SIMPLE-MINDEDNESS: MUTATION, REDUCTION, TILTING

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Abstract. This is an extended abstract for a talk given at the Oberwolfach workshop "Representation Theory of Quivers and Finite-Dimensional Algebras" (12-18 February, 2023). It is based on joint work with N. Broomhead, R. Coelho Simões, D. Ploog, J. Woolf and A. Zvonareva.

1. Setting

Throughout, D will be a Hom-finite, Krull-Schmidt, $\mathbf{k}$-linear triangulated category with shift functor $\Sigma\colon \mathsf{D} \to \mathsf{D}$. For simplicity, we will assume that $\mathbf{k}$ is an algebraically closed field. When D has a Serre functor, it will be denoted $\mathbb{S}\colon \mathsf{D} \to \mathsf{D}$. If X is a subcategory or collection of objects of D then ${}^{\perp}\mathsf{X} = \{d \in D \mid \mathrm{Hom}_{\mathsf{D}}(d,x) = 0 \,\forall x \in \mathsf{X}\}$ and $\mathsf{X}^{\perp} = \{d \in D \mid \mathrm{Hom}_{\mathsf{D}}(x,d) = 0 \,\forall x \in \mathsf{X}\}$.

2. Tilting and (co-)t-structures

A *torsion pair* in D is a pair of full subcategories (X,Y), each closed under direct summands, such that

(1) $\mathrm{Hom}_{\mathsf{D}}(x,y) = 0$ for each $x \in \mathsf{X}$ and $y \in \mathsf{Y}$;
(2) $\mathsf{D} = \mathsf{X} * \mathsf{Y} = \{d \in \mathsf{D} \mid \text{there exists a triangle } x \to d \to y \to \Sigma x \text{ with } x \in \mathsf{X} \text{ and } y \in \mathsf{Y}\}$.

A torsion pair (X,Y) is called a *t-structure* if $\Sigma\mathsf{X} \subset \mathsf{X}$ and $\Sigma^{-1}\mathsf{Y} \subset \mathsf{Y}$, and is called a *co-t-structure* if $\Sigma^{-1}\mathsf{X} \subset \mathsf{X}$ and $\Sigma\mathsf{Y} \subset \mathsf{Y}$. The subcategory X is called the *aisle* of the torsion pair and the subcategory Y is called the *co-aisle*. If (X,Y) then its *heart*, $\mathsf{H} = \mathsf{X} \cap \Sigma\mathsf{Y}$ is an abelian subcategory of D. A t-structure (X,Y) is called *bounded* if

$$\mathsf{D} = \bigcup_{i \geq j} \Sigma^i \mathsf{H} * \Sigma^{i-1}\mathsf{H} * \cdots * \Sigma^j \mathsf{H}.$$

Since $\mathsf{H} \subset \mathsf{X}$ and $\mathsf{H} \subset \Sigma\mathsf{Y}$ we have $\mathrm{Hom}_{\mathsf{D}}(\Sigma^i h_1, h_2) = 0$ for each $h_1, h_2 \in \mathsf{H}$ and $i > 0$.

A *torsion pair* in an abelian category H is a pair of full subcategories $(\mathcal{T},\mathcal{F})$ such that $\mathrm{Hom}_{\mathsf{H}}(t,f) = 0$ for each $t \in \mathcal{T}$ and $f \in \mathcal{F}$, and $\mathsf{H} = \mathcal{T} * \mathcal{F} = \{h \in \mathsf{H} \mid \text{there exists a short exact sequence } 0 \to t \to h \to f \to 0 \text{ with } t \in \mathcal{T} \text{ and } f \in \mathcal{F}\}$.

Theorem 2.1 ([6, Proposition 2.1])**.** *Let (X,Y) be a t-structure in D with heart H. Suppose $(\mathcal{T},\mathcal{F})$ is a torsion pair in H. Then $\big(\mathsf{X} * \Sigma^{-1}\mathcal{T}, \Sigma^{-1}(\mathcal{F} * \mathsf{Y})\big)$ is a t-structure in D with heart $\mathsf{K} = \mathcal{F} * \Sigma^{-1}\mathcal{T}$; see Figure 1.*

A subcategory X of D is *contravariantly finite* if each object $d \in \mathsf{D}$ admits a morphism $f\colon x_d \to d$ such that $\mathrm{Hom}_{\mathsf{D}}(x,f)\colon \mathrm{Hom}_{\mathsf{D}}(x,x_d) \twoheadrightarrow \mathrm{Hom}_{\mathsf{D}}(x,d)$ is surjective for each $x \in \mathsf{X}$.

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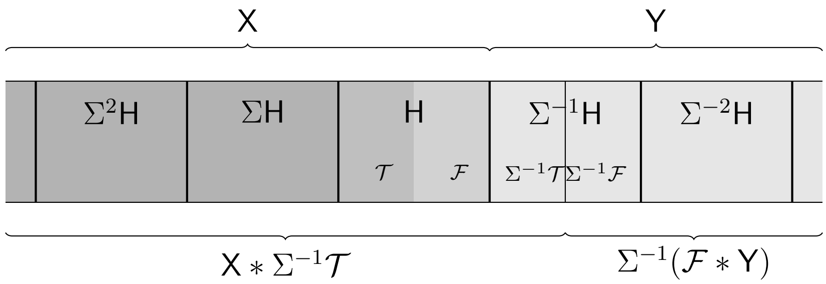

FIGURE 1. Schematic showing the t-structure (X, Y) and the Happel-Reiten-Smalø tilted t-structure $\big(\mathsf{X} * \Sigma^{-1}\mathcal{T}, \Sigma^{-1}(\mathcal{F} * \mathcal{Y})\big)$ at the torsion pair $(\mathcal{T}, \mathcal{F})$ in the heart $\mathsf{H} = \mathsf{X} \cap \Sigma\mathsf{Y}$.

Covariantly finite subcategories are defined dually and a subcategory is *functorially finite* if it is contravariantly and covariantly finite.

Theorem 2.2 ([4, Corollary 2.8])**.** *Suppose* D *is a Hom-finite, Krull-Schmidt, saturated triangulated category. Let* (X, Y) *be a bounded t-structure in* D *with heart* H*. The following are equivalent:*

(1) H *is contravariantly finite (resp. covariantly finite) in* D*.*
(2) H *has enough injectives (resp. projectives).*
(3) (X, Y) *has a right (resp. left) adjacent co-t-structure, i.e. there is a co-t-structure* $(\mathsf{Y}, \mathsf{Y}^{\perp})$ *(resp.* $({}^{\perp}\mathsf{X}, \mathsf{X})$*).*

Prototypical examples of saturated triangulated categories are $\mathsf{D}^b(\mathsf{mod}\,A)$ and $\mathsf{D}^b(\mathsf{coh}\,X)$, where A is a finite-dimensional algebra of finite global dimension and X is a smooth projective variety. A more technical version of Theorem 2.2 is true without the restriction that D is saturated in [4, Theorem 2.4].

Corollary 2.3. *Suppose* D *is a Hom-finite, Krull-Schmidt, saturated triangulated category. If* H *is functorially finite in* D *and* $(\mathcal{T}, \mathcal{F})$ *is a torsion pair in which* $\mathcal{T}$ *and* $\mathcal{F}$ *are functorially finite, then the HRS-tilted heart* $\mathsf{K} = \mathcal{F} * \Sigma^{-1}\mathcal{T}$ *is also functorially finite in* D*.*

Proof. If (X, Y) is a t-structure, then $({}^{\perp}\mathsf{X}, \mathsf{X})$ is a co-t-structure if and only if X is functorially finite in D. If H is functorially finite in D then so is the torsion class $\mathcal{T}$. In particular, by [13, Lemma 5.3], $\mathsf{X} * \Sigma^{-1}\mathcal{T}$ is functorially finite in D. Hence, $\big({}^{\perp}(\mathsf{X} * \Sigma^{-1}\mathcal{T}), \mathsf{X} * \Sigma^{-1}\mathcal{T}\big)$ is a co-t-structure in D. One argues similarly with the torsionfree class. $\square$

3. Simple-minded objects

Definition 3.1. A collection of objects S of D is an *orthogonal collection*, if for each $s_1, s_2 \in \mathsf{S}$ Schur's lemma holds, i.e.

$$\mathrm{Hom}_{\mathsf{D}}(s, t) = \begin{cases} \mathbf{k} & \text{if } s_1 \simeq s_2, \\ 0 & \text{otherwise.} \end{cases}$$

An orthogonal collection S is a *simple-minded collection* (*SMC*) [12] if

(1) it is an ∞-*orthogonal collection*, i.e. $\mathrm{Hom}_{\mathsf{D}}(\Sigma^i s_1, s_2) = 0$ for each $i > 0$ and $s_1, s_2 \in \mathsf{S}$, and,

(2) $\mathsf{D} = \bigcup_{i\geq j} \Sigma^i\langle\mathsf{S}\rangle * \Sigma^{i-1}\langle\mathsf{S}\rangle * \cdots * \Sigma^j\langle\mathsf{S}\rangle$, i.e. $\langle\mathsf{S}\rangle$ is the heart of a bounded t-structure in D.

For $w \geq 1$, S is a *w-simple-minded system* (*w-SMS*) [2, 11] if

(1) it is a *w-orthogonal collection*, i.e. $\mathrm{Hom}_{\mathsf{D}}(\Sigma^i s_1, s_2) = 0$ for each $1 \leq i \leq w-1$ and $s_1, s_2 \in \mathsf{S}$, and,
(2) $\mathsf{D} = \Sigma^{w-1}\langle\mathsf{S}\rangle * \cdots * \Sigma\langle\mathsf{S}\rangle * \langle\mathsf{S}\rangle$

Theorem 3.2 ([5, Theorem 3.3])**.** *Let S be an orthogonal collection of D and suppose $\mathsf{T} \subseteq \mathsf{S}$. Then $\langle\mathsf{T}\rangle$ is functorially finite in $\langle\mathsf{S}\rangle$.*

Remark 3.3. If S is a w-SMS in D, then, as a consequence of condition (2) in the definition, $\langle\mathsf{S}\rangle$ is functorially finite in D, see [3, Corollary 2.9]. By Theorem 3.2 it follows that if $\mathsf{T} \subseteq \mathsf{S}$ then $\langle\mathsf{T}\rangle$ is also functorially finite in D. In particular, functorial finiteness of the extension closure of a w-orthogonal collection is a necessary condition for that collection to occur as a subcollection of a w-SMS.

If S is an orthogonal collection and $\mathsf{T} \subseteq \mathsf{S}$ then $\big(\langle\mathsf{T}\rangle, \mathsf{T}^{\perp} \cap \langle\mathsf{S}\rangle\big)$ and $\big({}^{\perp}\mathsf{T} \cap \langle\mathsf{S}\rangle, \langle\mathsf{T}\rangle\big)$ are "torsion pairs" in $\langle\mathsf{S}\rangle$ with functorially finite "torsion class" and functorially finite "torsionfree class", respectively. In the case that S is an SMC, then the two "torsion pairs" above are genuine torsion pairs in the abelian sense.

4. Reduction and mutation

Let T be an orthogonal collection and U be a collection of objects of D. Provided that $\langle\mathsf{T}\rangle$ is functorially finite in D, we can define two mutation operations on U with respect to T. The *right mutation* of U at T is obtained by taking for each object $u \in \mathsf{U}$ a minimal right $\langle\mathsf{T}\rangle$-approximation $u_t \to \Sigma u$ and extending it a distinguished triangle,

$$t_u \to \Sigma u \to \mathsf{R}_{\mathsf{T}}(u) \to \Sigma t_u,$$

and setting $\mathsf{R}_{\mathsf{T}}(\mathsf{U}) = \{\mathsf{R}_{\mathsf{T}}(u) \mid u \in \mathsf{U}\}$. *Left mutation* is defined analogously, see [3] for precise details.

In analogy with [8] for cluster-tilting/silting mutation, in [3] a pair of collections of objects (U, V) is called a T-*mutation pair* if $\mathsf{U} = \mathsf{L}_{\mathsf{T}}(\mathsf{V})$ and $\mathsf{V} = \mathsf{R}_{\mathsf{T}}(\mathsf{U})$.

When T is a subcollection of a w-SMS, the extension closure $\langle\mathsf{T}\rangle$ is automatically functorially finite by Theorem 3.2 and as such mutation is always defined. However, if T is a subcollection of an SMC this is not automatic. This motivates the following definition, which permits us to discuss mutation of SMCs.

Definition 4.1. An SMC S in D is called *strong* if $\langle\mathsf{S}\rangle$ is functorially finite in D.

Theorem 4.2 ([3, Theorems 4.1 & 5.1])**.** *Suppose T is an orthogonal collection such that*

(1) $\langle\mathsf{T}\rangle$ is functorially finite in D; and,
(2) $\mathbb{S}\Sigma\mathsf{T} = \mathsf{T}$ or $\mathrm{Hom}_{\mathsf{D}}(\Sigma t_1, t_2) = 0$ for each $t_1, t_2 \in \mathsf{T}$.

Let Z be a subcategory of D such that (Z, Z) is an T-mutation pair satisfying,

(Z1) *Z is closed under extensions and direct summands;*
(Z2) *the cones in D of maps in Z lie in $\langle\mathsf{T}\rangle * \mathsf{Z}$; and*
(Z3) *the cocones in D of maps in Z lie in $\mathsf{Z} * \langle\mathsf{T}\rangle$.*

Then there is a functor $\langle 1\rangle\colon \mathsf{Z} \to \mathsf{Z}$ and for each morphism $f\colon x \to y$ in Z there is a diagram $x \xrightarrow{f} y \longrightarrow z_f \longrightarrow x\langle 1\rangle$ giving rise to a class of triangles Δ which makes D into a triangulated category.

The key point is that the shift functor $\langle 1\rangle\colon \mathsf{Z} \to \mathsf{Z}$ is defined via the right mutation formula with respect to T. In particular, if $\mathsf{T} = \{0\}$ then $\langle 1\rangle = \Sigma$.

This result allows one to obtain a reduction result for w-SMSs and SMCs analogous to the reduction results for w-cluster-tilting subcategories and silting subcategories obtained in [1, 7, 8]. We state the result fo w-SMSs and SMCs together. The result for w-SMSs is due to [3, Theorem 6.6] and the result for SMCs is due to [9, Theorem 3.1]. An alternative proof in the SMC case in the same spirit as the SMS case is given in [4, Theorem A.2] of the appendix to that article.

Theorem 4.3. *Let T be a w-orthogonal (resp. ∞-orthogonal) collection and*

$$\mathsf{Z} = \begin{cases} \{d \in \mathsf{D} \mid \mathrm{Hom}_{\mathsf{D}}(\Sigma^i t, d) = 0\, \forall t \in \mathsf{T} \textit{ and } 0 \leq i \leq w\} & \textit{if } \mathsf{T} \textit{ is } w\textit{-orthogonal;} \\ \mathsf{Z} = {}^{\perp}(\Sigma^{\leq 0}\mathsf{T}) \cap (\Sigma^{\geq 0}\mathsf{T})^{\perp}) & \textit{if } \mathsf{T} \textit{ is } \infty\textit{-orthogonal.} \end{cases}$$

Then, (Z, Z) is a T-mutation pair satisfying the hypotheses of Theorem 4.2. Moreover, there is bijection,

$$\{w\text{-SMSs (resp. SMCs) in } \mathsf{D} \text{ containing } \mathsf{T}\} \xleftrightarrow{1-1} \{w\text{-SMSs (resp. SMCs) in } \mathsf{Z}\}.$$

The key observation in this theorem is that a right mutation on the left-hand side of the bijection corresponds to a shift on the right-hand side of the bijection. Therefore, the question of whether the mutation of a w-SMS or an SMC is again a w-SMS or an SMC boils down to asking whether the shift of a w-SMS or an SMC is again a w-SMS or an SMC, which is tautologous. The following theorem recovers [10, Theorem 6.3] in the case of w-SMSs and generalises the SMC mutation theory for derived categories of finite-dimensional algebras of [12].

Theorem 4.4. *Let T be a w-orthogonal (resp. ∞-orthogonal) collection such that $\langle \mathsf{T}\rangle$ is functorially finite in D. Suppose (U, V) is a T-mutation pair. Then $\mathsf{U} \cup \mathsf{T}$ is a w-SMS (resp. strong SMC) if and only if $\mathsf{V} \cup \mathsf{T}$ is a w-SMS (resp. strong SMC).*

Remark 4.5. Corollary 2.3 says that tilting a functorially finite aisle (resp. coaisle) at a functorially finite torsion (resp. torsionfree) class produces another functorially finite aisle (resp. coaisle). That is, the property of having an adjacent co-t-structure is preserved by tilting at functorially finite torsion pairs providing a conceptual homological explanation behind the Koenig-Yang correspondences and their compatibility with mutation.

Tilting at torsion pairs whose torsion (resp. torsionfree) class is generated by a subset of simple objects is simple-minded mutation. In particular, simple tilts of length hearts with enough projectives and enough injectives produce length hearts with enough projectives and enough injectives. This means that "algebraic" hearts are well behaved within the space of Bridgeland stability conditions.

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