

EQUIVALENCE AFTER EXTENSION AND SCHUR COUPLING FOR FREDHOLM OPERATORS ON BANACH SPACES

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ABSTRACT. Schur coupling (SC) and equivalence after extension (EAE) are important relations for bounded operators on Banach spaces. It has been known for 30 years that the former implies the latter, but only recently Ter Horst, Messerschmidt, Ran and Roelands disproved the converse by constructing a pair of Fredholm operators which are EAE, but not SC.

Motivated by this result, we investigate when EAE and SC coincide for Fredholm operators. Fredholm operators which are EAE have the same Fredholm index. Surprisingly, we find that for each integer k and every pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$, either *no* pair of Fredholm operators of index k acting on \mathcal{X} and \mathcal{Y} , respectively, is SC, or *every* pair of this kind which is EAE is also SC. Consequently, whether EAE and SC coincide for Fredholm operators of index k depends only on the geometry of the underlying Banach spaces \mathcal{X} and \mathcal{Y} , not on the properties of the operators themselves.

We quantify this finding by introducing two numerical indices which capture the coincidence of EAE and SC and provide a number of examples illustrating the possible values of these indices. Notably, this includes an example showing that the above-mentioned result of Ter Horst *et al.*, which is based on a pair of essentially incomparable Banach spaces, does not extend to projectively incomparable Banach spaces.

1. INTRODUCTION

Equivalence after extension (EAE), matricial coupling (MC) and Schur coupling (SC) are three relations for bounded operators on Banach spaces that originate in the study of Wiener–Hopf integral operators [5] and have found numerous applications since. A key feature in many of these applications is that the three relations coincide. This observation led Bart and Tsekanovskii [9] to ask whether this is always true. They already knew that EAE and MC are equivalent [5, 7] and that SC implies EAE [8, 9], so their precise question was whether EAE implies SC.

Despite numerous results confirming this implication in special cases [8, 9, 6, 33, 54, 29, 32], recently Ter Horst, Messerschmidt, Ran and Roelands [31] showed that EAE does not in general imply SC. Their counterexample uses bounded operators defined on a pair of Banach spaces which is essentially incomparable, in which case EAE (and hence SC) can occur only for Fredholm operators, while SC additionally requires that the operators have index zero. By contrast, it is known that EAE

2010 *Mathematics Subject Classification.* Primary 47A62, 47B01; Secondary 46B03, 46B20, 47A53.

Key words and phrases. Equivalence after extension, Schur coupling, Fredholm operators, incomparable Banach spaces, Gowers–Maurey spaces.

This work is based on the research supported in part by the National Research Foundation of South Africa (Grant Numbers 90670 and 118513).

and SC coincide for Fredholm operators acting on isomorphic Banach spaces [30, Proposition 6.1(iv)].

These results motivated the present work, in which we study EAE and SC for Fredholm operators without imposing any restrictions on the underlying Banach spaces. Further justification for focussing on Fredholm operators comes from the prominent role this class plays in many applications of the theory, as the following studies from the last decade evidence: diffraction problems [15, 51]; Wiener–Hopf factorization [49, 28] and invertibility of Wiener–Hopf plus Hankel operators [16]; truncated Toeplitz operators [14, 38]; Riemann–Hilbert problems [13]; Helmholtz and Sylvester equations [48, 50] and [18], respectively; completeness theorems for integral and differential operators [35]; and problems concerning electrical networks [10].

Before we state our main results, let us introduce some notation and terminology, most of which is standard. We follow the convention that $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Let \mathcal{X} and \mathcal{Y} be Banach spaces, either real or complex, with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ denoting the scalar field. The term “operator” means a bounded linear map between Banach spaces, and $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ denotes the Banach space of all operators from \mathcal{X} to \mathcal{Y} . As usual, $\mathcal{B}(\mathcal{X}, \mathcal{X})$ is abbreviated $\mathcal{B}(\mathcal{X})$. This convention applies whenever we consider sets of operators: once a subset $\Sigma(\mathcal{X}, \mathcal{Y})$ of $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ has been defined, we write $\Sigma(\mathcal{X})$ instead of $\Sigma(\mathcal{X}, \mathcal{X})$.

The identity operator on a Banach space \mathcal{X} is denoted by $I_{\mathcal{X}}$, while the kernel and the range of an operator T are denoted by $\ker T$ and $\operatorname{ran} T$, respectively. Two Banach spaces \mathcal{X} and \mathcal{Y} are *isomorphic*, written $\mathcal{X} \cong \mathcal{Y}$, if $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ contains a bijection, called an *isomorphism*. The Banach Isomorphism Theorem ensures that the inverse of an isomorphism is automatically bounded.

Definition 1.1. Let $U \in \mathcal{B}(\mathcal{X})$ and $V \in \mathcal{B}(\mathcal{Y})$. We say that:

- (i) U and V are *equivalent after extension*, abbreviated EAE, if there exist Banach spaces \mathcal{X}_0 and \mathcal{Y}_0 and isomorphisms $E \in \mathcal{B}(\mathcal{Y} \oplus \mathcal{Y}_0, \mathcal{X} \oplus \mathcal{X}_0)$ and $F \in \mathcal{B}(\mathcal{X} \oplus \mathcal{X}_0, \mathcal{Y} \oplus \mathcal{Y}_0)$ such that

$$\begin{bmatrix} U & 0 \\ 0 & I_{\mathcal{X}_0} \end{bmatrix} = E \begin{bmatrix} V & 0 \\ 0 & I_{\mathcal{Y}_0} \end{bmatrix} F. \quad (1.1)$$

- (ii) U and V are *Schur coupled*, abbreviated SC, if there exist isomorphisms $A \in \mathcal{B}(\mathcal{X})$ and $D \in \mathcal{B}(\mathcal{Y})$ and operators $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that

$$U = A - BD^{-1}C \quad \text{and} \quad V = D - CA^{-1}B. \quad (1.2)$$

As noted above, whenever U and V are SC, they are also EAE. Motivated by many applications in which the converse implication holds true, Bart and Tsekanovskii asked the following question in [9].

Question 1.2. Under which conditions on the Banach spaces \mathcal{X} and \mathcal{Y} and/or on the operators U and V is it true that whenever the operators $U \in \mathcal{B}(\mathcal{X})$ and $V \in \mathcal{B}(\mathcal{Y})$ are EAE, they are also SC?

We shall address this question in the case where U and V are Fredholm operators. Before doing so, let us recall some basic facts about this class of operators. An operator $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is called a *Fredholm operator* if the quantities

$$\alpha(T) = \dim \ker T \quad \text{and} \quad \beta(T) = \dim \mathcal{Y} / \operatorname{ran} T$$

are both finite. The latter condition implies that $\text{ran } T$ is closed. As usual, we write $\Phi(\mathcal{X}, \mathcal{Y})$ for the subset of $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ consisting of Fredholm operators. The *index* of a Fredholm operator T is defined by

$$i(T) = \alpha(T) - \beta(T) \in \mathbb{Z},$$

and for $k \in \mathbb{Z}$, $\Phi_k(\mathcal{X}, \mathcal{Y})$ denotes the set of $T \in \Phi(\mathcal{X}, \mathcal{Y})$ such that $i(T) = k$.

In the 1990s, Bart and Tsekanovskii gave the following characterization of equivalence after extension for Fredholm operators; see [8, Theorem 4], and also [6, Theorem 6, page 211].

Theorem 1.3. *Let $U \in \mathcal{B}(\mathcal{X})$ and $V \in \mathcal{B}(\mathcal{Y})$ for some Banach spaces \mathcal{X} and \mathcal{Y} .*

- (i) *Suppose that U and V are EAE. Then U is a Fredholm operator if and only if V is a Fredholm operator.*
- (ii) *Suppose that U and V are Fredholm operators. Then U and V are EAE if and only if*

$$\alpha(U) = \alpha(V) \quad \text{and} \quad \beta(U) = \beta(V).$$

In particular, Fredholm operators which are EAE have the same index.

As a consequence, the following sets, defined for every $k \in \mathbb{Z}$ and every pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$, provide the natural setting in which to study Question 1.2 for Fredholm operators:

$$\begin{aligned} \text{EAE}_k(\mathcal{X}, \mathcal{Y}) &= \{(U, V) \in \Phi_k(\mathcal{X}) \times \Phi_k(\mathcal{Y}) : U \text{ and } V \text{ are EAE}\}, \\ \text{SC}_k(\mathcal{X}, \mathcal{Y}) &= \{(U, V) \in \Phi_k(\mathcal{X}) \times \Phi_k(\mathcal{Y}) : U \text{ and } V \text{ are SC}\}. \end{aligned} \quad (1.3)$$

In view of Theorem 1.3(ii), the former set can alternatively be written as

$$\text{EAE}_k(\mathcal{X}, \mathcal{Y}) = \{(U, V) \in \Phi_k(\mathcal{X}) \times \Phi_k(\mathcal{Y}) : \alpha(U) = \alpha(V)\}. \quad (1.4)$$

These sets are useful in our investigation because they allow us to express the statement that EAE and SC are equivalent for every pair of Fredholm operators of index k on \mathcal{X} and \mathcal{Y} , respectively, in the concise form $\text{SC}_k(\mathcal{X}, \mathcal{Y}) = \text{EAE}_k(\mathcal{X}, \mathcal{Y})$, where we note that the inclusion $\text{SC}_k(\mathcal{X}, \mathcal{Y}) \subseteq \text{EAE}_k(\mathcal{X}, \mathcal{Y})$ is always true because SC implies EAE.

Using this notation, we can easily state two important results that motivated our work. First, the answer to Question 1.2 is always affirmative for Fredholm operators of index 0 (see [8, Theorem 3] and [6, Theorem 5]). In the above notation, this simply means that

$$\text{SC}_0(\mathcal{X}, \mathcal{Y}) = \text{EAE}_0(\mathcal{X}, \mathcal{Y}) \quad (1.5)$$

for every pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$.

Second, we can state the seminal result of Ter Horst, Messerschmidt, Ran and Roelands [31] showing that there are pairs of Fredholm operators which are EAE, but not SC. This requires the following piece of terminology.

Definition 1.4. A pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$ is *essentially incomparable* if $I_{\mathcal{X}} - ST \in \Phi(\mathcal{X})$ for every $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$.

Theorem 1.5. (i) *Let $(\mathcal{X}, \mathcal{Y})$ be a pair of essentially incomparable Banach spaces. Then $U \in \mathcal{B}(\mathcal{X})$ and $V \in \mathcal{B}(\mathcal{Y})$ are SC if and only if $(U, V) \in \text{EAE}_0(\mathcal{X}, \mathcal{Y})$.*

- (ii) *There exists a pair of essentially incomparable Banach spaces $(\mathcal{X}, \mathcal{Y})$ such that*

$$\text{EAE}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset \quad \text{for every } k \in \mathbb{Z}.$$

Hence EAE and SC are not equivalent for Fredholm operators of non-zero index on such Banach spaces.

An example is given by $\mathcal{X} = \ell_p$ and $\mathcal{Y} = \ell_q$ for $1 \leq p < q < \infty$.

The significance of analyzing whether SC and EAE are equivalent for each value k of the Fredholm index separately will become clear from the next result, which is the first main outcome of our work. To state it concisely, we introduce a numerical index $\text{eae}(\mathcal{X}, \mathcal{Y})$ as follows: set $\mathbb{I}_\Phi(\mathcal{X}) = \{k \in \mathbb{Z} : \Phi_k(\mathcal{X}) \neq \emptyset\}$ and then define

$$\text{eae}(\mathcal{X}, \mathcal{Y}) = \begin{cases} 0 & \text{if } \mathbb{I}_\Phi(\mathcal{X}) \cap \mathbb{I}_\Phi(\mathcal{Y}) \cap \mathbb{N} = \emptyset, \\ \min \mathbb{I}_\Phi(\mathcal{X}) \cap \mathbb{I}_\Phi(\mathcal{Y}) \cap \mathbb{N} & \text{otherwise.} \end{cases} \quad (1.6)$$

Theorem 1.6. *Let $k \in \mathbb{Z}$, and let \mathcal{X} and \mathcal{Y} be Banach spaces.*

- (i) *$\text{EAE}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset$ if and only if k is a multiple of $\text{eae}(\mathcal{X}, \mathcal{Y})$.*
- (ii) *Suppose that k is a multiple of $\text{eae}(\mathcal{X}, \mathcal{Y})$. Then $\text{SC}_k(\mathcal{X}, \mathcal{Y}) = \text{EAE}_k(\mathcal{X}, \mathcal{Y})$ if and only if $\text{SC}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset$.*
- (iii) *Suppose that k is not a multiple of $\text{eae}(\mathcal{X}, \mathcal{Y})$. Then $\Phi_k(\mathcal{X}) = \emptyset$ or $\Phi_k(\mathcal{Y}) = \emptyset$, and consequently $\text{SC}_k(\mathcal{X}, \mathcal{Y}) = \text{EAE}_k(\mathcal{X}, \mathcal{Y}) = \emptyset$.*

The most remarkable part of Theorem 1.6 is the implication \Leftarrow in (ii) which, when written out, states that as soon as *one* pair of operators $(U, V) \in \Phi_k(\mathcal{X}) \times \Phi_k(\mathcal{Y})$ is SC, then EAE and SC are equivalent for *all* pairs $(U, V) \in \Phi_k(\mathcal{X}) \times \Phi_k(\mathcal{Y})$. In other words, equivalence of EAE and SC for Fredholm operators depends only on the geometry of the underlying Banach spaces \mathcal{X} and \mathcal{Y} and on the Fredholm index k , not on the Fredholm operators themselves.

In view of Theorem 1.6(ii), it would be of great interest to establish a counterpart of Theorem 1.6(i) for SC. In analogy with (1.6), we introduce the set

$$\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \{k \in \mathbb{Z} : \text{SC}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset\} \quad (1.7)$$

and the associated index

$$\text{sc}(\mathcal{X}, \mathcal{Y}) = \begin{cases} 0 & \text{if } \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) \cap \mathbb{N} = \emptyset, \\ \min \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) \cap \mathbb{N} & \text{otherwise.} \end{cases} \quad (1.8)$$

Combining Theorem 1.6(i) with the inclusion $\text{SC}_k(\mathcal{X}, \mathcal{Y}) \subseteq \text{EAE}_k(\mathcal{X}, \mathcal{Y})$, we see that $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) \subseteq \text{eae}(\mathcal{X}, \mathcal{Y})\mathbb{Z}$. By Theorem 1.6(ii)–(iii), our main question — whether EAE and SC are equivalent for every pair of Fredholm operators on \mathcal{X} and \mathcal{Y} , respectively — boils down to whether $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \text{eae}(\mathcal{X}, \mathcal{Y})\mathbb{Z}$. We address this question in the following proposition.

Proposition 1.7. *Let \mathcal{X} and \mathcal{Y} be Banach spaces. Then $\text{SC}_k(\mathcal{X}, \mathcal{Y}) = \text{EAE}_k(\mathcal{X}, \mathcal{Y})$ for every $k \in \mathbb{Z}$ if and only if $\text{sc}(\mathcal{X}, \mathcal{Y}) = \text{eae}(\mathcal{X}, \mathcal{Y})$.*

In general, $\text{sc}(\mathcal{X}, \mathcal{Y}) = n \text{eae}(\mathcal{X}, \mathcal{Y})$ for some $n \in \mathbb{N}_0$, and the following chain of inclusions holds:

$$\begin{aligned} \text{sc}(\mathcal{X}, \mathcal{Y})\mathbb{Z} &\subseteq \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \{k \in \mathbb{Z} : \text{SC}_k(\mathcal{X}, \mathcal{Y}) = \text{EAE}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset\} \\ &\subseteq \text{eae}(\mathcal{X}, \mathcal{Y})\mathbb{Z} = \{k \in \mathbb{Z} : \text{EAE}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset\} = \mathbb{I}_\Phi(\mathcal{X}) \cap \mathbb{I}_\Phi(\mathcal{Y}). \end{aligned} \quad (1.9)$$

Remark 1.8. The first part of Proposition 1.7 implies that the second inclusion in (1.9) is an equality if and only if $\text{sc}(\mathcal{X}, \mathcal{Y}) = \text{eae}(\mathcal{X}, \mathcal{Y})$, in which case the first inclusion is also an equality. In fact, we do not know any instances where the first inclusion in (1.9) is proper, and we conjecture that it may always be an equality; see Section 5 for a more detailed discussion of this question.

We conclude this overview of our main findings with some results that illustrate the values which the numerical indices $\text{eae}(\mathcal{X}, \mathcal{Y})$ and $\text{sc}(\mathcal{X}, \mathcal{Y})$ can take when various incomparability conditions are imposed on the Banach spaces \mathcal{X} and \mathcal{Y} . This work is motivated by, and closely related to, the seminal result of Ter Horst, Messerschmidt, Ran and Roelands that we stated in Theorem 1.5. We begin with a result whose first part is simply a restatement of Theorem 1.5(i), while its second part contains Theorem 1.5(ii) as a special case, corresponding to $k_0 = 1$.

Theorem 1.9. (i) *Let $(\mathcal{X}, \mathcal{Y})$ be a pair of essentially incomparable Banach spaces. Then $\text{sc}(\mathcal{X}, \mathcal{Y}) = 0$.*
(ii) *For every $k_0 \in \mathbb{N}_0$, there exists a pair of essentially incomparable Banach spaces $(\mathcal{X}, \mathcal{Y})$ such that $\text{eae}(\mathcal{X}, \mathcal{Y}) = k_0$.*

Theorem 1.9(i) immediately raises the question: can we weaken the hypothesis that the pair $(\mathcal{X}, \mathcal{Y})$ is essentially incomparable without losing the conclusion that $\text{sc}(\mathcal{X}, \mathcal{Y}) = 0$? The most obvious, very modest weakening would be to assume that \mathcal{X} and \mathcal{Y} are projectively incomparable in the following sense.

Definition 1.10. A pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$ is *projectively incomparable* if no infinite-dimensional, complemented subspace of \mathcal{X} is isomorphic to a complemented subspace of \mathcal{Y} .

However, it turns out that this hypothesis is too weak to imply that $\text{sc}(\mathcal{X}, \mathcal{Y}) = 0$, as our next result will show. It also contains some information about the possible values of the indices $\text{eae}(\mathcal{X}, \mathcal{Y})$ and $\text{sc}(\mathcal{X}, \mathcal{Y})$.

Theorem 1.11. (i) *For every $k_0 \in \mathbb{N}$, there exists a pair of projectively incomparable Banach spaces $(\mathcal{X}, \mathcal{Y})$ such that $\text{sc}(\mathcal{X}, \mathcal{Y}) = \text{eae}(\mathcal{X}, \mathcal{Y}) = k_0$.*
(ii) *For every $k_0 \in \mathbb{N}$, there exists a pair of projectively incomparable Banach spaces $(\mathcal{X}, \mathcal{Y})$ such that $\text{eae}(\mathcal{X}, \mathcal{Y}) = 1$ and $\text{sc}(\mathcal{X}, \mathcal{Y}) = k_0$.*

Theorem 1.11 is highly surprising because the difference between essential and projective incomparability is very subtle, as evidenced by the fact that it took nearly 30 years to find an example which distinguishes them. Indeed, Tarafdar [52, 53] asked in 1972 whether projective incomparability implies essential incomparability, having noted that the converse is true, but it was not until 2000 that Aiena and González answered this question by giving a counterexample (see [2, Proposition 3.7]). It relied on a sophisticated Banach space constructed by Gowers and Maurey [27]. To the best of our knowledge, no simpler examples have subsequently been found. We shall discuss the relationship between essential and projective incomparability in more detail in Section 2.

In view of Theorem 1.11, let us consider what may happen when \mathcal{X} and \mathcal{Y} are *not* projectively incomparable. Then they contain isomorphic, infinite-dimensional complemented subspaces; that is, \mathcal{X} and \mathcal{Y} admit decompositions of the form

$$\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \quad \text{and} \quad \mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2 \quad \text{with} \quad \mathcal{X}_2 \cong \mathcal{Y}_2, \quad (1.10)$$

where \mathcal{X}_2 and \mathcal{Y}_2 are infinite-dimensional. Our next result answers the following question. Suppose that the subspaces \mathcal{X}_1 , \mathcal{X}_2 and \mathcal{Y}_1 in (1.10) are pairwise essentially incomparable. When is $\text{eae}(\mathcal{X}, \mathcal{Y}) = \text{sc}(\mathcal{X}, \mathcal{Y})$? The statement of this result involves the greatest common divisor (gcd) of two quantities that could potentially both be 0, in which case the gcd is not defined. We fix this issue by adopting the convention that $\text{gcd}(0, 0) = 0$.

Proposition 1.12. *Let \mathcal{X} and \mathcal{Y} be Banach spaces that decompose as in (1.10), and suppose that each of the pairs $(\mathcal{X}_1, \mathcal{X}_2)$ and $(\mathcal{Y}_1, \mathcal{Y}_2)$ is essentially incomparable. Then*

$$\text{eae}(\mathcal{X}, \mathcal{Y}) = \text{gcd}(\text{eae}(\mathcal{X}_1, \mathcal{Y}_1), \text{eae}(\mathcal{X}_2, \mathcal{Y}_2)). \quad (1.11)$$

Suppose additionally that the pair $(\mathcal{X}_1, \mathcal{Y}_1)$ is essentially incomparable. Then

$$\text{sc}(\mathcal{X}, \mathcal{Y}) = \text{sc}(\mathcal{X}_2, \mathcal{Y}_2) = \text{eae}(\mathcal{X}_2, \mathcal{Y}_2). \quad (1.12)$$

In particular, EAE and SC coincide for all pairs of Fredholm operators on \mathcal{X} and \mathcal{Y} , i.e., $\text{eae}(\mathcal{X}, \mathcal{Y}) = \text{sc}(\mathcal{X}, \mathcal{Y})$, if and only if $\text{eae}(\mathcal{X}_2, \mathcal{Y}_2)$ divides $\text{eae}(\mathcal{X}_1, \mathcal{Y}_1)$.

Note that we do not demand that the isomorphic subspaces \mathcal{X}_2 and \mathcal{Y}_2 are infinite-dimensional in Proposition 1.12. Therefore part (i) of Theorem 1.9 appears as a special case of Proposition 1.12 corresponding to $\mathcal{X}_2 = \mathcal{Y}_2 = \{0\}$, while the case where \mathcal{X} and \mathcal{Y} are isomorphic is obtained by taking $\mathcal{X}_1 = \mathcal{Y}_1 = \{0\}$.

In analogy with Theorem 1.11, it turns out that the second part of Proposition 1.12 is no longer true if we replace the hypothesis that the pair $(\mathcal{X}_1, \mathcal{Y}_1)$ is essentially incomparable with the weaker hypothesis that it is projectively incomparable.

Theorem 1.13. (i) *For every $k_0 \in \mathbb{N}$, there exist infinite-dimensional Banach spaces \mathcal{X}_1 , \mathcal{Y}_1 and \mathcal{Z} such that:*

- (1) *The pair $(\mathcal{X}_1, \mathcal{Y}_1)$ is projectively incomparable.*
- (2) *The pairs $(\mathcal{X}_1, \mathcal{Z})$ and $(\mathcal{Y}_1, \mathcal{Z})$ are essentially incomparable.*
- (3) *$\text{eae}(\mathcal{Z}, \mathcal{Z}) = 0$.*
- (4) *The Banach spaces $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{Z}$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Z}$ satisfy*

$$\text{sc}(\mathcal{X}, \mathcal{Y}) = \text{eae}(\mathcal{X}, \mathcal{Y}) = k_0.$$

- (ii) *For every $k_0 \in \mathbb{N}$, there exist infinite-dimensional Banach spaces \mathcal{X}_1 , \mathcal{Y}_1 and \mathcal{Z} satisfying (1)–(3) above, and such that the Banach spaces $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{Z}$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Z}$ satisfy $\text{eae}(\mathcal{X}, \mathcal{Y}) = 1$ and $\text{sc}(\mathcal{X}, \mathcal{Y}) = k_0$.*

Remark 1.14. Let us compare and contrast Theorem 1.13 with Proposition 1.12. To align notation, note that $\mathcal{X}_2 = \mathcal{Y}_2 = \mathcal{Z}$. Theorem 1.13(2) implies that (1.11) holds true. However, (1.12) fails for the pair $(\mathcal{X}, \mathcal{Y})$ in both parts (i) and (ii) of Theorem 1.13 because they satisfy $\text{sc}(\mathcal{X}, \mathcal{Y}) = k_0 \neq 0 = \text{eae}(\mathcal{Z}, \mathcal{Z})$. This difference is due to the fact that (1.12) requires that the pair $(\mathcal{X}_1, \mathcal{Y}_1)$ is essentially incomparable, but we only know that it is projectively incomparable in Theorem 1.13.

Conclusion. Prior to this paper, at the level of general Banach spaces, the only definite answers to the question of whether EAE and SC coincide for Fredholm operators were that they do if the underlying spaces are isomorphic, and that there exist examples where they do not if the underlying spaces are essentially incomparable. We have shown that the result for essentially incomparable spaces does not carry over to projectively incomparable spaces (see Theorem 1.11), despite the fact that the difference between these two incomparability notions is very subtle.

In the case where the Banach spaces \mathcal{X} and \mathcal{Y} admit decompositions of the form (1.10) in which the subspaces \mathcal{X}_1 , \mathcal{Y}_1 and $\mathcal{X}_2 (\cong \mathcal{Y}_2)$ are pairwise essentially incomparable, the question of whether EAE and SC coincide for Fredholm operators is completely resolved in Proposition 1.12; the answer can be expressed in terms of the values of the Fredholm index of operators on the subspaces \mathcal{X}_1 , \mathcal{Y}_1 and \mathcal{X}_2 . Theorem 1.13 shows that, once again, this result does not carry over to projectively incomparable spaces; see Remark 1.14 for details.

The question that remains is whether one can always find a decomposition of the form (1.10) in which the subspaces \mathcal{X}_1 , \mathcal{Y}_1 and \mathcal{X}_2 are pairwise essentially incomparable. Unfortunately, this is not possible, even if we replace “essentially incomparable” with “projectively incomparable”, as we shall see in Corollary 2.8 and Proposition 2.10.

The above results rely on the remarkable observation in Theorem 1.6 that, for Banach spaces \mathcal{X} and \mathcal{Y} and $k \in \mathbb{Z}$, either no pair of operators $(U, V) \in \Phi_k(\mathcal{X}) \times \Phi_k(\mathcal{Y})$ is SC, or a pair of this kind which is SC exists, in which case the set of all such pairs that are SC is the same as the set of all such pairs that are EAE. This means that the question of whether EAE and SC coincide for Fredholm operators on \mathcal{X} and \mathcal{Y} reduces to determining the sets of indices for which pairs of Fredholm operators on \mathcal{X} and \mathcal{Y} with these particular indices that are EAE or SC, respectively, exist.

Our analysis of these sets led us to define the numerical indices $\text{eae}(\mathcal{X}, \mathcal{Y})$ and $\text{sc}(\mathcal{X}, \mathcal{Y})$ which satisfy that $\text{eae}(\mathcal{X}, \mathcal{Y}) = \text{sc}(\mathcal{X}, \mathcal{Y})$ if and only if EAE and SC coincide for all Fredholm operators on \mathcal{X} and \mathcal{Y} . We have computed their values in various cases; see Theorems 1.9, 1.11 and 1.13.

Remark 1.15. We are grateful to the referee for pointing out that the definitions of EAE and SC stated in Definition 1.1 may be formulated for certain classes of rings, using module actions instead of operators on Banach spaces. We consider this an exciting new direction of research that we would like to explore in future work. To the best of our knowledge, EAE and SC have not previously been investigated beyond the context of operators on Banach spaces, although Cvetković-Ilić, Djordjević and Rakočević [17] have studied Schur complements based on generalized inverses in C^* -algebras.

Organization. We conclude this introduction with a brief outline of how the remainder of this paper is organized. It consists of six sections, including the present.

In Section 2 we elaborate on the incomparability notions for Banach spaces introduced in Definitions 1.4 and 1.10, focussing on their connections with operator theory. Section 3 contains a characterization of when the set $\text{EAE}_k(\mathcal{X}, \mathcal{Y})$ is non-empty for Banach spaces \mathcal{X} and \mathcal{Y} and $k \in \mathbb{Z}$, and also the proof of Equation (1.11).

It turns out to be much more complicated to obtain a similar characterization for the non-emptiness of the set $\text{SC}_k(\mathcal{X}, \mathcal{Y})$, and only a partial analogue is obtained in Section 5, where we also prove Proposition 1.7 and the remainder of Proposition 1.12. These results rely strongly on a novel characterization of the existence of Schur-coupled Fredholm operators of a given index that we establish in Section 4. This characterization may be viewed as the fundamental new insight of the paper. Theorem 1.6 is also proved in Section 4.

Finally, in Section 6 we use some of the “exotic” Banach spaces constructed by Gowers and Maurey, together with ideas from subsequent work of Aiena, González and Ferenczi, to prove Theorems 1.9, 1.11 and 1.13.

2. INCOMPARABILITY NOTIONS FOR BANACH SPACES AND THEIR CONNECTION TO OPERATOR THEORY

The notions of essential and projective incomparability of a pair of Banach spaces will play a key role in the final section of this paper, where Theorems 1.9, 1.11 and 1.13 are proved. However, there are certain related notions and results that we shall require beforehand. For that reason, we survey the relevant material at this point.

The formal definitions of essential and projective incomparability were already given in Definitions 1.4 and 1.10, respectively. We refer to [1, Section 7.5] for a much more comprehensive treatment of them than we can give here. Indeed, we shall consider only the aspects that we require later, namely their relationship and certain connections to operator theory. This will involve a third incomparability notion, which is stronger, older and arguably more “natural” than the other two. It is defined as follows.

Definition 2.1. A pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$ is *totally incomparable* if no closed, infinite-dimensional subspace of \mathcal{X} embeds isomorphically into \mathcal{Y} .

Totally incomparable Banach spaces are clearly projectively incomparable, and it is well known and not hard to see that the converse fails; for instance, ℓ_2 embeds into $L_1[0, 1]$, but not complementably, so $L_1[0, 1]$ and ℓ_2 are projectively incomparable without being totally incomparable.

However, more is true, namely that essential incomparability lies between these two properties, in the sense that total incomparability implies essential incomparability, which in turn implies projective incomparability. The easiest way to explain this goes via the following operator-theoretic notion, which will also be useful elsewhere in this work.

Definition 2.2. An operator $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ between Banach spaces \mathcal{X} and \mathcal{Y} is *inessential* if $I_{\mathcal{X}} - ST \in \Phi(\mathcal{X})$ for every operator $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$.

We write $\mathcal{E}(\mathcal{X}, \mathcal{Y})$ for the collection of inessential operators from \mathcal{X} to \mathcal{Y} . It generalizes the ideal $\mathcal{K}(\mathcal{X}, \mathcal{Y})$ of compact operators in several ways. The following remark lists the main properties that we require.

- Remark 2.3.**
- (i) Every compact operator is inessential and, more generally, every strictly singular operator is inessential (see for instance [1, Theorems 7.36 and 7.44] or [46, §26.7.3]). We recall that by definition an operator is strictly singular if no restriction of it to an infinite-dimensional subspace of its domain is an isomorphic embedding.
 - (ii) The assignment \mathcal{E} is a closed operator ideal in the sense of Pietsch (see for instance [1, Theorem 7.5] or [46, Section 4.3]).
 - (iii) For every $k \in \mathbb{Z}$, the class of Fredholm operators of index k is stable under inessential perturbations in the sense that $U + T \in \Phi_k(\mathcal{X}, \mathcal{Y})$ whenever $U \in \Phi_k(\mathcal{X}, \mathcal{Y})$ and $T \in \mathcal{E}(\mathcal{X}, \mathcal{Y})$ (see for instance [1, Theorem 7.23]).

Comparing Definitions 1.4 and 2.2, we see that a pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$ is essentially incomparable if and only if every operator from \mathcal{X} to \mathcal{Y} is inessential. Both definitions display an obvious lack of symmetry, which raises the question what happens if we replace the condition that $I_{\mathcal{X}} - ST \in \Phi(\mathcal{X})$ with $I_{\mathcal{Y}} - TS \in \Phi(\mathcal{Y})$

in either of them? It turns out that it makes no difference because the following well-known elementary lemma implies that $I_{\mathcal{X}} - ST \in \Phi(\mathcal{X})$ if and only if $I_{\mathcal{Y}} - TS \in \Phi(\mathcal{Y})$.

Lemma 2.4. *Let $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ for some Banach spaces \mathcal{X} and \mathcal{Y} . Then $\ker(I_{\mathcal{X}} - ST) \cong \ker(I_{\mathcal{Y}} - TS)$. Moreover, if $\text{ran}(I_{\mathcal{X}} - ST)$ and $\text{ran}(I_{\mathcal{Y}} - TS)$ are closed, then $\mathcal{X}/\text{ran}(I_{\mathcal{X}} - ST) \cong \mathcal{Y}/\text{ran}(I_{\mathcal{Y}} - TS)$.*

Proof. The earliest mention of the first part of this result that we know of is [45, Satz 1], while both parts can be found in [44, Sätze 2.4-A-2.5-A]. Alternatively, the result follows from [6, page 211, properties 1 and 6], see also [7, Proposition 1], because the operators $I_{\mathcal{X}} - ST$ and $I_{\mathcal{Y}} - TS$ are Schur coupled via $A = I_{\mathcal{X}}$, $B = S$, $C = T$ and $D = I_{\mathcal{Y}}$. \square

Using Definition 2.2 and Remark 2.3(i), we can explain the relationship between total, essential and projective incomparability of a pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$ as follows.

Remark 2.5. (i) If \mathcal{X} and \mathcal{Y} are totally incomparable, then clearly every operator between \mathcal{X} and \mathcal{Y} is strictly singular and therefore inessential, so \mathcal{X} and \mathcal{Y} are essentially incomparable.
(ii) If \mathcal{X} and \mathcal{Y} are essentially incomparable, then they are also projectively incomparable. Indeed, suppose contrapositively that \mathcal{X} and \mathcal{Y} contain isomorphic, complemented infinite-dimensional subspaces. Then it is easy to construct operators $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that $I_{\mathcal{X}} - ST$ is a projection with infinite-dimensional kernel and therefore not a Fredholm operator. Hence \mathcal{X} and \mathcal{Y} are not essentially incomparable. We refer to [1, Theorem 7.69] and the paragraph following Definition 7.102 for further details.

The fact that \mathcal{E} is an operator ideal has the following important consequence, which we shall use repeatedly without further reference. Suppose that we express an operator $T: \mathcal{X}_1 \oplus \mathcal{X}_2 \rightarrow \mathcal{Y}_1 \oplus \mathcal{Y}_2$ between two direct sums of Banach spaces as an operator-valued matrix in the usual way, that is,

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad \text{where } T_{ij} \in \mathcal{B}(\mathcal{X}_j, \mathcal{Y}_i) \quad \text{for } i, j \in \{1, 2\}.$$

Then T is inessential if and only if T_{11} , T_{12} , T_{21} and T_{22} are inessential.

We conclude this section by answering a natural question about a pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$ which is *not* projectively incomparable. This material will not play any role in the remainder of the paper; we have included it simply because the question is very natural in our context.

Negating the definition of projective incomparability, we see that \mathcal{X} and \mathcal{Y} admit decompositions of the form (1.10), where the isomorphic subspaces \mathcal{X}_2 and \mathcal{Y}_2 are infinite-dimensional. If the subspaces \mathcal{X}_1 and \mathcal{Y}_1 fail to be projectively incomparable, then they contain isomorphic, complemented, infinite-dimensional subspaces. One may wonder whether *all* such subspaces can somehow be “transferred” to \mathcal{X}_2 and \mathcal{Y}_2 , respectively, leading to the following question.

Question 2.6. Let \mathcal{X} and \mathcal{Y} be Banach spaces which are not projectively incomparable. Is it always possible to find decompositions of the form (1.10), where the subspaces \mathcal{X}_1 and \mathcal{Y}_1 are projectively incomparable?

The answer to this question is “no”. We shall present two short examples showing this. In the first, we consider Banach spaces which are c_0 -direct sums of certain sequences of finite-dimensional Banach spaces. The formal definition is as follows. The c_0 -direct sum of a sequence $(\mathcal{X}_n)_{n \in \mathbb{N}}$ of Banach spaces is

$$\left(\bigoplus_{n=1}^{\infty} \mathcal{X}_n \right)_{c_0} = \{ (x_n)_{n \in \mathbb{N}} : x_n \in \mathcal{X}_n \ (n \in \mathbb{N}) \text{ and } \|x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty \},$$

endowed with the pointwise vector-space operations and with the norm given by $\|(x_n)\| = \sup_{n \in \mathbb{N}} \|x_n\|$.

Bourgain, Casazza, Lindenstrauss and Tzafriri [12, §8] classified the complemented subspaces of $(\bigoplus_{n=1}^{\infty} \mathcal{X}_n)_{c_0}$ in certain cases, including the following result.

Theorem 2.7. *Let $\mathcal{Z} = (\bigoplus_{n=1}^{\infty} \ell_1^n)_{c_0}$ or $\mathcal{Z} = (\bigoplus_{n=1}^{\infty} \ell_2^n)_{c_0}$, and let \mathcal{W} be a complemented, infinite-dimensional subspace of \mathcal{Z} . Then either $\mathcal{W} \cong c_0$ or $\mathcal{W} \cong \mathcal{Z}$.*

Corollary 2.8. *Suppose that $\mathcal{X} = (\bigoplus_{n=1}^{\infty} \ell_1^n)_{c_0}$ and $\mathcal{Y} = (\bigoplus_{n=1}^{\infty} \ell_2^n)_{c_0}$ are decomposed as in (1.10), with $\mathcal{X}_2 \cong \mathcal{Y}_2$ infinite-dimensional. Then $\mathcal{X}_1 \cong \mathcal{X}$ and $\mathcal{Y}_1 \cong \mathcal{Y}$. In particular \mathcal{X}_1 and \mathcal{Y}_1 both contain a complemented subspace isomorphic to c_0 , so they are not projectively incomparable.*

Proof. This follows immediately from Theorem 2.7 because \mathcal{X} and \mathcal{Y} are not isomorphic to each other or to c_0 , so we must have $\mathcal{X}_2 \cong \mathcal{Y}_2 \cong c_0$. \square

Our second example is similar, but uses only reflexive Banach spaces. It relies on the following well-known, important properties of $L_p[0, 1]$ for $1 < p < \infty$.

Theorem 2.9. *Let $p \in (1, \infty)$.*

- (i) *The Banach space $L_p[0, 1]$ is primary; that is, if it is decomposed into a direct sum of two closed subspaces, then (at least) one of them is isomorphic to $L_p[0, 1]$.*
- (ii) *Let $q \in (1, \infty)$. Then $L_p[0, 1]$ contains a complemented subspace which is isomorphic to $L_q[0, 1]$ if and only if $q = p$ or $q = 2$.*

Proof. (i). This is shown in [4].

(ii). This follows easily from [3, Theorem 6.4.21]. \square

Proposition 2.10. *Let $\mathcal{X} = L_p[0, 1]$ and $\mathcal{Y} = L_q[0, 1]$ for distinct $p, q \in (1, 2) \cup (2, \infty)$, and suppose that \mathcal{X} and \mathcal{Y} are decomposed as in (1.10), with $\mathcal{X}_2 \cong \mathcal{Y}_2$ infinite-dimensional. Then $\mathcal{X}_1 \cong \mathcal{X}$ and $\mathcal{Y}_1 \cong \mathcal{Y}$. In particular \mathcal{X}_1 and \mathcal{Y}_1 both contain a complemented subspace isomorphic to $L_2[0, 1]$, so they are not projectively incomparable.*

Proof. Since $L_p[0, 1]$ is primary, either $\mathcal{X}_1 \cong \mathcal{X}$ or $\mathcal{X}_2 \cong \mathcal{X}$. However, the latter is impossible by Theorem 2.9(ii) because $\mathcal{X}_2 \cong \mathcal{Y}_2$, which is a complemented subspace of $L_q[0, 1]$, where $q \notin \{2, p\}$. Therefore $\mathcal{X}_1 \cong \mathcal{X}$. The proof that $\mathcal{Y}_1 \cong \mathcal{Y}$ is similar.

Another application of Theorem 2.9(ii) shows that $L_p[0, 1]$ and $L_q[0, 1]$ both contain a complemented subspace which is isomorphic to $L_2[0, 1]$. \square

3. NON-EMPTINESS OF THE SET $\text{EAE}_k(\mathcal{X}, \mathcal{Y})$

The main purpose of this section is to prove the following characterization of the integers k for which the set $\text{EAE}_k(\mathcal{X}, \mathcal{Y})$ defined in (1.3) is non-empty. This is a

natural starting point for our investigation because when $\text{EAE}_k(\mathcal{X}, \mathcal{Y})$ is empty, it is obviously equal to its subset $\text{SC}_k(\mathcal{X}, \mathcal{Y})$.

Proposition 3.1. *The following three conditions are equivalent for every pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$ and every $k \in \mathbb{Z}$:*

- (i) $\Phi_k(\mathcal{X}) \neq \emptyset$ and $\Phi_k(\mathcal{Y}) \neq \emptyset$.
- (ii) $\text{EAE}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset$.
- (iii) $k \in \text{eae}(\mathcal{X}, \mathcal{Y})\mathbb{Z}$.

The proof of Proposition 3.1 is relatively simple, using only (1.4) and two lemmas that follow from basic Fredholm theory. Both of these lemmas are almost certainly known to specialists; we include their (short) proofs for completeness.

Lemma 3.2. *Let $T \in \Phi(\mathcal{X}, \mathcal{Y})$ for some infinite-dimensional Banach spaces \mathcal{X} and \mathcal{Y} . Then, for every $m \in \mathbb{N}_0 \cap [i(T), \infty)$, there exists a finite-rank operator $R \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that $\alpha(T + R) = m$.*

Proof. We consider three cases:

Case 1. If $\alpha(T) = m$, then we can simply take $R = 0$.

Case 2. If $\alpha(T) > m$, set $n = \alpha(T) - m \in \mathbb{N}$, and note that $\beta(T) = \alpha(T) - i(T) \geq n$, so we can find an n -dimensional subspace \mathcal{Z} of \mathcal{Y} such that $\mathcal{Z} \cap \text{ran } T = \{0\}$. Take an operator $A \in \mathcal{B}(\ker T, \mathcal{Y})$ with range \mathcal{Z} and a bounded linear projection P of \mathcal{X} onto $\ker T$, and define $R = AP \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then $\ker(T + R) = \ker A$, which has dimension $\alpha(T) - \dim \text{ran } A = m$.

Case 3. If $\alpha(T) < m$, choose an m -dimensional subspace \mathcal{W} of \mathcal{X} such that $\ker T \subseteq \mathcal{W}$, and let $P \in \mathcal{B}(\mathcal{X})$ be a projection onto \mathcal{W} . Then $\ker T(I_{\mathcal{X}} - P) = \mathcal{W}$, so $R = -TP$ has the required property. \square

Remark 3.3. The condition that $m \geq i(T)$ is necessary in Lemma 3.2 because

$$\alpha(T + R) \geq i(T + R) = i(T)$$

for every finite-rank operator $R \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$.

Lemma 3.4. *For every Banach space \mathcal{X} , the set*

$$\mathbb{I}_{\Phi}(\mathcal{X}) = \{k \in \mathbb{Z} : \Phi_k(\mathcal{X}) \neq \emptyset\}$$

is an ideal of \mathbb{Z} .

Proof. The result is clear if \mathcal{X} is finite-dimensional because $\mathbb{I}_{\Phi}(\mathcal{X}) = \{0\}$ in this case. In general the set $\mathbb{I}_{\Phi}(\mathcal{X})$ contains 0 because $I_{\mathcal{X}} \in \Phi_0(\mathcal{X})$. Moreover, it is closed under addition and under multiplication by positive integers because the Index Theorem implies that $ST \in \Phi_{k+m}(\mathcal{X})$ and $T^n \in \Phi_{mn}(\mathcal{X})$ for $S \in \Phi_k(\mathcal{X})$ and $T \in \Phi_m(\mathcal{X})$ whenever $k, m \in \mathbb{Z}$ and $n \in \mathbb{N}$.

It remains to show that $-k \in \mathbb{I}_{\Phi}(\mathcal{X})$ whenever $k \in \mathbb{I}_{\Phi}(\mathcal{X})$. Suppose that $\Phi_k(\mathcal{X}) \neq \emptyset$ for some $k \in \mathbb{Z}$. If $k \geq 0$, Lemma 3.2 implies that $\Phi_k(\mathcal{X})$ contains a surjection T . Since $\ker T$ is finite-dimensional, it follows that T has a right inverse, which must be a Fredholm operator of index $-k$. A similar argument works for $k \leq 0$, except that we find that $\Phi_k(\mathcal{X})$ contains an injection which has a left inverse. \square

Corollary 3.5. *Let \mathcal{X} and \mathcal{Y} be Banach spaces. Then $\mathbb{I}_{\Phi}(\mathcal{X}) \cap \mathbb{I}_{\Phi}(\mathcal{Y}) = \text{eae}(\mathcal{X}, \mathcal{Y})\mathbb{Z}$.*

Proof. Lemma 3.4 implies that $\mathbb{I}_\Phi(\mathcal{X}) \cap \mathbb{I}_\Phi(\mathcal{Y})$ is an ideal of \mathbb{Z} , which is a principal ideal domain, so $\mathbb{I}_\Phi(\mathcal{X}) \cap \mathbb{I}_\Phi(\mathcal{Y}) = n\mathbb{Z}$ for some $n \in \mathbb{N}_0$. It is clear from the definition (1.6) of $\text{eae}(\mathcal{X}, \mathcal{Y})$ that $n = \text{eae}(\mathcal{X}, \mathcal{Y})$. \square

Remark 3.6. Elaborating on these ideas, we obtain an alternative formula for $\text{eae}(\mathcal{X}, \mathcal{Y})$, which will be useful later. Indeed, for every Banach space \mathcal{X} , the ideal $\mathbb{I}_\Phi(\mathcal{X})$ has a unique non-negative generator, which we shall denote by $\gamma(\mathcal{X})$; in other words, $\gamma(\mathcal{X}) \in \mathbb{N}_0$ is the unique number such that $\mathbb{I}_\Phi(\mathcal{X}) = \gamma(\mathcal{X})\mathbb{Z}$.

It follows from elementary number theory that, for every pair of Banach spaces \mathcal{X} and \mathcal{Y} , the ideal $\mathbb{I}_\Phi(\mathcal{X}) \cap \mathbb{I}_\Phi(\mathcal{Y}) = \gamma(\mathcal{X})\mathbb{Z} \cap \gamma(\mathcal{Y})\mathbb{Z}$ is generated by the lowest common multiple (lcm) of $\gamma(\mathcal{X})$ and $\gamma(\mathcal{Y})$, provided that we set $\text{lcm}(n, 0) = \text{lcm}(0, n) = 0$ for every $n \in \mathbb{Z}$. Combining this result with Corollary 3.5, we see that

$$\text{eae}(\mathcal{X}, \mathcal{Y}) = \text{lcm}(\gamma(\mathcal{X}), \gamma(\mathcal{Y})). \quad (3.1)$$

Proof of Proposition 3.1. Conditions (i) and (iii) are equivalent because

$$(\Phi_k(\mathcal{X}) \neq \emptyset \text{ and } \Phi_k(\mathcal{Y}) \neq \emptyset) \iff k \in \mathbb{I}_\Phi(\mathcal{X}) \cap \mathbb{I}_\Phi(\mathcal{Y}) \iff k \in \text{eae}(\mathcal{X}, \mathcal{Y})\mathbb{Z},$$

where the final bi-implication follows from Corollary 3.5.

We complete the proof by showing that conditions (i) and (ii) are also equivalent. Here, the implication (ii) \Rightarrow (i) is obvious because $\text{EAE}_k(\mathcal{X}, \mathcal{Y}) \subseteq \Phi_k(\mathcal{X}) \times \Phi_k(\mathcal{Y})$ by definition. Conversely, suppose that $\Phi_k(\mathcal{X})$ and $\Phi_k(\mathcal{Y})$ are both non-empty. If \mathcal{X} or \mathcal{Y} is finite-dimensional, then necessarily $k = 0$, in which case $\text{EAE}_k(\mathcal{X}, \mathcal{Y})$ is non-empty because it contains $(I_{\mathcal{X}}, I_{\mathcal{Y}})$. Otherwise we may apply Lemma 3.2 to find operators $U \in \Phi_k(\mathcal{X})$ and $V \in \Phi_k(\mathcal{Y})$ such that $\alpha(U) = \alpha(V)$. This implies that $\text{EAE}_k(\mathcal{X}, \mathcal{Y})$ is non-empty because it contains (U, V) by (1.4). \square

Remark 3.7. The index $\text{eae}(\mathcal{X}, \mathcal{Y}) \in \mathbb{N}_0$ satisfies: $\text{eae}(\mathcal{X}, \mathcal{Y}) = 0$ if and only if $\Phi_k(\mathcal{X}) = \emptyset$ for every $k \in \mathbb{N}$ or $\Phi_k(\mathcal{Y}) = \emptyset$ for every $k \in \mathbb{N}$.

Indeed, the implication \Leftarrow is immediate from the definition of $\text{eae}(\mathcal{X}, \mathcal{Y})$. Conversely, suppose that $\Phi_k(\mathcal{X}) \neq \emptyset$ and $\Phi_m(\mathcal{Y}) \neq \emptyset$ for some $k, m \in \mathbb{N}$. Then the Index Theorem implies that $\Phi_n(\mathcal{X}) \neq \emptyset$ and $\Phi_n(\mathcal{Y}) \neq \emptyset$ for every common multiple $n \in \mathbb{N}$ of k and m , so $\text{eae}(\mathcal{X}, \mathcal{Y}) \geq 1$. (Alternatively, this follows from (3.1) because $\text{lcm}(\gamma(\mathcal{X}), \gamma(\mathcal{Y})) = 0$ if and only if $\gamma(\mathcal{X}) = 0$ or $\gamma(\mathcal{Y}) = 0$.)

Proposition 3.1 naturally raises the question: what are the possible values of the index $\text{eae}(\mathcal{X}, \mathcal{Y}) \in \mathbb{N}_0$? Obviously, $\text{eae}(\mathcal{X}, \mathcal{Y}) = 0$ if \mathcal{X} or \mathcal{Y} is finite-dimensional. Theorems 1.9(ii), 1.11(i) and 1.13(i) show that all non-negative integers can be realized as $\text{eae}(\mathcal{X}, \mathcal{Y})$ for infinite-dimensional Banach spaces \mathcal{X} and \mathcal{Y} satisfying various additional conditions. At this stage, let us use the index $\gamma(\mathcal{X})$ from Remark 3.6 to verify that all non-negative integers can be realized as $\text{eae}(\mathcal{X}, \mathcal{X})$. Although this example may appear simpler than the three above-mentioned theorems, ultimately they all rely on the same family of “exotic” Banach spaces constructed by Gowers and Maurey in [27].

Example 3.8. We claim that, for every $k_0 \in \mathbb{N}_0$, there exists an infinite-dimensional Banach space \mathcal{X}_{k_0} such that $\gamma(\mathcal{X}_{k_0}) = k_0$. Consequently $\text{eae}(\mathcal{X}_{k_0}, \mathcal{X}_{k_0}) = k_0$ by (3.1), and Proposition 3.1 shows that $\text{EAE}_k(\mathcal{X}_{k_0}, \mathcal{X}_{k_0}) \neq \emptyset$ if and only if $k \in k_0\mathbb{Z}$.

To verify this claim for $k_0 = 0$, we require an infinite-dimensional Banach space on which all Fredholm operators have index 0. Gowers and Maurey constructed such a Banach space in [26]. We shall encounter another space with this property in Theorem 6.8.

For $k_0 = 1$, any Banach space which is isomorphic to its hyperplanes satisfies the claim. Virtually every infinite-dimensional Banach space known prior to 1990 has this property.

Finally, to see that the claim is true for every $k_0 \geq 2$, we use a family of Banach spaces which Gowers and Maurey constructed in [27, §(4.3)]. These Banach spaces will play a key role in Section 6, where Theorem 6.1 summarizes their main properties; we refer to (iv) for the particular result required at this point.

We conclude this section with two easy observations. The first is that the index $\gamma(\mathcal{X})$, and therefore the associated quantity $\text{eae}(\mathcal{X}, \mathcal{Y})$, is an isomorphic invariant in the following precise sense.

Lemma 3.9. *Let \mathcal{X}_1 and \mathcal{X}_2 be isomorphic Banach spaces. Then $\gamma(\mathcal{X}_1) = \gamma(\mathcal{X}_2)$. Consequently, if $(\mathcal{Y}_1, \mathcal{Y}_2)$ is another pair of isomorphic Banach spaces, then*

$$\text{eae}(\mathcal{X}_1, \mathcal{Y}_1) = \text{eae}(\mathcal{X}_2, \mathcal{Y}_2).$$

Proof. Let $R \in \mathcal{B}(\mathcal{X}_1, \mathcal{X}_2)$ be an isomorphism. For each $k \in \mathbb{I}_\Phi(\mathcal{X}_1)$, we can take $T \in \Phi_k(\mathcal{X}_1)$. The Index Theorem implies that $RTR^{-1} \in \Phi_k(\mathcal{X}_2)$, so $k \in \mathbb{I}_\Phi(\mathcal{X}_2)$. This proves that $\mathbb{I}_\Phi(\mathcal{X}_1) \subseteq \mathbb{I}_\Phi(\mathcal{X}_2)$. The opposite inclusion follows by interchanging \mathcal{X}_1 and \mathcal{X}_2 . Hence the ideals $\mathbb{I}_\Phi(\mathcal{X}_1)$ and $\mathbb{I}_\Phi(\mathcal{X}_2)$ are equal, so they must have the same non-negative generator; that is, $\gamma(\mathcal{X}_1) = \gamma(\mathcal{X}_2)$.

The final clause is an immediate consequence of (3.1). \square

Our second easy observation will be the key ingredient in the proof of the first part of Proposition 1.12, as we shall show immediately after it.

Lemma 3.10. *Let $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ be a Banach space.*

- (i) $\gamma(\mathcal{X})$ divides $\gcd(\gamma(\mathcal{X}_1), \gamma(\mathcal{X}_2))$.
- (ii) Suppose that \mathcal{X}_1 and \mathcal{X}_2 are essentially incomparable. Then

$$\gamma(\mathcal{X}) = \gcd(\gamma(\mathcal{X}_1), \gamma(\mathcal{X}_2)). \quad (3.2)$$

Proof. (i). Suppose that $k_j \in \mathbb{I}_\Phi(\mathcal{X}_j)$ for $j \in \{1, 2\}$, and take $T_j \in \Phi_{k_j}(\mathcal{X}_j)$. Then we have

$$\begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \in \Phi_{k_1+k_2}(\mathcal{X}),$$

so $k_1 + k_2 \in \mathbb{I}_\Phi(\mathcal{X})$. This shows that

$$\mathbb{I}_\Phi(\mathcal{X}_1) + \mathbb{I}_\Phi(\mathcal{X}_2) \subseteq \mathbb{I}_\Phi(\mathcal{X}). \quad (3.3)$$

Recall that we have defined $\gcd(0, 0) = 0$. This ensures that the formula $m\mathbb{Z} + n\mathbb{Z} = \gcd(m, n)\mathbb{Z}$ holds true for all values of $m, n \in \mathbb{Z}$. Using it, we can rewrite the identity (3.3) as $\gcd(\gamma(\mathcal{X}_1), \gamma(\mathcal{X}_2))\mathbb{Z} \subseteq \gamma(\mathcal{X})\mathbb{Z}$, which proves (i).

(ii). Suppose that the subspaces \mathcal{X}_1 and \mathcal{X}_2 are essentially incomparable. For each $k \in \mathbb{I}_\Phi(\mathcal{X})$, we can take

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \Phi_k(\mathcal{X}).$$

By hypothesis, T_{12} and T_{21} are inessential, so Remark 2.3(iii) implies that

$$\Phi_k(\mathcal{X}) \ni T - \begin{bmatrix} 0 & T_{12} \\ T_{21} & 0 \end{bmatrix} = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix},$$

which in turn means that $T_{11} \in \Phi(\mathcal{X}_1)$ and $T_{22} \in \Phi(\mathcal{X}_2)$ with

$$k = i(T_{11}) + i(T_{22}) \in \mathbb{I}_\Phi(\mathcal{X}_1) + \mathbb{I}_\Phi(\mathcal{X}_2).$$

Hence $\mathbb{I}_\Phi(\mathcal{X}_1) + \mathbb{I}_\Phi(\mathcal{X}_2) = \mathbb{I}_\Phi(\mathcal{X})$, and (3.2) follows. \square

Proof of Equation (1.11). Since the pairs $(\mathcal{X}_1, \mathcal{X}_2)$ and $(\mathcal{Y}_1, \mathcal{Y}_2)$ are essentially incomparable, Lemma 3.10(ii) implies that $\gamma(\mathcal{X}) = \gcd(\gamma(\mathcal{X}_1), \gamma(\mathcal{X}_2))$ and $\gamma(\mathcal{Y}) = \gcd(\gamma(\mathcal{Y}_1), \gamma(\mathcal{Y}_2))$, where we observe that $\gamma(\mathcal{X}_2) = \gamma(\mathcal{Y}_2) = \text{eae}(\mathcal{X}_2, \mathcal{Y}_2)$ because $\mathcal{X}_2 \cong \mathcal{Y}_2$. Therefore, applying (3.1) twice and using the standard identity

$$\text{lcm}(\gcd(a, c), \gcd(b, c)) = \gcd(\text{lcm}(a, b), c) \quad (a, b, c \in \mathbb{Z}),$$

we obtain

$$\begin{aligned} \text{eae}(\mathcal{X}, \mathcal{Y}) &= \text{lcm}(\gcd(\gamma(\mathcal{X}_1), \gamma(\mathcal{X}_2)), \gcd(\gamma(\mathcal{Y}_1), \gamma(\mathcal{X}_2))) \\ &= \gcd(\text{lcm}(\gamma(\mathcal{X}_1), \gamma(\mathcal{Y}_1)), \gamma(\mathcal{X}_2)) = \gcd(\text{eae}(\mathcal{X}_1, \mathcal{Y}_1), \text{eae}(\mathcal{X}_2, \mathcal{Y}_2)), \end{aligned}$$

as required. \square

4. THE KEY TECHNICAL THEOREM AND THE DEDUCTION OF THEOREM 1.6 FROM IT

The main aim of this section is to prove the following theorem, which will be the key tool in the remainder of our investigation, and is arguably the most important new insight in the paper, despite the somewhat technical nature of conditions (i)–(ii). Notably, Theorem 1.6 is an easy consequence of it (using also Proposition 3.1), as we shall show at the end of this section.

Theorem 4.1. *The following four conditions are equivalent for every pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$ and every $k \in \mathbb{Z}$:*

- (i) *There exist operators $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that $I_{\mathcal{X}} - ST \in \Phi_k(\mathcal{X})$.*
- (ii) *There exist operators $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that $I_{\mathcal{Y}} - TS \in \Phi_k(\mathcal{Y})$.*
- (iii) *$\text{EAE}_k(\mathcal{X}, \mathcal{Y}) = \text{SC}_k(\mathcal{X}, \mathcal{Y})$, and this set is non-empty.*
- (iv) *$\text{SC}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset$.*

The proof of Theorem 4.1 involves two lemmas. The first of these reformulates SC in a way that is much closer to condition (i) above.

Lemma 4.2. *Let $U \in \mathcal{B}(\mathcal{X})$ and $V \in \mathcal{B}(\mathcal{Y})$ for some Banach spaces \mathcal{X} and \mathcal{Y} . Then U and V are SC if and only if there are isomorphisms $M \in \mathcal{B}(\mathcal{X})$ and $N \in \mathcal{B}(\mathcal{Y})$ and operators $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that*

$$UM = I_{\mathcal{X}} - ST \quad \text{and} \quad VN = I_{\mathcal{Y}} - TS. \quad (4.1)$$

Proof. This is a straightforward verification. On the one hand, if the operators A, B, C and D satisfy (1.2), then $M = A^{-1}$, $N = D^{-1}$, $S = BD^{-1}$ and $T = CA^{-1}$ satisfy (4.1), and on the other, if M, N, S and T satisfy (4.1), then $A = M^{-1}$, $B = SN^{-1}$, $C = TM^{-1}$ and $D = N^{-1}$ satisfy (1.2). \square

The second lemma can be viewed as a technical refinement of Lemma 3.2. Its proof is an adaption of the proof of [30, Lemma 5.10].

Lemma 4.3. *Let \mathcal{X} and \mathcal{Y} be Banach spaces, and suppose that $I_{\mathcal{X}} - S_1T_1 \in \Phi_k(\mathcal{X})$ for some $k \in \mathbb{Z} \setminus \{0\}$ and some operators $S_1 \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T_1 \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then, for every $m \in \mathbb{N}_0 \cap [k, \infty)$, there are operators $S_2 \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T_2 \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that $S_1T_1 - S_2T_2$ is a finite-rank operator and $I_{\mathcal{X}} - S_2T_2 \in \Phi_k(\mathcal{X})$ with $\alpha(I_{\mathcal{X}} - S_2T_2) = m$.*

Proof. We begin by observing that \mathcal{X} must be infinite-dimensional because it admits a Fredholm operator of non-zero index. We can therefore apply Lemma 3.2 to find a finite-rank operator $R \in \mathcal{B}(\mathcal{X})$ such that

$$\alpha(I_{\mathcal{X}} - S_1 T_1 - R) = m. \quad (4.2)$$

Take a finite-dimensional subspace \mathcal{W} of \mathcal{X} such that $\text{ran } R \subseteq \mathcal{W}$ and $\dim \mathcal{W} = n|k|$ for some $n \in \mathbb{N}$, and let $R_0 \in \mathcal{B}(\mathcal{X}, \mathcal{W})$ denote the operator R regarded as a map into \mathcal{W} . Furthermore, let $J \in \mathcal{B}(\mathcal{W}, \mathcal{X})$ be the inclusion map.

Lemma 2.4 shows that $I_{\mathcal{Y}} - T_1 S_1 \in \Phi_k(\mathcal{Y})$. This implies that $\Phi_{n|k|}(\mathcal{Y})$ is non-empty by Lemma 3.4, and therefore $\mathcal{Y} \cong \mathcal{Y} \oplus \mathcal{W}$ by [30, Proposition 4.2]. Take an isomorphism $L \in \mathcal{B}(\mathcal{Y}, \mathcal{Y} \oplus \mathcal{W})$. Then the operators

$$S_2 = [S_1 \quad J] L \in \mathcal{B}(\mathcal{Y}, \mathcal{X}) \quad \text{and} \quad T_2 = L^{-1} \begin{bmatrix} T_1 \\ R_0 \end{bmatrix} \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$$

satisfy $S_2 T_2 = S_1 T_1 + R$. It follows that $S_1 T_1 - S_2 T_2 = -R$ is a finite-rank operator, and $I_{\mathcal{X}} - S_2 T_2 \in \Phi_k(\mathcal{X})$ because finite-rank perturbations do not change the Fredholm index. Finally, (4.2) shows that $\alpha(I_{\mathcal{X}} - S_2 T_2) = m$. \square

Proof of Theorem 4.1. Lemma 2.4 shows that conditions (i) and (ii) are equivalent.

(i) \Rightarrow (iii). For $k = 0$, (1.5) shows that $\text{EAE}_0(\mathcal{X}, \mathcal{Y}) = \text{SC}_0(\mathcal{X}, \mathcal{Y})$, and this set is non-empty because it contains $(I_{\mathcal{X}}, I_{\mathcal{Y}})$.

Hence it suffices to consider the case $k \neq 0$. Suppose that $I_{\mathcal{X}} - S_1 T_1 \in \Phi_k(\mathcal{X})$ for some operators $S_1 \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T_1 \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then obviously $\Phi_k(\mathcal{X})$ is non-empty, and $\Phi_k(\mathcal{Y})$ is also non-empty by Lemma 2.4, so $\text{EAE}_k(\mathcal{X}, \mathcal{Y})$ is non-empty by Proposition 3.1.

Suppose that $(U, V) \in \text{EAE}_k(\mathcal{X}, \mathcal{Y})$, so that $\alpha(U) = \alpha(V)$ by (1.4). Call this number m , and note that $m \geq k$. Our strategy is to modify the operators S_1 and T_1 to obtain a pair for which we can construct isomorphisms $M \in \mathcal{B}(\mathcal{X})$ and $N \in \mathcal{B}(\mathcal{Y})$ such that (4.1) is satisfied.

We begin by applying Lemma 4.3 to find operators $S_2 \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T_2 \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that $S_1 T_1 - S_2 T_2$ has finite rank and $I_{\mathcal{X}} - S_2 T_2 \in \Phi_k(\mathcal{X})$ with $\alpha(I_{\mathcal{X}} - S_2 T_2) = m$. Then $\beta(I_{\mathcal{X}} - S_2 T_2) = m - k = \beta(U)$, so $\text{ran}(I_{\mathcal{X}} - S_2 T_2)$ and $\text{ran } U$ are closed subspaces of the same finite codimension in \mathcal{X} , and therefore we can take an isomorphism $A \in \mathcal{B}(\mathcal{X})$ such that

$$A[\text{ran}(I_{\mathcal{X}} - S_2 T_2)] = \text{ran } U. \quad (4.3)$$

Lemma 2.4 implies that $\beta(I_{\mathcal{Y}} - T_2 S_2) = \beta(I_{\mathcal{X}} - S_2 T_2) = m - k = \beta(V)$, so we can also find an isomorphism $B \in \mathcal{B}(\mathcal{Y})$ such that

$$B[\text{ran}(I_{\mathcal{Y}} - T_2 S_2)] = \text{ran } V. \quad (4.4)$$

Set $S_3 = A S_2 B^{-1} \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T_3 = B T_2 A^{-1} \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and observe that these operators satisfy

$$I_{\mathcal{X}} - S_3 T_3 = A(I_{\mathcal{X}} - S_2 T_2)A^{-1}. \quad (4.5)$$

This implies that $\alpha(I_{\mathcal{X}} - S_3 T_3) = \alpha(I_{\mathcal{X}} - S_2 T_2) = m = \alpha(U)$, which is finite, so we can take an isomorphism $M_1 \in \mathcal{B}(\ker(I_{\mathcal{X}} - S_3 T_3), \ker U)$. Choose closed subspaces \mathcal{X}_1 and \mathcal{X}_2 of \mathcal{X} such that $\mathcal{X} = \ker(I_{\mathcal{X}} - S_3 T_3) \oplus \mathcal{X}_1$ and $\mathcal{X} = \ker U \oplus \mathcal{X}_2$, and let

$$R: x \mapsto (I_{\mathcal{X}} - S_3 T_3)x, \quad \mathcal{X}_1 \rightarrow \text{ran}(I_{\mathcal{X}} - S_3 T_3), \quad \text{and} \quad U_0: x \mapsto Ux, \quad \mathcal{X}_2 \rightarrow \text{ran } U,$$

be the restrictions of $I_{\mathcal{X}} - S_3T_3$ and U , respectively. The choices of \mathcal{X}_1 and \mathcal{X}_2 imply that R and U_0 are isomorphisms.

Using (4.5) and (4.3), we see that $\text{ran}(I_{\mathcal{X}} - S_3T_3) = \text{ran } U$, so we can define an isomorphism $M_2 \in \mathcal{B}(\mathcal{X}_1, \mathcal{X}_2)$ by $M_2 = U_0^{-1}R$, and therefore

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} : \mathcal{X} = \ker(I_{\mathcal{X}} - S_3T_3) \oplus \mathcal{X}_1 \rightarrow \ker U \oplus \mathcal{X}_2 = \mathcal{X}$$

is an isomorphism. For $x \in \ker(I_{\mathcal{X}} - S_3T_3)$, we have $Mx \in \ker U$, so $UMx = 0 = (I_{\mathcal{X}} - S_3T_3)x$; and for $x \in \mathcal{X}_1$, we have $UMx = UU_0^{-1}Rx = (I_{\mathcal{X}} - S_3T_3)x$. This shows that $UM = I_{\mathcal{X}} - S_3T_3$ because $\mathcal{X} = \ker(I_{\mathcal{X}} - T_3S_3) + \mathcal{X}_1$.

Lemma 2.4 implies that $\alpha(I_{\mathcal{Y}} - T_3S_3) = \alpha(I_{\mathcal{X}} - S_3T_3) = m = \alpha(V)$, and combining the identity

$$I_{\mathcal{Y}} - T_3S_3 = B(I_{\mathcal{Y}} - T_2S_2)B^{-1}$$

with (4.4), we deduce that $\text{ran}(I_{\mathcal{Y}} - T_3S_3) = \text{ran } V$. Therefore we can repeat the constructions from the previous paragraphs to obtain an isomorphism $N \in \mathcal{B}(\mathcal{Y})$ such that $VN = I_{\mathcal{Y}} - T_3S_3$. Now the conclusion that U and V are SC follows from Lemma 4.2.

The implication (iii) \Rightarrow (iv) is clear.

(iv) \Rightarrow (i). Suppose that $(U, V) \in \text{SC}_k(\mathcal{X}, \mathcal{Y})$. Then Lemma 4.2 implies that there are operators $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and an isomorphism $M \in \mathcal{B}(\mathcal{X})$ such that $UM = I_{\mathcal{X}} - ST$. We have $UM \in \Phi_k(\mathcal{X})$ because $U \in \Phi_k(\mathcal{X})$ and M is an isomorphism, and consequently (i) is satisfied. \square

Proof of Theorem 1.6. (i) is simply a restatement of the equivalence of conditions (ii) and (iii) in Proposition 3.1.

(ii). Take $k \in \text{eae}(\mathcal{X}, \mathcal{Y})\mathbb{Z}$. By (i), we have $\text{EAE}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset$. In view of this, the implication \Rightarrow is clear, while the converse follows from Theorem 4.1 (specifically, the implication (iv) \Rightarrow (iii)).

(iii). Proposition 3.1 shows that $\Phi_k(\mathcal{X}) = \emptyset$ or $\Phi_k(\mathcal{Y}) = \emptyset$ for $k \in \mathbb{Z} \setminus \text{eae}(\mathcal{X}, \mathcal{Y})\mathbb{Z}$. Since

$$\text{SC}_k(\mathcal{X}, \mathcal{Y}) \subseteq \text{EAE}_k(\mathcal{X}, \mathcal{Y}) \subseteq \Phi_k(\mathcal{X}) \times \Phi_k(\mathcal{Y}),$$

we see that $\text{SC}_k(\mathcal{X}, \mathcal{Y}) = \text{EAE}_k(\mathcal{X}, \mathcal{Y}) = \emptyset$ in this case. \square

We conclude this section by showing how Theorem 4.1 can be deduced from results obtained in [30]. To this end, take $(U, V) \in \text{EAE}_k(\mathcal{X}, \mathcal{Y})$. Translating the conclusion of [30, Proposition 5.9] about what is called ‘‘the Banach space operator problem’’ in [30] to the setting of EAE and SC, as explained in [30, Section 3], we see that U and V are SC if and only if there exist operators $B_1 \in \mathcal{B}(\text{ran } U, \text{ran } V)$ and $B_2 \in \mathcal{B}(\text{ran } V, \text{ran } U)$ such that

$$I_{\text{ran } V} - B_1B_2 \in \Phi_k(\text{ran } V). \quad (4.6)$$

Alternative proof of Theorem 4.1. As before, Lemma 2.4 shows that conditions (i) and (ii) are equivalent, and the implication (iii) \Rightarrow (iv) is trivial.

(ii) \Rightarrow (iii). Suppose that (ii) is satisfied, so that $I_{\mathcal{Y}} - TS \in \Phi_k(\mathcal{Y})$ for some operators $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and take $(U, V) \in \text{EAE}_k(\mathcal{X}, \mathcal{Y})$. We must show that $(U, V) \in \text{SC}_k(\mathcal{X}, \mathcal{Y})$, which by the result from [30] stated above amounts to finding operators $B_1 \in \mathcal{B}(\text{ran } U, \text{ran } V)$ and $B_2 \in \mathcal{B}(\text{ran } V, \text{ran } U)$ which satisfy (4.6).

Take finite-dimensional subspaces \mathcal{X}_1 and \mathcal{Y}_1 of \mathcal{X} and \mathcal{Y} , respectively, such that

$$\mathcal{X} = \text{ran } U \oplus \mathcal{X}_1 \quad \text{and} \quad \mathcal{Y} = \text{ran } V \oplus \mathcal{Y}_1, \quad (4.7)$$

and decompose the operators S and T accordingly; that is,

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix},$$

where $S_{11} \in \mathcal{B}(\text{ran } V, \text{ran } U)$, $S_{12} \in \mathcal{B}(\mathcal{Y}_1, \text{ran } U)$, $S_{21} \in \mathcal{B}(\text{ran } V, \mathcal{X}_1)$, $S_{22} \in \mathcal{B}(\mathcal{Y}_1, \mathcal{X}_1)$, $T_{11} \in \mathcal{B}(\text{ran } U, \text{ran } V)$, $T_{12} \in \mathcal{B}(\mathcal{X}_1, \text{ran } V)$, $T_{21} \in \mathcal{B}(\text{ran } U, \mathcal{Y}_1)$ and $T_{22} \in \mathcal{B}(\mathcal{X}_1, \mathcal{Y}_1)$. Define

$$S_1 = \begin{bmatrix} S_{11} & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{Y}, \mathcal{X}) \quad \text{and} \quad T_1 = \begin{bmatrix} T_{11} & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{X}, \mathcal{Y}).$$

Since \mathcal{X}_1 and \mathcal{Y}_1 are finite-dimensional, the operators $S_2 = S - S_1$ and $T_2 = T - T_1$ have finite rank, and

$$\Phi_k(\mathcal{Y}) \ni I_{\mathcal{Y}} - TS = I_{\mathcal{Y}} - T_1 S_1 - (T_1 S_2 + T_2 S_1 + T_2 S_2),$$

where $T_1 S_2 + T_2 S_1 + T_2 S_2$ is a finite-rank operator. Hence

$$\Phi_k(\mathcal{Y}) \ni I_{\mathcal{Y}} - T_1 S_1 = \begin{bmatrix} I_{\text{ran } V} - T_{11} S_{11} & 0 \\ 0 & I_{\mathcal{Y}_1} \end{bmatrix},$$

which in turn implies that $I_{\text{ran } V} - T_{11} S_{11} \in \Phi_k(\text{ran } V)$. Therefore the operators $B_1 = T_{11}$ and $B_2 = S_{11}$ satisfy (4.6).

(iv) \Rightarrow (ii). Suppose that $\text{SC}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset$, and take $(U, V) \in \text{SC}_k(\mathcal{X}, \mathcal{Y})$. Then, by the result from [30] stated above, we can find operators $B_1 \in \mathcal{B}(\text{ran } U, \text{ran } V)$ and $B_2 \in \mathcal{B}(\text{ran } V, \text{ran } U)$ which satisfy (4.6). As before, take finite-dimensional subspaces \mathcal{X}_1 and \mathcal{Y}_1 of \mathcal{X} and \mathcal{Y} , respectively, such that (4.7) is satisfied, and define

$$S = \begin{bmatrix} B_2 & 0 \\ 0 & 0 \end{bmatrix} : \mathcal{Y} = \text{ran } V \oplus \mathcal{Y}_1 \rightarrow \text{ran } U \oplus \mathcal{X}_1 = \mathcal{X}$$

and

$$T = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \mathcal{X} = \text{ran } U \oplus \mathcal{X}_1 \rightarrow \text{ran } V \oplus \mathcal{Y}_1 = \mathcal{Y}.$$

Then

$$I_{\mathcal{Y}} - TS = \begin{bmatrix} I_{\text{ran } V} - B_1 B_2 & 0 \\ 0 & I_{\mathcal{Y}_1} \end{bmatrix} \in \Phi_k(\mathcal{Y}),$$

which shows that (ii) is satisfied. \square

5. NON-EMPTINESS OF $\text{SC}_k(\mathcal{X}, \mathcal{Y})$ AND THE PROOFS OF PROPOSITIONS 1.7 AND 1.12

The aim of this section is to investigate the set of integers k for which there exist Schur-coupled operators $U \in \Phi_k(\mathcal{X})$ and $V \in \Phi_k(\mathcal{Y})$; that is, $\text{SC}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset$. We follow a similar strategy to the one successfully employed in Section 3, beginning with a partial analogue of Lemma 3.4 for the set $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \{k \in \mathbb{Z} : \text{SC}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset\}$ defined in (1.7). As we shall see, the situation for SC is considerably more complicated than for EAE, primarily due to the difficulty of analyzing the technical conditions (i)–(ii) in Theorem 4.1. In particular, we have been unable to obtain an exact counterpart of Lemma 3.4 for SC because we do not know if the set $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ is always closed under addition.

Lemma 5.1. *The set $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ has the following properties for every pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$:*

- (i) $0 \in \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$.
- (ii) $\text{sc}(\mathcal{X}, \mathcal{Y}) \in \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$.
- (iii) $km \in \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ for every $k \in \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ and $m \in \mathbb{Z}$.
- (iv) Suppose that $\text{sc}(\mathcal{X}, \mathcal{Y}) \in \{0, \text{eae}(\mathcal{X}, \mathcal{Y})\}$. Then $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \text{sc}(\mathcal{X}, \mathcal{Y})\mathbb{Z}$.

Proof. The first two properties are easy to verify. Indeed, (i) follows from the fact that $(I_{\mathcal{X}}, I_{\mathcal{Y}}) \in \text{SC}_0(\mathcal{X}, \mathcal{Y})$, while (ii) follows from (i) if $\text{sc}(\mathcal{X}, \mathcal{Y}) = 0$, and otherwise from the definition (1.8) of $\text{sc}(\mathcal{X}, \mathcal{Y})$.

However, the proof of (iii) requires more work. Take $k \in \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ and $m \in \mathbb{Z}$. By (i), we may suppose that $k \neq 0$ and $m \neq 0$. Since $\text{SC}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset$, Theorem 4.1 implies that we can find operators $S_1 \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T_1 \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that $I_{\mathcal{X}} - S_1 T_1 \in \Phi_k(\mathcal{X})$. We claim that there exist operators $S_m \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T_m \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that

$$I_{\mathcal{X}} - S_m T_m \in \Phi_{km}(\mathcal{X}). \quad (5.1)$$

Once we have established this claim, the conclusion will follow from another application of Theorem 4.1.

We prove the claim by considering three different cases: $m \geq 2$, $m = -1$ and $m \leq -2$. (Note that the case $m = 1$ is already covered by the choice of S_1 and T_1 .)

Case 1. For $m \geq 2$, we can apply the Binomial Theorem because $I_{\mathcal{X}}$ and $S_1 T_1$ commute. It shows that

$$(I_{\mathcal{X}} - S_1 T_1)^m = I_{\mathcal{X}} + \sum_{j=1}^m \binom{m}{j} (-S_1 T_1)^j = I_{\mathcal{X}} - S_1 T_1 \sum_{j=1}^m \binom{m}{j} (-S_1 T_1)^{j-1},$$

so the Index Theorem implies that the operators $S_m = S_1 \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T_m = T_1 \sum_{j=1}^m \binom{m}{j} (-S_1 T_1)^{j-1} \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ satisfy (5.1).

Case 2. For $m = -1$, we consider the cases $k > 0$ and $k < 0$ separately. For $k > 0$, Lemma 4.3 implies that we can find operators $U \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $V \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that $S_1 T_1 - UV$ is a finite-rank operator and $I_{\mathcal{X}} - UV \in \Phi_k(\mathcal{X})$ with $\alpha(I_{\mathcal{X}} - UV) = k$. Then $\beta(I_{\mathcal{X}} - UV) = 0$, so $I_{\mathcal{X}} - UV$ is a surjective Fredholm operator, and therefore it has a right inverse $R \in \Phi_{-k}(\mathcal{X})$. Consequently

$$I_{\mathcal{X}} = (I_{\mathcal{X}} - UV)R = R - UVR,$$

which implies that $S_{-1} = -U \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T_{-1} = VR \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ satisfy (5.1) because $I_{\mathcal{X}} - S_{-1} T_{-1} = R \in \Phi_{-k}(\mathcal{X})$.

The argument for $k < 0$ is very similar. In this case, we can apply Lemma 4.3 to find $U \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $V \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that $S_1 T_1 - UV$ is a finite-rank operator and $I_{\mathcal{X}} - UV \in \Phi_k(\mathcal{X})$ with $\alpha(I_{\mathcal{X}} - UV) = 0$. Then, being an injective Fredholm operator, $I_{\mathcal{X}} - UV$ has a left inverse $L \in \Phi_{-k}(\mathcal{X})$, which implies that the operators $S_{-1} = -LU \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T_{-1} = V \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ satisfy

$$I_{\mathcal{X}} - S_{-1} T_{-1} = L(I_{\mathcal{X}} - UV) + (LU)V = L \in \Phi_{-k}(\mathcal{X}).$$

Case 3. Finally, for $m \leq -2$, we apply the argument from Case 1 to the $-m^{\text{th}}$ power of the operator $I_{\mathcal{X}} - S_{-1} T_{-1}$, where S_{-1} and T_{-1} are the operators found in Case 2, to conclude that the operators $S_m = S_{-1} \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T_m = T_{-1} \sum_{j=1}^{-m} \binom{-m}{j} (-S_{-1} T_{-1})^{j-1} \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ satisfy (5.1).

(iv). First, if $\text{sc}(\mathcal{X}, \mathcal{Y}) = 0$, then the definition (1.8) of $\text{sc}(\mathcal{X}, \mathcal{Y})$ implies that $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) \cap \mathbb{N} = \emptyset$, and therefore $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \{0\} = \text{sc}(\mathcal{X}, \mathcal{Y})\mathbb{Z}$ by (i) and (iii).

Second, suppose that $\text{eae}(\mathcal{X}, \mathcal{Y}) = \text{sc}(\mathcal{X}, \mathcal{Y})$, and call this number k_0 . Then, by (ii), (iii) and Corollary 3.5, we have

$$k_0\mathbb{Z} \subseteq \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) \subseteq \mathbb{I}_{\Phi}(\mathcal{X}) \cap \mathbb{I}_{\Phi}(\mathcal{Y}) = k_0\mathbb{Z},$$

which shows that $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = k_0\mathbb{Z}$, as required. \square

Proof of Proposition 1.7. We begin by verifying the chain of inclusions (1.9), which we restate here for ease of reference:

$$\begin{aligned} \text{sc}(\mathcal{X}, \mathcal{Y})\mathbb{Z} &\subseteq \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \{k \in \mathbb{Z} : \text{SC}_k(\mathcal{X}, \mathcal{Y}) = \text{EAE}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset\} \\ &\subseteq \text{eae}(\mathcal{X}, \mathcal{Y})\mathbb{Z} = \{k \in \mathbb{Z} : \text{EAE}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset\} = \mathbb{I}_{\Phi}(\mathcal{X}) \cap \mathbb{I}_{\Phi}(\mathcal{Y}). \end{aligned}$$

The first inclusion is immediate from Lemma 5.1(ii)–(iii), while the equality of the second and third set in the first line follows by combining the definition of $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ with the equivalence of conditions (iii) and (iv) in Theorem 4.1. Proposition 3.1 shows that the three sets in the second line are equal (the equality of the first and last of these sets was also recorded in Corollary 3.5), and finally the inclusion at the beginning of the second line follows because the final set in the first line is trivially contained in the second set in the second line.

Next, to prove the first claim of Proposition 1.7, suppose that $\text{EAE}_k(\mathcal{X}, \mathcal{Y}) = \text{SC}_k(\mathcal{X}, \mathcal{Y})$ for every $k \in \mathbb{Z}$. Then

$$\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \{k \in \mathbb{Z} : \text{EAE}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset\} = \mathbb{I}_{\Phi}(\mathcal{X}) \cap \mathbb{I}_{\Phi}(\mathcal{Y}),$$

where the final equality follows from (1.9). Hence the definitions (1.6) and (1.8) show that $\text{eae}(\mathcal{X}, \mathcal{Y}) = \text{sc}(\mathcal{X}, \mathcal{Y})$.

Conversely, suppose that $\text{eae}(\mathcal{X}, \mathcal{Y}) = \text{sc}(\mathcal{X}, \mathcal{Y})$, and call this number k_0 . Then the inclusions in (1.9) are in fact equalities, so $\text{EAE}_k(\mathcal{X}, \mathcal{Y}) = \text{SC}_k(\mathcal{X}, \mathcal{Y}) (\neq \emptyset)$ for every $k \in \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = k_0\mathbb{Z}$. On the other hand, Theorem 1.6(iii) shows that the identity $\text{EAE}_k(\mathcal{X}, \mathcal{Y}) = \text{SC}_k(\mathcal{X}, \mathcal{Y}) (= \emptyset)$ is also true for every $k \notin \text{eae}(\mathcal{X}, \mathcal{Y})\mathbb{Z} = k_0\mathbb{Z}$.

Finally, we verify that $\text{sc}(\mathcal{X}, \mathcal{Y}) = n \text{eae}(\mathcal{X}, \mathcal{Y})$ for some $n \in \mathbb{N}_0$. Set $k_0 = \text{sc}(\mathcal{X}, \mathcal{Y}) \in \mathbb{N}_0$. Then Lemma 5.1(ii) shows that $\emptyset \neq \text{SC}_{k_0}(\mathcal{X}, \mathcal{Y}) \subseteq \text{EAE}_{k_0}(\mathcal{X}, \mathcal{Y})$, so $k_0 = n \text{eae}(\mathcal{X}, \mathcal{Y})$ for some $n \in \mathbb{N}_0$ by Proposition 3.1 and the fact that k_0 and $\text{eae}(\mathcal{X}, \mathcal{Y})$ are both non-negative. \square

Remark 5.2. To illustrate the applicability of our work thus far, let us explain how it leads to an explicit algorithm for deciding whether EAE and SC are equivalent for all pairs of Fredholm operators on a given pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$.

- (i) Find, if possible, the least $k \in \mathbb{N}$ such that $\Phi_k(\mathcal{X}) \neq \emptyset$ and $\Phi_k(\mathcal{Y}) \neq \emptyset$. This is $k_0 = \text{eae}(\mathcal{X}, \mathcal{Y})$.
- (ii) If no such $k \in \mathbb{N}$ exists, then $\text{eae}(\mathcal{X}, \mathcal{Y}) = 0 = \text{sc}(\mathcal{X}, \mathcal{Y})$, and EAE and SC are equivalent for all pairs of Fredholm operators on \mathcal{X} and \mathcal{Y} by Proposition 1.7. More precisely, we have

$$\text{SC}_k(\mathcal{X}, \mathcal{Y}) = \text{EAE}_k(\mathcal{X}, \mathcal{Y}) \begin{cases} \neq \emptyset & \text{for } k = 0 \\ = \emptyset & \text{for } k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

- (iii) Otherwise choose any pair of Fredholm operators $(U, V) \in \Phi_{k_0}(\mathcal{X}) \times \Phi_{k_0}(\mathcal{Y})$ with $\alpha(U) = \alpha(V)$ (and hence $\beta(U) = \beta(V)$), and decide whether U and V are SC.

- (1) If U and V are SC, then Proposition 1.7 implies that EAE and SC are equivalent for all pairs of Fredholm operators on \mathcal{X} and \mathcal{Y} , and

$$\text{SC}_k(\mathcal{X}, \mathcal{Y}) = \text{EAE}_k(\mathcal{X}, \mathcal{Y}) \begin{cases} \neq \emptyset & \text{for } k \in k_0\mathbb{Z} \\ = \emptyset & \text{for } k \in \mathbb{Z} \setminus k_0\mathbb{Z}. \end{cases}$$

- (2) Otherwise EAE and SC are evidently not equivalent for all pairs of Fredholm operators on \mathcal{X} and \mathcal{Y} , as (U, V) is a concrete example of a pair which is EAE, but not SC.

Our next lemma is the counterpart of Lemma 3.9 for SC, showing that the set $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$, and hence the associated index $\text{sc}(\mathcal{X}, \mathcal{Y})$, is an isomorphic invariant.

Lemma 5.3. *Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1$ and \mathcal{Y}_2 be Banach spaces satisfying $\mathcal{X}_1 \cong \mathcal{X}_2$ and $\mathcal{Y}_1 \cong \mathcal{Y}_2$. Then $\mathbb{I}_{\text{SC}}(\mathcal{X}_1, \mathcal{Y}_1) = \mathbb{I}_{\text{SC}}(\mathcal{X}_2, \mathcal{Y}_2)$, and therefore $\text{sc}(\mathcal{X}_1, \mathcal{Y}_1) = \text{sc}(\mathcal{X}_2, \mathcal{Y}_2)$.*

Proof. Take isomorphisms $R \in \mathcal{B}(\mathcal{X}_1, \mathcal{X}_2)$ and $S \in \mathcal{B}(\mathcal{Y}_1, \mathcal{Y}_2)$. For each $k \in \mathbb{I}_{\text{SC}}(\mathcal{X}_1, \mathcal{Y}_1)$, we can find $U \in \Phi_k(\mathcal{X}_1)$ and $V \in \Phi_k(\mathcal{Y}_1)$ which are SC, so that there exist isomorphisms $A \in \mathcal{B}(\mathcal{X}_1)$ and $D \in \mathcal{B}(\mathcal{Y}_1)$ and operators $B \in \mathcal{B}(\mathcal{Y}_1, \mathcal{X}_1)$ and $C \in \mathcal{B}(\mathcal{X}_1, \mathcal{Y}_1)$ such that (1.2) is satisfied. The Index Theorem implies that $RUR^{-1} \in \Phi_k(\mathcal{X}_2)$ and $SVS^{-1} \in \Phi_k(\mathcal{Y}_2)$, and it is easy to check that they are SC, using the operators $RAR^{-1} \in \mathcal{B}(\mathcal{X}_2)$, $SDS^{-1} \in \mathcal{B}(\mathcal{Y}_2)$, $RBS^{-1} \in \mathcal{B}(\mathcal{Y}_2, \mathcal{X}_2)$ and $SCR^{-1} \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_2)$ to verify (1.2). This implies that $k \in \mathbb{I}_{\text{SC}}(\mathcal{X}_2, \mathcal{Y}_2)$, so $\mathbb{I}_{\text{SC}}(\mathcal{X}_1, \mathcal{Y}_1) \subseteq \mathbb{I}_{\text{SC}}(\mathcal{X}_2, \mathcal{Y}_2)$. The opposite inclusion follows by interchanging \mathcal{X}_1 and \mathcal{X}_2 , and \mathcal{Y}_1 and \mathcal{Y}_2 .

The final statement is immediate from the definition (1.8) of sc . \square

As another consequence of Lemma 5.1, we obtain the following variant of Proposition 3.1 for SC.

Corollary 5.4. *Let \mathcal{X} and \mathcal{Y} be Banach spaces. Then the set $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ is closed under addition if and only if $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \text{sc}(\mathcal{X}, \mathcal{Y})\mathbb{Z}$.*

Proof. The implication \Leftarrow is obvious because the set $\text{sc}(\mathcal{X}, \mathcal{Y})\mathbb{Z}$ is closed under addition. Conversely, suppose that $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ is closed under addition. Then, in view of Lemma 5.1(i) and (iii), it is an ideal of \mathbb{Z} , so $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = m\mathbb{Z}$ for some $m \in \mathbb{N}_0$. Combining this identity with the definition (1.8) of $\text{sc}(\mathcal{X}, \mathcal{Y})$, we conclude that $m = \text{sc}(\mathcal{X}, \mathcal{Y})$. \square

Corollary 5.4 is not entirely satisfactory because we have been unable to answer the following question.

Question 5.5. Is the set $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ closed under addition for every pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$?

We know that the answer to this question is “yes” in certain cases because Lemma 5.1(iv) shows that $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ is closed under addition if $\text{sc}(\mathcal{X}, \mathcal{Y}) = 0$ or $\text{sc}(\mathcal{X}, \mathcal{Y}) = \text{eae}(\mathcal{X}, \mathcal{Y})$. We can also obtain a positive answer to it by imposing suitable conditions on the Banach spaces \mathcal{X} and \mathcal{Y} . To state this result precisely, we require the following additional notation and terminology.

Definition 5.6. Let \mathcal{X} and \mathcal{Y} be Banach spaces.

- (i) Set $\mathcal{G}_{\mathcal{Y}}(\mathcal{X}) = \{ST : S \in \mathcal{B}(\mathcal{Y}, \mathcal{X}), T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})\}$.

- (ii) We say that a subset Σ of $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is *essentially closed under addition* if, for every pair of operators $U, V \in \Sigma$, there exists an inessential operator $R \in \mathcal{E}(\mathcal{X}, \mathcal{Y})$ such that $U + V - R \in \Sigma$.

Proposition 5.7. *Let \mathcal{X} and \mathcal{Y} be Banach spaces, and suppose that at least one of the sets $\mathcal{G}_{\mathcal{Y}}(\mathcal{X})$ and $\mathcal{G}_{\mathcal{X}}(\mathcal{Y})$ is essentially closed under addition. Then $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ is closed under addition.*

Proof. Suppose that $\mathcal{G}_{\mathcal{Y}}(\mathcal{X})$ is essentially closed under addition, and take $k_1, k_2 \in \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$. Then by Theorem 4.1, we can find operators $S_j \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T_j \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that $I_{\mathcal{X}} - S_j T_j \in \Phi_{k_j}(\mathcal{X})$ for $j = 1, 2$. The Index Theorem shows that

$$\Phi_{k_1+k_2}(\mathcal{X}) \ni (I_{\mathcal{X}} - S_1 T_1)(I_{\mathcal{X}} - S_2 T_2) = I_{\mathcal{X}} - [S_1 T_1(I_{\mathcal{X}} - S_2 T_2) + S_2 T_2]. \quad (5.2)$$

Both of the operators $S_1 T_1(I_{\mathcal{X}} - S_2 T_2)$ and $S_2 T_2$ belong to $\mathcal{G}_{\mathcal{Y}}(\mathcal{X})$, so by the hypothesis, we can find operators $S_3 \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$, $T_3 \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $R \in \mathcal{E}(\mathcal{X})$ such that $S_1 T_1(I_{\mathcal{X}} - S_2 T_2) + S_2 T_2 = S_3 T_3 + R$. Combining (5.2) with Remark 2.3(iii), we deduce that

$$\Phi_{k_1+k_2}(\mathcal{X}) \ni I_{\mathcal{X}} - [S_1 T_1(I_{\mathcal{X}} - S_2 T_2) + S_2 T_2] + R = I_{\mathcal{X}} - S_3 T_3,$$

and therefore $\text{SC}_{k_1+k_2}(\mathcal{X}, \mathcal{Y}) \neq \emptyset$ by another application of Theorem 4.1. This shows that $k_1 + k_2 \in \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$, as required.

The case where $\mathcal{G}_{\mathcal{X}}(\mathcal{Y})$ is essentially closed under addition is similar, just using condition (ii) in Theorem 4.1 instead of condition (i). \square

Remark 5.8. The set $\mathcal{G}_{\mathcal{Y}}(\mathcal{X})$ is closed under addition (without the need for any inessential perturbations) if the Banach space \mathcal{Y} contains a complemented subspace isomorphic to $\mathcal{Y} \oplus \mathcal{Y}$. This result is “folklore”; it can for instance be found in [39, the paragraph following Definition 3.6]. Most “classical” Banach spaces \mathcal{Y} satisfy the even stronger condition that $\mathcal{Y} \cong \mathcal{Y} \oplus \mathcal{Y}$. The two conditions are not equivalent because Gowers and Maurey [27, §(4.4)] have constructed a Banach space \mathcal{Y} which is isomorphic to its cube $\mathcal{Y} \oplus \mathcal{Y} \oplus \mathcal{Y}$, but not to its square $\mathcal{Y} \oplus \mathcal{Y}$. Hence \mathcal{Y} contains a complemented subspace isomorphic to $\mathcal{Y} \oplus \mathcal{Y}$ without being isomorphic to it.

There are infinite-dimensional Banach spaces \mathcal{Y} which do not contain any complemented subspaces isomorphic to $\mathcal{Y} \oplus \mathcal{Y}$. James’ quasi-reflexive Banach space, which will feature prominently in our next example, was the first space shown to have this property.

Example 5.9. The purpose of this example is to show that the converse of Proposition 5.7 fails; that is, we shall construct Banach spaces \mathcal{X} and \mathcal{Y} such that $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ is closed under addition, but neither $\mathcal{G}_{\mathcal{Y}}(\mathcal{X})$ nor $\mathcal{G}_{\mathcal{X}}(\mathcal{Y})$ are essentially closed under addition. This construction relies heavily on the quasi-reflexive James spaces \mathcal{J}_p for $1 < p < \infty$. These Banach spaces originate in James’ paper [34], where only the case $p = 2$ was considered. Subsequently, Edelstein and Mityagin [19] observed that James’ methods and results carry over to arbitrary $p \in (1, \infty)$. We require the following specific facts about this family of Banach spaces:

- (i) \mathcal{J}_p is isomorphic to its hyperplanes for every $p \in (1, \infty)$, so $\gamma(\mathcal{J}_p) = 1$.
- (ii) $\mathcal{B}(\mathcal{J}_q, \mathcal{J}_p) = \mathcal{K}(\mathcal{J}_q, \mathcal{J}_p)$ for $1 < p < q < \infty$ by [43, Theorem 4.5], and therefore \mathcal{J}_p and \mathcal{J}_q are essentially incomparable whenever $p, q \in (1, \infty)$ are distinct.

- (iii) $\mathcal{K}(\mathcal{J}_p) = \mathcal{E}(\mathcal{J}_p) \subseteq \mathcal{W}(\mathcal{J}_p)$ for every $p \in (1, \infty)$ by [37, Proposition 4.9], where $\mathcal{W}(\mathcal{J}_p)$ denotes the ideal of weakly compact operators on \mathcal{J}_p .

Berkson and Porta [11, page 18] for $p = 2$ and Edelstein and Mityagin [19] for general $p \in (1, \infty)$ observed that $\mathcal{W}(\mathcal{J}_p)$ has codimension 1 in $\mathcal{B}(\mathcal{J}_p)$, so we have a unital algebra homomorphism $\varphi: \mathcal{B}(\mathcal{J}_p) \rightarrow \mathbb{K}$ with $\ker \varphi = \mathcal{W}(\mathcal{J}_p)$. We shall in fact require the amplification of this homomorphism to the 2×2 matrices, that is, the unital algebra homomorphism $\varphi_2: M_2(\mathcal{B}(\mathcal{J}_p)) \rightarrow M_2(\mathbb{K})$ given by

$$\varphi_2 \left(\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right) = \begin{bmatrix} \varphi(R_{11}) & \varphi(R_{12}) \\ \varphi(R_{21}) & \varphi(R_{22}) \end{bmatrix}.$$

We are now ready to begin our construction: choose distinct numbers $p, q \in (1, \infty)$, and set $\mathcal{X} = \mathcal{J}_p \oplus \mathcal{J}_p \oplus \mathcal{J}_q$ and $\mathcal{Y} = \mathcal{J}_p \oplus \mathcal{J}_q \oplus \mathcal{J}_q$.

First, we observe that $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ is closed under addition. This follows immediately from the fact that $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \mathbb{Z}$. Indeed, for each $k \in \mathbb{Z}$, (i) implies that we can take $R \in \Phi_k(\mathcal{J}_p)$. Then the operators

$$S = \begin{bmatrix} I_{\mathcal{J}_p} - R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{Y}, \mathcal{X}) \quad \text{and} \quad T = \begin{bmatrix} I_{\mathcal{J}_p} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$$

satisfy

$$I_{\mathcal{X}} - ST = \begin{bmatrix} R & 0 & 0 \\ 0 & I_{\mathcal{J}_p} & 0 \\ 0 & 0 & I_{\mathcal{J}_q} \end{bmatrix} \in \Phi_k(\mathcal{X}),$$

so $\text{SC}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset$ by Theorem 4.1, and therefore $k \in \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$, as desired.

Second, we shall show the set $\mathcal{G}_{\mathcal{Y}}(\mathcal{X})$ is not essentially closed under addition. Assume the contrary, and consider the operators

$$U = \begin{bmatrix} I_{\mathcal{J}_p} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_{\mathcal{J}_p} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{\mathcal{J}_p} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{X})$$

and

$$V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{\mathcal{J}_p} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ I_{\mathcal{J}_p} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & I_{\mathcal{J}_p} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{X}).$$

They both belong to $\mathcal{G}_{\mathcal{Y}}(\mathcal{X})$ as the indicated factorizations show. Therefore, by the hypothesis, we can find operators $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that $U + V - ST \in \mathcal{E}(\mathcal{X})$. By (ii), we can write $S = S_1 + S_2$ and $T = T_1 + T_2$, where

$$S_1 = \begin{bmatrix} S_{11} & 0 & 0 \\ S_{21} & 0 & 0 \\ 0 & S_{32} & S_{33} \end{bmatrix} \quad \text{and} \quad T_1 = \begin{bmatrix} T_{11} & T_{12} & 0 \\ 0 & 0 & T_{23} \\ 0 & 0 & T_{33} \end{bmatrix},$$

and $S_2 \in \mathcal{E}(\mathcal{Y}, \mathcal{X})$ and $T_2 \in \mathcal{E}(\mathcal{X}, \mathcal{Y})$. Since \mathcal{E} is an operator ideal, we deduce that

$$\begin{aligned} \mathcal{E}(\mathcal{J}_p \oplus \mathcal{J}_p) \ni \begin{bmatrix} I_{\mathcal{J}_p} & 0 & 0 \\ 0 & I_{\mathcal{J}_p} & 0 \end{bmatrix} (U + V - S_1 T_1) \begin{bmatrix} I_{\mathcal{J}_p} & 0 \\ 0 & I_{\mathcal{J}_p} \\ 0 & 0 \end{bmatrix} \\ = \begin{bmatrix} I_{\mathcal{J}_p} & 0 \\ 0 & I_{\mathcal{J}_p} \end{bmatrix} - \begin{bmatrix} S_{11} T_{11} & S_{11} T_{12} \\ S_{21} T_{11} & S_{21} T_{12} \end{bmatrix}. \end{aligned}$$

Hence, applying the algebra homomorphism φ_2 from (iii), we obtain

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \varphi_2 \left(\begin{bmatrix} I_{\mathcal{J}_p} & 0 \\ 0 & I_{\mathcal{J}_p} \end{bmatrix} \right) = \varphi_2 \left(\begin{bmatrix} S_{11} T_{11} & S_{11} T_{12} \\ S_{21} T_{11} & S_{21} T_{12} \end{bmatrix} \right) \\ = \begin{bmatrix} \varphi(S_{11})\varphi(T_{11}) & \varphi(S_{11})\varphi(T_{12}) \\ \varphi(S_{21})\varphi(T_{11}) & \varphi(S_{21})\varphi(T_{12}) \end{bmatrix}. \end{aligned}$$

However, this is impossible because the diagonal entries imply that $\varphi(S_{11})$, $\varphi(T_{11})$, $\varphi(S_{21})$ and $\varphi(T_{12})$ are all non-zero, but then the off-diagonal entries $\varphi(S_{11})\varphi(T_{12})$ and $\varphi(S_{21})\varphi(T_{11})$ are also non-zero. This contradiction proves that $\mathcal{G}_{\mathcal{X}}(\mathcal{Y})$ cannot be essentially closed under addition.

Finally, a similar argument with \mathcal{X} and \mathcal{Y} interchanged shows that the set $\mathcal{G}_{\mathcal{X}}(\mathcal{Y})$ is not essentially closed under addition.

Note. We are grateful to the referee for pointing out that the above construction will also work if we replace the James spaces \mathcal{J}_p and \mathcal{J}_q with their “long” counterparts $\mathcal{J}_p(\omega_1)$ and $\mathcal{J}_q(\omega_1)$. The space $\mathcal{J}_2(\omega_1)$ was originally defined by Edgar [21]. Kania and Kochanek [36, Section 3] studied $\mathcal{J}_p(\omega_1)$ for general $p \in (1, \infty)$, showing in particular that $\mathcal{B}(\mathcal{J}_p(\omega_1))$ contains a unique maximal ideal of codimension 1.

Remark 5.10. The purpose of this remark is to summarize our knowledge about the values of $k \in \mathbb{Z}$ for which the equation $\text{SC}_k(\mathcal{X}, \mathcal{Y}) = \text{EAE}_k(\mathcal{X}, \mathcal{Y})$ holds true for a given pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$, and explain how this problem is related to Question 5.5. Recall from (1.9) and Theorem 1.6(iii) that

$$\begin{aligned} \text{SC}_k(\mathcal{X}, \mathcal{Y}) = \text{EAE}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset \quad \text{for } k \in \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}), \\ \text{SC}_k(\mathcal{X}, \mathcal{Y}) = \text{EAE}_k(\mathcal{X}, \mathcal{Y}) = \emptyset \quad \text{for } k \in \mathbb{Z} \setminus \text{eae}(\mathcal{X}, \mathcal{Y})\mathbb{Z}. \end{aligned} \quad (5.3)$$

We now split in two cases, beginning with the case where the answer to Question 5.5 is affirmative, so that the set $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ is closed under addition. Then Corollary 5.4 shows that $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \text{sc}(\mathcal{X}, \mathcal{Y})\mathbb{Z}$ and

$$\emptyset = \text{SC}_k(\mathcal{X}, \mathcal{Y}) \neq \text{EAE}_k(\mathcal{X}, \mathcal{Y}) \quad \text{for } k \in \text{eae}(\mathcal{X}, \mathcal{Y})\mathbb{Z} \setminus \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}),$$

where we have applied Proposition 3.1 to conclude that $\text{EAE}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset$. Together with (5.3), this covers all possible values of $k \in \mathbb{Z}$. It follows in particular that EAE and SC coincides for all pairs of Fredholm operators $(U, V) \in \Phi(\mathcal{X}) \times \Phi(\mathcal{Y})$ if and only if $\text{eae}(\mathcal{X}, \mathcal{Y}) = \text{sc}(\mathcal{X}, \mathcal{Y})$, as we have already seen in Proposition 1.7.

Otherwise, when the answer to Question 5.5 is negative, so that $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ fails to be closed under addition, Proposition 1.7 implies that $\text{sc}(\mathcal{X}, \mathcal{Y}) = n \text{eae}(\mathcal{X}, \mathcal{Y})$ for some $n \geq 2$, and

$$\emptyset = \text{SC}_k(\mathcal{X}, \mathcal{Y}) \neq \text{EAE}_k(\mathcal{X}, \mathcal{Y})$$

for every $k \in \{\pm m \text{eae}(\mathcal{X}, \mathcal{Y}) : 1 \leq m < n\}$ because $k = \text{sc}(\mathcal{X}, \mathcal{Y})$ is the smallest positive number for which $\text{SC}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset$, and $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ is closed under sign

changes. However, since $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ fails to be closed under addition, there must be some number $k = m \text{eae}(\mathcal{X}, \mathcal{Y})$, where $m \in \mathbb{N} \cap (n, \infty) \setminus n\mathbb{N}$, for which

$$\text{SC}_k(\mathcal{X}, \mathcal{Y}) = \text{EAE}_k(\mathcal{X}, \mathcal{Y}) \neq \emptyset.$$

We conclude this section with a couple of results about direct sums which will be useful in the proof of Proposition 1.12, as well as in the next section.

Lemma 5.11. *Let \mathcal{X} and \mathcal{Y} be Banach spaces, and suppose that \mathcal{Y} is isomorphic to a complemented subspace of \mathcal{X} . Then*

$$\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \mathbb{I}_{\Phi}(\mathcal{Y}) \subseteq \mathbb{I}_{\Phi}(\mathcal{X}), \quad (5.4)$$

and consequently $\text{sc}(\mathcal{X}, \mathcal{Y}) = \text{eae}(\mathcal{X}, \mathcal{Y})$.

Proof. In view of Lemmas 3.9 and 5.3, we may suppose that $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$ for some Banach space \mathcal{Z} .

The inclusion $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) \subseteq \mathbb{I}_{\Phi}(\mathcal{Y})$ is clear because $\text{SC}_k(\mathcal{X}, \mathcal{Y}) \subseteq \Phi_k(\mathcal{X}) \times \Phi_k(\mathcal{Y})$. Conversely, for $k \in \mathbb{I}_{\Phi}(\mathcal{Y})$, we can take $R \in \Phi_k(\mathcal{Y})$. Then the operators

$$S = \begin{bmatrix} I_{\mathcal{Y}} - R \\ 0 \end{bmatrix} : \mathcal{Y} \rightarrow \mathcal{Y} \oplus \mathcal{Z} = \mathcal{X} \quad \text{and} \quad T = [I_{\mathcal{Y}} \quad 0] : \mathcal{X} = \mathcal{Y} \oplus \mathcal{Z} \rightarrow \mathcal{Y}$$

satisfy

$$I_{\mathcal{X}} - ST = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & I_{\mathcal{Z}} \end{bmatrix} - \begin{bmatrix} I_{\mathcal{Y}} - R & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & I_{\mathcal{Z}} \end{bmatrix} \in \Phi_k(\mathcal{X}), \quad (5.5)$$

so $k \in \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$ by Theorem 4.1. This shows that $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \mathbb{I}_{\Phi}(\mathcal{Y})$, while the inclusion $\mathbb{I}_{\Phi}(\mathcal{Y}) \subseteq \mathbb{I}_{\Phi}(\mathcal{X})$ is an immediate consequence of (5.5). (Alternatively, the latter inclusion follows easily from Lemma 3.10(i).)

Finally, we have $\text{sc}(\mathcal{X}, \mathcal{Y}) = \text{eae}(\mathcal{X}, \mathcal{Y})$ because (5.4) shows that $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \mathbb{I}_{\Phi}(\mathcal{X}) \cap \mathbb{I}_{\Phi}(\mathcal{Y})$. \square

Lemma 5.12. *Let $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$ be Banach spaces. Then:*

- (i) $\mathbb{I}_{\text{SC}}(\mathcal{X}_2, \mathcal{Y}_2) \subseteq \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$.
- (ii) *Suppose that each of the pairs $(\mathcal{X}_1, \mathcal{Y}_1)$, $(\mathcal{X}_1, \mathcal{Y}_2)$ and $(\mathcal{Y}_1, \mathcal{X}_2)$ is essentially incomparable. Then $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \mathbb{I}_{\text{SC}}(\mathcal{X}_2, \mathcal{Y}_2)$.*

Proof. (i). For each $k \in \mathbb{I}_{\text{SC}}(\mathcal{X}_2, \mathcal{Y}_2)$, we can take Schur-coupled operators $U \in \Phi_k(\mathcal{X}_2)$ and $V \in \Phi_k(\mathcal{Y}_2)$. Choose isomorphisms $A \in \mathcal{B}(\mathcal{X}_2)$ and $D \in \mathcal{B}(\mathcal{Y}_2)$ and operators $B \in \mathcal{B}(\mathcal{Y}_2, \mathcal{X}_2)$ and $C \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_2)$ such that (1.2) is satisfied. Then it is easy to see that the operators

$$\begin{bmatrix} I_{\mathcal{X}_1} & 0 \\ 0 & U \end{bmatrix} \in \Phi_k(\mathcal{X}) \quad \text{and} \quad \begin{bmatrix} I_{\mathcal{Y}_1} & 0 \\ 0 & V \end{bmatrix} \in \Phi_k(\mathcal{Y})$$

are Schur-coupled via

$$\begin{bmatrix} I_{\mathcal{X}_1} & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} I_{\mathcal{X}_1} & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I_{\mathcal{Y}_1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}$$

and

$$\begin{bmatrix} I_{\mathcal{Y}_1} & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} I_{\mathcal{Y}_1} & 0 \\ 0 & D \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} I_{\mathcal{X}_1} & 0 \\ 0 & A^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix},$$

so we conclude that $k \in \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$.

(ii). Suppose that $k \in \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$. By Theorem 4.1, we can find operators

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \in \mathcal{B}(\mathcal{Y}, \mathcal{X}) \quad \text{and} \quad T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$$

such that $I_{\mathcal{X}} - ST \in \Phi_k(\mathcal{X})$. The hypothesis implies that $S_{11}, T_{11}, S_{12}, T_{12}, S_{21}$ and T_{21} are inessential. Since \mathcal{E} is an operator ideal, it follows that the operator

$$ST - \begin{bmatrix} 0 & 0 \\ 0 & S_{22}T_{22} \end{bmatrix}$$

is inessential, and therefore, using Remark 2.3(iii), we obtain

$$\begin{aligned} I_{\mathcal{X}} - ST \in \Phi_k(\mathcal{X}) &\iff \begin{bmatrix} I_{\mathcal{X}_1} & 0 \\ 0 & I_{\mathcal{X}_2} - S_{22}T_{22} \end{bmatrix} \in \Phi_k(\mathcal{X}) \\ &\iff I_{\mathcal{X}_2} - S_{22}T_{22} \in \Phi_k(\mathcal{X}_2), \end{aligned}$$

so $k \in \mathbb{I}_{\text{SC}}(\mathcal{X}_2, \mathcal{Y}_2)$ by another application of Theorem 4.1. \square

We can now complete the proof of Proposition 1.12.

Proof of Proposition 1.12. We have already proved Equation (1.11) on page 14.

To prove (1.12), suppose that each of the pairs $(\mathcal{X}_1, \mathcal{X}_2)$, $(\mathcal{Y}_1, \mathcal{Y}_2)$ and $(\mathcal{X}_1, \mathcal{Y}_1)$ is essentially incomparable. Then the hypothesis of Lemma 5.12(ii) is satisfied because $\mathcal{X}_2 \cong \mathcal{Y}_2$, so $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \mathbb{I}_{\text{SC}}(\mathcal{X}_2, \mathcal{Y}_2)$, and therefore $\text{sc}(\mathcal{X}, \mathcal{Y}) = \text{sc}(\mathcal{X}_2, \mathcal{Y}_2)$. Moreover, Lemma 5.11 implies that $\text{sc}(\mathcal{X}_2, \mathcal{Y}_2) = \text{eae}(\mathcal{X}_2, \mathcal{Y}_2)$, which completes the proof of (1.12).

Combining (1.11) and (1.12), we see that $\text{eae}(\mathcal{X}, \mathcal{Y}) = \text{sc}(\mathcal{X}, \mathcal{Y})$ if and only if $\text{gcd}(\text{eae}(\mathcal{X}_1, \mathcal{Y}_1), \text{eae}(\mathcal{X}_2, \mathcal{Y}_2)) = \text{eae}(\mathcal{X}_2, \mathcal{Y}_2)$, which is equivalent to saying that $\text{eae}(\mathcal{X}_2, \mathcal{Y}_2)$ divides $\text{eae}(\mathcal{X}_1, \mathcal{Y}_1)$. \square

6. THE GOWERS–MAUREY–AIENA–GONZÁLEZ–FERENCZI CYCLE OF IDEAS AND THE PROOFS OF THEOREMS 1.9, 1.11 AND 1.13

The Banach space that Aiena and González used in [2] to show that projective incomparability does not imply essential incomparability is the so-called “shift space” constructed by Gowers and Maurey in [27, §(4.2)]. Refining the approach of Aiena and González, Ferenczi [22, Section 4] has more recently used this space to prove that there is no largest proper operator ideal, thereby solving a famous open problem going back to Pietsch’s monograph [46].

The proofs of Theorems 1.11 and 1.13 are inspired by this body of work. However, the shift space itself will not suffice for our purposes; we need to work with a larger family of “higher-order shift spaces” which Gowers and Maurey also constructed in [27]. We shall now give a brief introduction to this family.

Following the terminology introduced in [27, page 549], for two infinite subsets $A = \{a_1 < a_2 < \dots\}$ and $B = \{b_1 < b_2 < \dots\}$ of \mathbb{N} , we define the associated *spread* $S_{A,B}$ to be the linear map on c_{00} determined by

$$S_{A,B}e_j = \begin{cases} e_{b_k} & \text{if } j = a_k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

where $(e_n)_{n \in \mathbb{N}}$ denotes the standard unit vector (Hamel) basis for c_{00} . Let $k_0 \in \mathbb{N}_0$, and set $\mathcal{M}_{k_0} = \{[k_0m + 1, \infty) \cap \mathbb{N} : m \in \mathbb{N}_0\}$. Then $\mathcal{S}_{k_0} = \{S_{A,B} : A, B \in \mathcal{M}_{k_0}\}$ is a “proper set of spreads” as defined in [27, page 549], but since we do not need the precise definition of this term in the sequel, we omit the details. The

important point is that, by [27, Theorem 5], \mathcal{S}_{k_0} induces a Banach space, which we shall call the k_0 -fold Gowers–Maurey shift space and denote by $\mathcal{X}_{\text{GM}}(k_0)$. (It is denoted $X(\mathcal{S}_{k_0})$ in [27].)

As already mentioned, Gowers and Maurey defined and investigated this family of Banach spaces in [27]. More precisely, they studied the space $\mathcal{X}_{\text{GM}}(0)$ in [27, §(4.1)], $\mathcal{X}_{\text{GM}}(1)$ in [27, §(4.2)] and $\mathcal{X}_{\text{GM}}(2)$ in [27, §(4.3)], before outlining the general case of $\mathcal{X}_{\text{GM}}(k_0)$ for $k_0 \geq 3$ in the final paragraph of [27, §(4.3)]. The following theorem summarizes the results from [27] that we require, together with the necessary notation and terminology.

Theorem 6.1 (Gowers and Maurey). *Let $k_0 \in \mathbb{N}_0$.*

- (i) *The Banach space $\mathcal{X}_{\text{GM}}(k_0)$ has a normalized Schauder basis $(e_n)_{n \in \mathbb{N}}$ which admits an isometric k_0 -fold right shift operator $R_{k_0} \in \mathcal{B}(\mathcal{X}_{\text{GM}}(k_0))$ given by $R_{k_0}e_n = e_{n+k_0}$ for every $n \in \mathbb{N}$, with left inverse $L_{k_0} \in \mathcal{B}(\mathcal{X}_{\text{GM}}(k_0))$ given by $L_{k_0}e_n = 0$ for $n \leq k_0$ and $L_{k_0}e_n = e_{n-k_0}$ for $n > k_0$.*
- (ii) *The Banach space $\mathcal{X}_{\text{GM}}(k_0)$ satisfies a lower f -estimate for the function $f(t) = \log_2(t+1)$; that is,*

$$\log_2(n+1) \left\| \sum_{k=1}^n x_k \right\| \geq \sum_{k=1}^n \|x_k\|$$

for every $n \in \mathbb{N}$ and vectors $x_1, \dots, x_n \in \mathcal{X}_{\text{GM}}(k_0)$ which are consecutive in the sense that there are integers $0 \leq m_0 < m_1 < \dots < m_n$ such that $x_k \in \text{span}\{e_j : m_{k-1} < j \leq m_k\}$ for each $1 \leq k \leq n$.

- (iii) *The Banach space $\mathcal{X}_{\text{GM}}(k_0)$ is indecomposable; that is, every complemented subspace of $\mathcal{X}_{\text{GM}}(k_0)$ is either finite-dimensional or finite-codimensional.*
- (iv) *The index γ introduced in Remark 3.6 is given by $\gamma(\mathcal{X}_{\text{GM}}(k_0)) = k_0$, and $\mathcal{X}_{\text{GM}}(k_0)$ is not isomorphic to any of its subspaces of infinite codimension. Therefore a closed subspace \mathcal{W} of $\mathcal{X}_{\text{GM}}(k_0)$ is isomorphic to $\mathcal{X}_{\text{GM}}(k_0)$ if and only if*

$$\dim \mathcal{X}_{\text{GM}}(k_0)/\mathcal{W} \in k_0\mathbb{N}_0.$$

- (v) *The Banach space $\mathcal{X}_{\text{GM}}(k_0)$ contains no unconditional basic sequences.*

Proof. Parts (i) and (ii) follow from [27, Theorem 5] and the definitions and conventions that it relies on.

For $k_0 = 0$, parts (iii)–(v) are all derived in [27, §(4.1)]. (Note in this context that $L_0 = R_0 = I_{\mathcal{X}_{\text{GM}}(0)}$.) Hence it remains to consider $k_0 \in \mathbb{N}$.

(iii). This result is contained in the proof of [27, Theorem 13] for $k_0 = 1$, with [27, Remarks, page 559] explaining how to generalize that proof to arbitrary $k_0 \geq 2$.

(iv). This result is a restatement of [27, Theorem 16] for $k_0 = 1$. For $k_0 \geq 2$, it follows from [27, Theorem 19] and [27, Remarks, page 559].

(v). This result is proved in the final paragraph on [27, page 567]. \square

Corollary 6.2. *Let \mathcal{Y} be a Banach space with an unconditional basis. Then, for every $k_0 \in \mathbb{N}_0$, $\mathcal{X}_{\text{GM}}(k_0)$ and \mathcal{Y} are totally incomparable.*

Proof. As explained in the comment after [42, Problem 1.d.5], every closed, infinite-dimensional subspace of \mathcal{Y} contains an unconditional basic sequence. Hence Theorem 6.1(v) implies that no such subspace embeds isomorphically into $\mathcal{X}_{\text{GM}}(k_0)$. \square

Using these results, we can easily prove Theorem 1.9.

Proof of Theorem 1.9. (i). We must show that $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \{0\}$. This was already proved in [31, Theorem 2.1(2)], but we would like to point out that it is also an almost immediate consequence of Theorem 4.1. Indeed, the essential incomparability of \mathcal{X} and \mathcal{Y} means that $I_{\mathcal{X}} - ST \in \Phi(\mathcal{X})$ for every $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and Remark 2.3(iii) shows that $i(I_{\mathcal{X}} - ST) = 0$. Therefore condition (i) in Theorem 4.1 is satisfied only for $k = 0$, so $\text{SC}_k(\mathcal{X}, \mathcal{Y}) = \emptyset$ for every $k \in \mathbb{Z} \setminus \{0\}$.

(ii). Set $\mathcal{X} = \mathcal{X}_{\text{GM}}(k_0)$, and let \mathcal{Y} be a Banach space which has an unconditional basis and is isomorphic to its hyperplanes, so that $\gamma(\mathcal{Y}) = 1$. (For instance, $\mathcal{Y} = \ell_2$ has these properties.) Corollary 6.2 shows that \mathcal{X} and \mathcal{Y} are totally incomparable and therefore essentially incomparable. Moreover, we have $\gamma(\mathcal{X}) = k_0$ by Theorem 6.1(iv), so $\text{eae}(\mathcal{X}, \mathcal{Y}) = \text{lcm}(k_0, 1) = k_0$ by (3.1). \square

While the above proof did not involve any ideas from [2] or [22], the proofs of Theorems 1.11 and 1.13 will, namely in the shape of part (ii) of the following lemma.

Lemma 6.3. *Let $k_0 \in \mathbb{N}$.*

- (i) *Suppose that \mathcal{X}_1 and \mathcal{Y}_1 are essentially incomparable Banach spaces with unconditional bases and that \mathcal{Y}_2 is a closed, infinite-dimensional and infinite-codimensional subspace of $\mathcal{X}_{\text{GM}}(k_0)$. Then the Banach spaces $\mathcal{X}_1 \oplus \mathcal{X}_{\text{GM}}(k_0)$ and $\mathcal{Y}_1 \oplus \mathcal{Y}_2$ are projectively incomparable.*
- (ii) *The Banach space $\mathcal{X}_{\text{GM}}(k_0)$ contains a closed, infinite-dimensional and infinite-codimensional subspace \mathcal{Y}_2 such that $\text{SC}_{k_0}(\mathcal{X}_{\text{GM}}(k_0), \mathcal{Y}_2) \neq \emptyset$.*

Remark 6.4. Lemma 6.3(i) is also true for $\mathcal{X}_1 = \{0\}$ (even though it may be debatable whether this space has an unconditional basis). This observation will be important in the proofs of Theorems 1.11(i) and 1.13(i). The conscientious reader can check that the proof which we are about to present remains valid for $\mathcal{X}_1 = \{0\}$.

Proof of Lemma 6.3(i). To unify notation, set $\mathcal{X}_2 = \mathcal{X}_{\text{GM}}(k_0)$. The proof is by contradiction, so assume that $\mathcal{X}_1 \oplus \mathcal{X}_2$ contains an infinite-dimensional, complemented subspace \mathcal{W} which is isomorphic to a complemented subspace \mathcal{Z} of $\mathcal{Y}_1 \oplus \mathcal{Y}_2$. Corollary 6.2 shows that each of the pairs $(\mathcal{X}_1, \mathcal{X}_2)$ and $(\mathcal{Y}_1, \mathcal{Y}_2)$ is totally incomparable, so a theorem of Edelstein and Wojtaszczyk (see [20, Theorem 3.5], or [42, Theorem 2.c.13] for an exposition) implies that $\mathcal{W} \cong \mathcal{W}_1 \oplus \mathcal{W}_2$ and $\mathcal{Z} \cong \mathcal{Z}_1 \oplus \mathcal{Z}_2$, where \mathcal{W}_j and \mathcal{Z}_j are complemented subspaces of \mathcal{X}_j and \mathcal{Y}_j , respectively, for $j \in \{1, 2\}$. Take an isomorphism

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} : \mathcal{W}_1 \oplus \mathcal{W}_2 \rightarrow \mathcal{Z}_1 \oplus \mathcal{Z}_2.$$

The hypothesis that \mathcal{X}_1 and \mathcal{Y}_1 are essentially incomparable implies that the operator U_{11} is inessential because essential incomparability clearly passes to complemented subspaces. Moreover, U_{12} and U_{21} are inessential by Corollary 6.2 and Remark 2.5(i). Consequently

$$\begin{bmatrix} 0 & 0 \\ 0 & U_{22} \end{bmatrix} = U - \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & 0 \end{bmatrix}$$

is an inessential perturbation of the isomorphism U and hence a Fredholm operator. This implies that U_{22} is a Fredholm operator and that \mathcal{W}_1 is finite-dimensional, so \mathcal{W}_2 must be infinite-dimensional. Since it is complemented in \mathcal{X}_2 , Theorem 6.1(iii) shows that \mathcal{W}_2 has finite codimension in \mathcal{X}_2 .

Choose a closed subspace \mathcal{W}_3 of finite codimension in \mathcal{W}_2 such that

$$\mathcal{W}_3 \cap \ker U_{22} = \{0\} \quad \text{and} \quad \dim \mathcal{X}_2/\mathcal{W}_3 \in k_0\mathbb{N}_0.$$

Then $\mathcal{W}_3 \cong \mathcal{X}_2$ by Theorem 6.1(iv), and the restriction of U_{22} to \mathcal{W}_3 is an isomorphic embedding into $\mathcal{Z}_2 \subseteq \mathcal{Y}_2$. However, another application of Theorem 6.1(iv) shows that no such embedding exists because \mathcal{Y}_2 has infinite codimension in \mathcal{X}_2 . \square

In order to prove the second part of Lemma 6.3, we require two lemmas. The statement of the first of these involves the following standard piece of terminology. An operator $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ (where \mathcal{X} and \mathcal{Y} can be any Banach spaces) is *bounded below* if there exists $\varepsilon > 0$ such that $\|Tx\| \geq \varepsilon\|x\|$ for every $x \in \mathcal{X}$. This is equivalent to saying that T is injective and has closed range, or in other words that T is an isomorphic embedding.

Lemma 6.5. *For every $k_0 \in \mathbb{N}$, the operator $I_{\mathcal{X}_{\text{GM}}(k_0)} - L_{k_0} \in \mathcal{B}(\mathcal{X}_{\text{GM}}(k_0))$ is injective, but not bounded below. Consequently its range is not closed in $\mathcal{X}_{\text{GM}}(k_0)$.*

Proof. The proof is a simple variant of an argument given by Ferenczi in the text preceding [22, Proposition 16]. First, to show that $I_{\mathcal{X}_{\text{GM}}(k_0)} - L_{k_0}$ is injective, suppose that $x = \sum_{j=1}^{\infty} a_j e_j \in \ker(I_{\mathcal{X}_{\text{GM}}(k_0)} - L_{k_0})$. Then we have

$$0 = \sum_{j=1}^{\infty} a_j e_j - \sum_{j=1}^{\infty} a_{j+k_0} e_j,$$

so $a_j = a_{j+k_0}$ for each $j \in \mathbb{N}$. By induction, we deduce that $a_j = a_{j+mk_0}$ for each $m \in \mathbb{N}$. Keeping j fixed and letting $m \rightarrow \infty$, we have $a_{j+mk_0} \rightarrow 0$, so $a_j = 0$. Since this is true for every $j \in \mathbb{N}$, we conclude that $x = 0$.

Second, to verify that $I_{\mathcal{X}_{\text{GM}}(k_0)} - L_{k_0}$ is not bounded below, we consider the vector $w_n = \sum_{j=1}^n e_j \in \mathcal{X}_{\text{GM}}(k_0)$ for $n > k_0$. Theorem 6.1(ii) implies that

$$\log_2(n+1) \|w_n\| \geq \sum_{j=1}^n \|e_j\| = n,$$

while

$$\|(I_{\mathcal{X}_{\text{GM}}(k_0)} - L_{k_0})w_n\| = \left\| \sum_{j=n-k_0+1}^n e_j \right\| \leq k_0,$$

so

$$\frac{\|(I_{\mathcal{X}_{\text{GM}}(k_0)} - L_{k_0})w_n\|}{\|w_n\|} \leq \frac{k_0 \log_2(n+1)}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Consequently $I_{\mathcal{X}_{\text{GM}}(k_0)} - L_{k_0}$ is not bounded below. \square

The other lemma that we require originates in the work of Lebow and Schechter [40, Theorem 5.4].

Lemma 6.6. *Let \mathcal{X} and \mathcal{Y} be Banach spaces, and suppose that $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is an operator whose range is not closed. Then, for every $\varepsilon > 0$, there exists a nuclear operator $B \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that $\|B\| < \varepsilon$ and the closure of the range of the operator $A - B$ has infinite codimension in \mathcal{Y} .*

Proof. One can prove this lemma by following the steps of the proof of [40, Theorem 5.4], starting in line 4 with the hypothesis that the range of the operator A is not closed. The only modification required is that to ensure that the nuclear

operator $B = \sum_{k=1}^{\infty} (A'y'_k) \otimes y_k$ has norm less than ε , we must replace the third inequality in [40, Equation (5.4)] with the estimate $\|A'y'_k\| < \varepsilon/2^k a_k$. \square

Proof of Lemma 6.3(ii). Combining Lemmas 6.5 and 6.6, we can find a nuclear operator $B \in \mathcal{B}(\mathcal{X}_{\text{GM}}(k_0))$ such that the closed subspace

$$\mathcal{Y}_2 = \overline{\text{ran}(I_{\mathcal{X}_{\text{GM}}(k_0)} - L_{k_0} - B)}$$

has infinite codimension in $\mathcal{X}_{\text{GM}}(k_0)$.

To show that $\text{SC}_{k_0}(\mathcal{X}_{\text{GM}}(k_0), \mathcal{Y}_2) \neq \emptyset$, let $T \in \mathcal{B}(\mathcal{X}_{\text{GM}}(k_0), \mathcal{Y}_2)$ denote the operator $I_{\mathcal{X}_{\text{GM}}(k_0)} - L_{k_0} - B$ regarded as a map into \mathcal{Y}_2 , and let $S \in \mathcal{B}(\mathcal{Y}_2, \mathcal{X}_{\text{GM}}(k_0))$ be the natural inclusion map. Then we have $I_{\mathcal{X}_{\text{GM}}(k_0)} - ST = L_{k_0} + B \in \Phi_{k_0}(\mathcal{X}_{\text{GM}}(k_0))$ because $L_{k_0} \in \Phi_{k_0}(\mathcal{X}_{\text{GM}}(k_0))$ and B is compact. This shows that condition (i) in Theorem 4.1 is satisfied, and the conclusion follows from condition (iv).

Finally, we observe that \mathcal{Y}_2 must be infinite-dimensional because otherwise T would be a finite-rank operator, in which case $i(I_{\mathcal{X}_{\text{GM}}(k_0)} - ST) = 0 \neq k_0$. \square

Remark 6.7. Lemma 6.6 does not follow from the statement of [40, Theorem 5.4] itself. In fact, Lebow and Schechter could have concluded their proof of [40, Theorem 5.4] after its first four lines by invoking the well-known fact that if $\beta(A) < \infty$ for an operator A between Banach spaces, then A has closed range.

However, as we have seen, the remainder of their proof is very useful for our purposes because it establishes the stronger conclusion stated in Lemma 6.6 that we required to prove Lemma 6.3(ii). More precisely, what we needed was that we can perturb the operator A by an inessential operator B to obtain that $\overline{\text{ran}(A - B)}$ has infinite codimension in \mathcal{Y} . We did not need that the perturbation B can be chosen to be nuclear and have arbitrarily small norm; we chose to state those facts simply because they follow automatically from the proof.

We remark that both Aiena–González and Ferenczi cite [40, Theorem 5.4] in their work, but as far as we can see, that result does not suffice to give their conclusions. Like us, they appear to rely on the stronger statement given in Lemma 6.6.

Proof of Theorem 1.11. (i). Set $\mathcal{X} = \mathcal{X}_{\text{GM}}(k_0)$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$, where \mathcal{Y}_1 is a Banach space which is isomorphic to its hyperplanes and has an unconditional basis (so for instance we can take $\mathcal{Y}_1 = \ell_2$ or $\mathcal{Y}_1 = c_0$), and \mathcal{Y}_2 is the closed, infinite-dimensional and infinite-codimensional subspace of \mathcal{X} constructed in Lemma 6.3(ii). Then \mathcal{X} and \mathcal{Y} are projectively incomparable, as observed in Remark 6.4. Moreover, Theorem 6.1(iv) shows that $\gamma(\mathcal{X}) = k_0$, while $\gamma(\mathcal{Y}) = 1$ because \mathcal{Y}_1 being isomorphic to its hyperplanes implies that the same is true for \mathcal{Y} . Therefore $\text{eae}(\mathcal{X}, \mathcal{Y}) = \text{lcm}(k_0, 1) = k_0$ by (3.1).

In view of Proposition 1.7, this means that $\text{sc}(\mathcal{X}, \mathcal{Y})$ is a multiple of k_0 . Hence, to show that $\text{sc}(\mathcal{X}, \mathcal{Y}) = k_0$, it will suffice to show that $\text{SC}_{k_0}(\mathcal{X}, \mathcal{Y}) \neq \emptyset$, which follows by combining Lemma 5.12(i) with the fact that $\text{SC}_{k_0}(\mathcal{X}, \mathcal{Y}_2) \neq \emptyset$.

(ii). This proof is a slightly more elaborate variant of the proof of (i) that we have just given. We begin by choosing two distinct spaces \mathcal{X}_1 and \mathcal{Y}_1 from the family $\{\ell_p : 1 \leq p < \infty\} \cup \{c_0\}$, so that \mathcal{X}_1 and \mathcal{Y}_1 are isomorphic to their hyperplanes, have unconditional bases and are totally incomparable (as observed in [42, page 75], for instance). Set $\mathcal{X}_2 = \mathcal{X}_{\text{GM}}(k_0)$, and let \mathcal{Y}_2 be the subspace of \mathcal{X}_2 constructed in Lemma 6.3(ii), as in the first part of the proof. Then $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$ are projectively incomparable by Lemma 6.3(i), and $\gamma(\mathcal{X}) = \gamma(\mathcal{Y}) = 1$ because \mathcal{X}_1 and \mathcal{Y}_1 are isomorphic to their hyperplanes, so $\text{eae}(\mathcal{X}, \mathcal{Y}) = 1$ by (3.1).

It remains to verify that $\text{sc}(\mathcal{X}, \mathcal{Y}) = k_0$. A previous observation and Corollary 6.2 show that each of the pairs $(\mathcal{X}_1, \mathcal{Y}_1)$, $(\mathcal{X}_1, \mathcal{Y}_2)$ and $(\mathcal{X}_2, \mathcal{Y}_1)$ is totally incomparable and therefore essentially incomparable, so $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \mathbb{I}_{\text{SC}}(\mathcal{X}_2, \mathcal{Y}_2)$ by Lemma 5.12(ii). On the one hand, $k_0 \in \mathbb{I}_{\text{SC}}(\mathcal{X}_2, \mathcal{Y}_2)$ because $\text{SC}_{k_0}(\mathcal{X}_2, \mathcal{Y}_2) \neq \emptyset$, so Lemma 5.1(iii) implies that $k_0\mathbb{Z} \subseteq \mathbb{I}_{\text{SC}}(\mathcal{X}_2, \mathcal{Y}_2)$. On the other, we have $\mathbb{I}_{\text{SC}}(\mathcal{X}_2, \mathcal{Y}_2) \subseteq \mathbb{I}_{\Phi}(\mathcal{X}_2) = k_0\mathbb{Z}$, where the inclusion is obvious and the equality follows from Theorem 6.1(iv). Hence $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \mathbb{I}_{\text{SC}}(\mathcal{X}_2, \mathcal{Y}_2) = k_0\mathbb{Z}$, and the conclusion follows. \square

The proofs of the two parts of Theorem 1.13 are somewhat more complicated variants of the proofs of the corresponding parts of Theorem 1.11 given above. They involve one additional ingredient, namely Gowers' solution to Banach's hyperplane problem, which was the first infinite-dimensional Banach space shown not to be isomorphic to its hyperplanes (see [25], as well as [27, §(5.1)] for further results). The following result summarizes the properties of this space that we require.

Theorem 6.8 (Gowers). *There exists an infinite-dimensional, reflexive Banach space \mathcal{X}_{G} with an unconditional basis such that \mathcal{X}_{G} fails to be isomorphic to any proper subspace of itself.*

Proof. The only part of this statement that Gowers did not prove explicitly in [25] is that \mathcal{X}_{G} is reflexive. We believe that this fact is known to specialists, but as we have been unable to locate a proof of it in the literature, we outline one here. Since \mathcal{X}_{G} is not isomorphic to its hyperplanes, it cannot contain any complemented subspace which is isomorphic to its hyperplanes, so in particular no complemented subspace of \mathcal{X}_{G} is isomorphic to c_0 or ℓ_1 . Hence, no subspace of \mathcal{X}_{G} is isomorphic to c_0 by Sobczyk's Theorem (see, e.g., [42, Theorem 2.f.5]) or to ℓ_1 by a much more recent theorem of Finol and Wójtcowicz [23]. (This result was previously stated without proof in [41].) Therefore, a classical result of James (see [34], or [42, Theorem 1.c.12(a)] for an exposition) shows that \mathcal{X}_{G} is reflexive. \square

Proof of Theorem 1.13. (i). Following the same approach as in the proof of Theorem 1.11(i), but using different notation, we define $\mathcal{X}_1 = \mathcal{X}_{\text{GM}}(k_0)$ and $\mathcal{Y}_1 = c_0 \oplus \mathcal{Y}_2$, where \mathcal{Y}_2 is the subspace of \mathcal{X}_1 constructed in Lemma 6.3(ii). Then, as shown in the proof of Theorem 1.11(i), \mathcal{X}_1 and \mathcal{Y}_1 are projectively incomparable, so (1) holds, and

$$\text{eae}(\mathcal{X}_1, \mathcal{Y}_1) = k_0. \quad (6.1)$$

Let $\mathcal{Z} = \mathcal{X}_{\text{G}}$ be the Banach space from Theorem 6.8. Then $\gamma(\mathcal{Z}) = 0$, so $\text{eae}(\mathcal{Z}, \mathcal{Z}) = 0$, which verifies (3). Moreover, Corollary 6.2 shows that \mathcal{Z} is totally incomparable with \mathcal{X}_1 , and therefore also with \mathcal{Y}_2 . Since every closed subspace of c_0 contains an isomorphic copy of c_0 , while \mathcal{Z} is reflexive, \mathcal{Z} and c_0 are also totally incomparable, and therefore \mathcal{Z} and \mathcal{Y}_1 are essentially incomparable. This shows that (2) is satisfied.

It remains to verify (4). By (2), we can apply (1.11) to calculate $\text{eae}(\mathcal{X}, \mathcal{Y})$ for $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{Z}$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Z}$. Using (6.1), we obtain

$$\text{eae}(\mathcal{X}, \mathcal{Y}) = \text{gcd}(\text{eae}(\mathcal{X}_1, \mathcal{Y}_1), \text{eae}(\mathcal{Z}, \mathcal{Z})) = \text{gcd}(k_0, 0) = k_0. \quad (6.2)$$

Finally, we combine Lemma 5.12(i) with the fact that $\text{SC}_{k_0}(\mathcal{X}_1, \mathcal{Y}_2) \neq \emptyset$ to deduce that $\text{SC}_{k_0}(\mathcal{X}, \mathcal{Y}) \neq \emptyset$. In view of (6.2) and Proposition 1.7, this implies that $\text{sc}(\mathcal{X}, \mathcal{Y}) = k_0$, as we already saw in the proof of Theorem 1.11(i).

(ii). As above, let $\mathcal{Z} = \mathcal{X}_{\text{G}}$ be the Banach space from Theorem 6.8 and set $\mathcal{Y}_1 = c_0 \oplus \mathcal{Y}_2$, where \mathcal{Y}_2 is the subspace of $\mathcal{X}_{\text{GM}}(k_0)$ from Lemma 6.3(ii), but

now define $\mathcal{X}_1 = \ell_1 \oplus \mathcal{X}_{\text{GM}}(k_0)$. Then \mathcal{X}_1 and \mathcal{Y}_1 are projectively incomparable by Lemma 6.3(i). We showed in the first part of the proof that \mathcal{Y}_1 and \mathcal{Z} are essentially incomparable; a similar argument gives the same conclusion for \mathcal{X}_1 and \mathcal{Z} . Hence conditions (1)–(3) are satisfied.

Arguing as before, we see that the Banach spaces

$$\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{Z} = \ell_1 \oplus \mathcal{X}_{\text{GM}}(k_0) \oplus \mathcal{Z} \quad \text{and} \quad \mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Z} = c_0 \oplus \mathcal{Y}_2 \oplus \mathcal{Z}$$

satisfy $\gamma(\mathcal{X}) = \gamma(\mathcal{Y}) = 1$, so that $\text{eae}(\mathcal{X}, \mathcal{Y}) = 1$, and we also have $\text{SC}_{k_0}(\mathcal{X}, \mathcal{Y}) \neq \emptyset$, which implies that $k_0\mathbb{Z} \subseteq \mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y})$. To verify the opposite inclusion, we observe that each of the pairs (ℓ_1, c_0) , $(\ell_1, \mathcal{Y}_2 \oplus \mathcal{Z})$ and $(c_0, \mathcal{X}_{\text{GM}}(k_0) \oplus \mathcal{Z})$ is essentially incomparable, so Lemma 5.12(ii) shows that

$$\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = \mathbb{I}_{\text{SC}}(\mathcal{X}_{\text{GM}}(k_0) \oplus \mathcal{Z}, \mathcal{Y}_2 \oplus \mathcal{Z}) \subseteq \mathbb{I}_{\Phi}(\mathcal{X}_{\text{GM}}(k_0) \oplus \mathcal{Z}) = k_0\mathbb{Z},$$

where the final equality follows from Corollary 6.2 and Lemma 3.10(ii). Hence we have $\mathbb{I}_{\text{SC}}(\mathcal{X}, \mathcal{Y}) = k_0\mathbb{Z}$, and therefore $\text{sc}(\mathcal{X}, \mathcal{Y}) = k_0$. \square

Acknowledgements. We are grateful to Tomasz Kania for having brought the reference [23] to our attention.

This work is based on research supported in part by the National Research Foundation of South Africa (NRF) and the DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS). Any opinion, finding and conclusion or recommendation expressed in this material is that of the authors and the NRF and CoE-MaSS do not accept any liability in this regard.

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