G) \leq |M| + t_\Delta - 2|M| = t_\Delta - \alpha'(G_\Delta).$$

The proof is completed. $\qquad\square$

3 Structure of minimum mitigating sets

The mitigating sets constructed in the previous section all have the property that they only contain edges meeting vertices of degree $\Delta(G)$ or $\Delta(G) - 1$. It is natural to ask whether minimum-size mitigating sets (i.e. sets of $\mathrm{es}_{\chi'}(G)$ edges whose removal reduces the chromatic index) necessarily have this property. We will show that the answer is "yes" in the sense that every graph has a minimum-size mitigating set with this property, but "no" in the sense that some graphs also have minimum-size mitigating sets that do not satisfy this property.

Theorem 7. *Let G be any graph. Then there exists a mitigating set S for G of size $\mathrm{es}_{\chi'}(G)$ such that if $e = uv \in S$ then*

$$\max\{d(u), d(v)\} \geq \Delta(G) - 1. \tag{1}$$

Proof. Let T be any mitigating set of G of size $\mathrm{es}_{\chi'}(G)$, and suppose that T contains an edge $e = uv$ which does not satisfy (1). We will adapt the proof of Vizing's theorem to show that there is some edge uv' with $d(v') \geq \Delta(G) - 1$, such that $T \cup \{uv'\} \setminus \{uv\}$ is also a mitigating set for G. By a sequence of such replacements, we eventually obtain a set S with the required properties.

Write $G' = G \setminus T$. Then, by minimality of T, $\chi'(G') = \chi'(G) - 1$ but $\chi'(G' \cup \{uv\}) = \chi'(G)$. Write k for $\chi'(G')$; note that $k \geq \Delta(G) - 1$. Fix a k-edge-coloring of G'. For any two colors c_1 and c_2, we say that a path is a $c_1 c_2$-*path* if its colors alternate between c_1 and c_2 (starting with either color).

We will construct a sequence $v = v_0, v_1, \ldots, v_r$ of neighbors of u such that $d_G(v_r) \geq d_{G'}(v_r) \geq k$. Since $d_{G'}(u) < d_G(u) < k$, there is a color c which is not used at u, and

similarly there is a colour c_0 which is not used at v_0. However, c_0 must be used at u, and c must be used at v_0, since otherwise we could color uv_0 with one of these colors to obtain a k-edge-coloring of $G' \cup \{uv\}$. Also, there must be a cc_0-path from u to v_0, since otherwise we could swap colours in the cc_0 path meeting u and then color uv_0 with c_0 to obtain a k-edge-coloring of $G' \cup \{uv\}$. Choose the neighbor v_1 such that uv_1 has color c_0.

Suppose we have constructed a sequence $v_0, \ldots, v_i$ of distinct neighbors of u, and a sequence of colors $c_0, \ldots, c_{i-1}$ such that for each $j < i$ the color c is used at v_j, the color c_j is not used at v_j, there is a cc_j-path from u to v_j, and uv_{j+1} has color c_j.

Suppose also that $d_{G'}(v_i) < k$. Then there is some color c_i not used at v_i. The color c_i must be used at u, and c must be used at v_i, since otherwise we could recolor uv_i with one of these colors, and give uv_j color c_j for each $j < i$ to obtain a k-edge-coloring of $G' \cup \{uv\}$.

For each i, consider a maximal cc_i-path P_i meeting u. Since u does not meet c, the first edge of P_i must be the unique edge of color c_i meeting u, and each subsequent edge is uniquely determined since the coloring is proper and P_i is maximal. Thus P_i is uniquely determined. If P_i does not meet v_i, then by swapping colors in P_i and giving uv_j color c_j for each $j \le i$ we obtain a k-edge-coloring of $G' \cup \{uv\}$. Thus v_i lies on P_i, and since v_i only uses one of the two colors it must be the end vertex of P_i. In particular, since $v_i \ne v_j$ it follows that $c_i \ne c_j$ for each $j < i$. Choose the neighbor v_{i+1} such that uv_{i+1} has color c_i; this ensures v_{i+1} is distinct from $v_0, \ldots, v_i$.

Thus the sequence can be extended until we have $d_{G'}(v_r) \ge k \ge \Delta - 1$. Now we may remove the edge uv_r and give uv_j color c_j for each $j < r$, to obtain a k-edge-coloring of $G' \cup \{uv\} \setminus \{uv_r\}$. Thus $T \cup \{uv_r\} \setminus \{uv\}$ is also a mitigating set of G, but has fewer edges violating (1). By choosing S to minimise the number of edges violating (1), we must have (1) holding for all edges of S, as required. $\qquad\square$

To complete this section, we show the following.

Proposition 8. *There exists a graph G having a mitigating set S of size $\mathrm{es}_{\chi'}(G)$ and an edge $uv \in S$ which does not satisfy (1).*

Proof. Construct a nine-vertex graph G' as follows. Let $V(G') = A \cup B \cup C \cup \{u, v\}$, with $|A| = |C| = 2$ and $|B| = 3$. Add all edges between u and A, between A and B, between B and C, and between C and v. Now define a graph G by adding two edges to G': the edges uv and xy, where x and y may be any two vertices in $A \cup B \cup C$ that are not adjacent in G'. (There are several possible choices for the second edge, giving three non-isomorphic possibilities for G.)

Figure 4 shows a 4-edge-colouring for G'. Since $\Delta(G) = 5$ we clearly have $\chi'(G) \ge 5$, and indeed the colouring shown may be extended to a 5-edge-colouring of G by using a fifth color on the two extra edges.
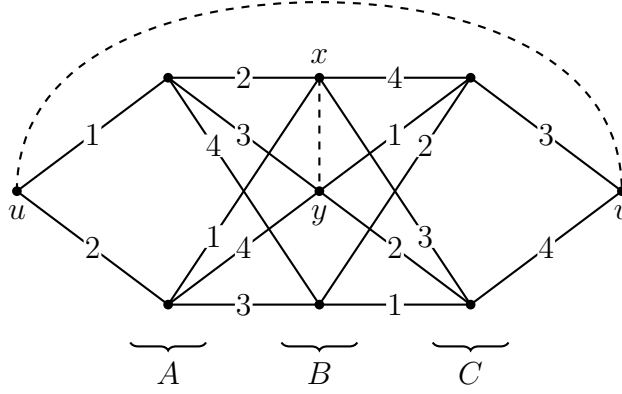
Figure 4. The graphs G' (solid edges) and one choice of G (solid and dashed edges), with a 4-edge-coloring of G.

The set $S = \{uv, xy\}$ thus shows that $\mathrm{es}_{\chi'}(G) \leq 2$, and since uv does not satisfy (1), it is sufficient to show that S is a minimum mitigating set, i.e. $\mathrm{es}_{\chi'}(G) = 2$.

Suppose that this is not the case, so that $\chi'(G \setminus \{e\}) = 4$ for some $e \in E(G)$. In order for this to happen, certainly we need $\Delta(G \setminus \{e\}) \leq 4$, and so we must have $e = xy$. Therefore $G \setminus \{e\} = G' \cup \{uv\}$ would have to be 4-edge-colorable. Take any 4-edge-coloring of G' with colors c_1, c_2, c_3, c_4; let c_1, c_2 be the colors used at u. Since the vertices in A have degree 4, each color is used the same number of times on edges meeting A. Consequently c_1 and c_2 are used once on edges between A and B, and c_3 and c_4 are used twice on edges between A and B. Similarly, since each color must be used the same number of times on edges meeting B, c_1 and c_2 are used twice on edges between B and C, and c_3 and c_4 are used once on edges between B and C. Finally, since each color must be used the same number of times on edges meeting C, the colors used at v must be c_3 and c_4. Thus there is no way to extend any 4-edge-coloring of G' to a 4-edge-coloring of $G' \cup \{uv\}$, which completes the proof. $\square$

Remark 4. The same proof works with $|A| = |C| = k$ and $|B| = 2k - 1$ to give an infinite sequence of graphs with this property.

Acknowledgements

References

[1] S. Akbari, A. Beikmohammadi, B. Brešar, T. Gologranc, M. M. Habibollahi, N. Movarraei, On the chromatic edge stability index of graphs. European J. Combin., 111 (2023), 103690, 16 pp.

[2] S. Akbari, A. Beikmohammadi, S. Klavžar, N. Movarraei, On the chromatic vertex stability number of graphs, European J. Combin., 102 (2022), 103504, 7 pp.

[3] S. Akbari, S. Klavžar, N. Movarraei, M. Nahvi, Nordhaus-Gaddum and other bounds for the chromatic edge-stability number, European J. Combin., 84 (2020), 103042, 8 pp.

[4] A.G. Chetwynd and A.J.W. Hilton, Regular graphs of high degree are 1-factorizable, Proc. London Math. Soc., 50 (1985), 193–206.

[5] J.-C. Fournier. Coloration des aretes dun graphe. Cahiers du CERO (Bruxelles), 15 (1973), 311–314.

[6] D. Kőnig, Gráok és alkalmazásuk a determinánsok és a halmazok elméletére, Matematikai és Természettudományi Értesitő, 34 (1916), 104–119.

[7] W. Staton, Edge deletions and the chromatic number, Ars Combin. 10 (1980), 103–106.

[8] M. Stiebitz, D. Scheide, B. Toft, L. M. Favrholdt, Graph edge coloring: Vizing's Theorem and Goldberg's Conjecture, New Jersey, Wiley (2012).

[9] V. G. Vizing, On an estimate of the chromatic class of a p-graph, Diskret. Analiz., 3 (1964), 25–30.