# ELEMENTARY LIE ALGEBRAS AND LIE A-ALGEBRAS

David A. Towers
Department of Mathematics, Lancaster University
Lancaster LA1 4YF, England
and
Vicente R. Varea [1]
Department of Mathematics, University of Zaragoza
Zaragoza, 50009 Spain

**Abstract**

A finite-dimensional Lie algebra $L$ over a field $F$ is called elementary if each of its subalgebras has trivial Frattini ideal; it is an $A$-algebra if every nilpotent subalgebra is abelian. The present paper is primarily concerned with the classification of elementary Lie algebras. In particular, we provide a complete list in the case when $F$ is algebraically closed and of characteristic different from 2,3, reduce the classification over fields of characteristic 0 to the description of elementary semisimple Lie algebras, and identify the latter in the case when $F$ is the real field. Additionally it is shown that over fields of characteristic 0 every elementary Lie algebra is almost algebraic; in fact, if $L$ has no non-zero semisimple ideals, then it is elementary if and only if it is an almost algebraic $A$-algebra.

*Keywords:* Lie algebra, elementary, $E$-algebra, $A$-algebra, almost algebraic, ad-semisimple

## 1 Introduction

Throughout this paper $L$ will denote a finite-dimensional Lie algebra over a field $F$. The Frattini ideal of $L$, $\phi(L)$, is the largest ideal of $L$ contained in

all maximal subalgebras of $L$. The Lie algebra $L$ is called $\phi$-*free* if $\phi(L) = 0$, and *elementary* if $\phi(B) = 0$ for every subalgebra $B$ of $L$. Elementary Lie algebras were introduced by Stitzinger [24] and Towers [27] by analogy to the definition of an elementary group given earlier by Bechtell [3]. An interesting property of an elementary Lie algebra is that it splits over each of its ideals, see [27].

The class of elementary Lie algebras is closely related to the class of Lie algebras all of whose nilpotent subalgebras are abelian (called $A$-algebras) and to the class of Lie algebras $L$ such that $\phi(B) \leq \phi(L)$ for all subalgebras $B$ of $L$ (called $E$-algebras). $A$-algebras have been studied by Drensky [7], Sheina [23], Premet and Semenov [21] and Dallmer [6]. Since the Frattini ideal of a nilpotent Lie algebra $L$ is just the derived subalgebra of $L$, every elementary Lie algebra is an $A$-algebra. $E$-algebras were introduced by Stitzinger in [24]. He proved that $L$ is an $E$-algebra if and only if $L/\phi(L)$ is elementary. A Lie algebra $L$ is called *strongly solvable* if $L^2$ is nilpotent. Stitizinger also proved in [24] that if $L$ is strongly solvable then $L$ is an $E$-algebra. In this paper it is shown that over a perfect field the converse also holds

For algebraically closed fields of characteristic zero, elementary Lie algebras were determined by Towers in [27]. This classification is shown to remain true for any algebraically closed field of characteristic different from two or three.

Following Jacobson [15], we say that a linear Lie algebra $L \leq \mathrm{gl}(V)$ is *almost algebraic* if $L$ contains the nilpotent and semisimple Jordan components of its elements. Every algebraic Lie algebra is almost algebraic. An abstract Lie algebra $L$ is called almost algebraic if $\mathrm{ad}L \leq \mathrm{gl}(L)$ is almost algebraic. Recently, Zhao and Lu have proved in [28] that every almost-algebraic $A$-algebra is elementary, whenever the ground field is algebraically closed of characteristic zero. In this paper we prove that every elementary Lie algebra is almost algebraic, provided that $\mathrm{char}(F) = 0$.

The final section of the paper is devoted to classifying the real elementary simple Lie algebras.

We will denote algebra direct sums by $\oplus$, direct sums of the vector space structure alone by $\dot{+}$, and semidirect products by $\rtimes$. The *nilradical* of $L$ will be denoted by $N(L)$, whilst $Asoc(L)$ will denote the sum (necessarily direct) of the minimal abelian ideals of $L$.

## 2 The solvable case

Over a field of characteristic zero every solvable Lie algebra is strongly solvable, by Lie's Theorem. This fails in characteristic $p$ for every $p > 0$ (see [22], page 96). However, we have the following result.

**Proposition 2.1** *Let $L$ be an elementary solvable Lie algebra over a perfect field $F$. Then $L$ is strongly solvable.*

*Proof.* Let $L$ be a minimal counter-example. As the hypotheses are sub-algebra closed, every proper subalgebra of $L$ is strongly solvable, and so $L$ has the structure described in Theorem 4 of [5]. Thus, $L = A \rtimes B$, where $A$ is the unique minimal ideal of $L$, dim $A \geq 2$, $A^2 = 0$, $B = M \dotplus Fx$ with $M^2 = 0$, and either $M$ is a minimal ideal of $B$, or $B$ is the three-dimensional Heisenberg algebra.

Pick any $m \in M$ and put $C = A + Fm$. Then $C$ is $\phi$-free, so $A \subseteq N(C) = \text{Asoc}(C)$ by Theorem 7.4 of [26], and $A$ is completely reducible as an $Fm$-module. Write $A = \oplus_{i=1}^r A_i$, where $A_i$ is an irreducible $Fm$-module for $1 \leq i \leq r$. Then the minimum polynomial of the restriction of ad $m$ to $A_i$ is irreducible for each $i$, and so $\{(\text{ad}m)|_A : m \in M\}$ is a set of commuting semisimple operators. Let $\Omega$ be the algebraic closure of $F$ and put $A_\Omega = A \otimes_F \Omega$, and so on. As $F$ is perfect, $\{(\text{ad}m)|_{A_\Omega} : m \in M\}$ is a set of simultaneously diagonalizable linear maps. So, we can decompose $A_\Omega$ into

$$(A_\Omega)_{\alpha_i} = \{a \in A_\Omega : [a, m] = \alpha_i(m)a \ \ \forall m \in M\},$$

where $1 \leq i \leq s$.

Suppose first that $M$ is a minimal ideal of $B$. Then $M_\Omega$ has a basis $m_1, \ldots, m_t$ of eigenvectors of ad $x$ with corresponding eigenvalues $\beta_1, \ldots, \beta_t$. Let $0 \neq a_i \in (A_\Omega)_{\alpha_i}$. Then

$$[x, a_i] \ \ = \ \ \sum_{k=1}^s a'_k \quad \text{where } a'_k \in (A_\Omega)_{\alpha_k}.$$

But now

$$
\begin{aligned}
0 \ \ &= \ \ [[a_i, m_j], x] + [[m_j, x], a_i] + [[x, a_i], m_j] \\
&= \ \ \alpha_i(m_j)[a_i, x] + \beta_j[m_j, a_i] + \sum_{k=1}^s [a'_k, m_j] \\
&= \ \ -\sum_{k=1}^s \alpha_i(m_j)a'_k - \beta_j\alpha_i(m_j)a_i + \sum_{k=1}^s \alpha_k(m_j)a'_k
\end{aligned}
$$

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Hence

$$\beta_j \alpha_i(m_j)a_i = \sum_{k=1}^{s}(\alpha_k(m_j) - \alpha_i(m_j))a'_k$$

This yields that $\beta_j \alpha_i(m_j)a_i = 0$ and therefore either $[x, m_j] = 0$ or $[m_j, a_i] = 0$ for all $1 \le i \le r$. The former is impossible, since it implies that $0$ is a characteristic root of $(\text{ad}x)|_M$, whence $B$ is two-dimensional abelian and $L^2$ is nilpotent. The latter is also impossible, since then $[M, A] = 0$ and $L^2 \subseteq A \oplus M$, which is abelian.

Hence $B$ is the three-dimensional Heisenberg algebra. But then $B$ is a non-abelian nilpotent subalgebra of $L$ and hence not $\phi$-free. This contradiction establishes the result.

A Lie algebra $L$ is called an $E$-algebra if $\phi(B) \le \phi(L)$ for all subalgebras $B$ of $L$. Groups with the analogous property are called $E$-groups by Bechtell. Stitzinger in [24] proved that a Lie algebra is an $E$-algebra if and only if $L/\phi(L)$ is elementary. He also proved that every strongly solvable Lie algebra over an arbitrary field is an $E$-algebra. Next, we prove the converse of this result, provided that the ground field is perfect.

**Corollary 2.2** *Let $F$ be perfect. Then every solvable $E$-algebra is strongly solvable.*

*Proof.* Let $L$ be a solvable $E$-algebra. If $L$ is $\phi$-free, then $L$ is elementary, by [24], and $L^2$ is nilpotent, by Proposition 2.1. So suppose that $\phi(L) \ne 0$. Then $L/\phi(L)$ is a solvable elementary Lie algebra, and so $L^2/\phi(L) = (L/\phi(L))^2$ is nilpotent. But then $L^2$ is nilpotent, by Theorem 5 of [2].

**Lemma 2.3** *Let $L$ be a Lie algebra over any field $F$ and let $A$ be a minimal ideal of $L$ with $[L^2, A] = 0$. Then $A \subseteq \text{Asoc}(C)$ for every subalgebra $C$ of $L$ containing $A$.*

*Proof.* We have $A^3 = [A^2, A] = 0$. Minimality of $A$ implies that $A$ is abelian. Moreover, since $[L^2, A] = 0$, we have that $(\text{ad}x)|_A$ is $C$-linear for every $x \in L$. This implies that the sum of the irreducible $C$-submodules of $A$ is invariant under $L$, and thus that it coincides with $A$. The result follows.

If $A$ is a subset of $L$ we denote by $C_L(A)$ the centraliser of $A$ in $L$. Now we give a construction of elementary solvable Lie algebras.

**Proposition 2.4** *Let $F$ be an arbitrary field. Let $A$ be a vector space of finite dimension and let $B$ be an abelian completely reducible subalgebra of $\mathrm{gl}(A)$. Then the semidirect product $A \rtimes B$ is an elementary almost-algebraic Lie algebra.*

*Proof.* Put $L = A \rtimes B$. Then $L$ is strongly solvable and hence an $E$-algebra. But $A \leq \mathrm{Asoc}(L) \leq C_L(A) = A$, so $L$ is $\phi$-free by Theorem 7.3 of [26]. It follows that $L$ is elementary, and $A = A_1 \oplus \cdots \oplus A_n$ where $A_i$ is a minimal ideal of $L$ for $1 \leq i \leq n$.

In order to prove that $L$ is almost algebraic, let $x \in L$, $x \notin A$. By Lemma 2.3 we have that $A = \mathrm{Asoc}(A + Fx)$ and so the action of $x$ on $A$ is semisimple. Let $A = E_1 \oplus \cdots \oplus E_k$, where $E_i$ is an irreducible $Fx$-module. Then $[E_i, x] = 0$ or $E_i$ for each $1 \leq i \leq k$, so we can write $A = C \oplus D$, where $[C, x] = 0$, $[D, x] = D$. Put $x = c + d + b$, where $c \in C, d \in D, b \in B$. Then $[D, b] = [D, x] = D$, so there is a $y \in D$ such that $[y, b] = d$. Consider the automorphism $e^{\mathrm{ad}y}$ of $L$. We have $e^{\mathrm{ad}y}(b) = (1 + \mathrm{ad}y)(b) = b + [y, b] = b + d$, so $b + d \in e^{\mathrm{ad}y}(B)$. Since $e^{\mathrm{ad}y}(B)$ is abelian, it follows that $\mathrm{ad}(b + d)$ is semisimple. Now $(\mathrm{ad}c)^2 L = [[L, c], c] \subseteq A^2 = 0$, whence $\mathrm{ad}c$ is nilpotent. This yields that $L$ is almost algebraic. The proof is complete.

An elementary Lie algebra which can be constructed as in Proposition 2.4 will be called of *type I*. Next, we show that every elementary solvable Lie algebra over a perfect field can be constructed as an algebra direct sum of an abelian Lie algebra and a Lie algebra of type I.

We say that $L$ is *metabelian* if $L^2$ is abelian. We denote the centre of $L$ by $Z(L)$.

**Theorem 2.5** *Let $F$ be perfect. For a solvable Lie algebra $L$, the following statements are equivalent:*

1. *$L$ is elementary,*

2. *$L$ is $\phi$-free and strongly solvable,*

3. *$L$ is $\phi$-free and metabelian,*

4. *$L = \mathrm{Asoc}(L) \rtimes B$, where $B$ is an abelian subalgebra of $L$,*

5. *$L \cong A \oplus E$, where $A$ is an abelian Lie algebra and $E$ is an elementary Lie algebra of type I.*

*Proof.* (1)⇒(2): This follows from Proposition 2.1.

(2)⇒(3): Let $L$ be $\phi$-free and strongly solvable. By Theorems 7.3 and 7.4 of [26], we have that $L = \mathrm{Asoc}(L) \rtimes B$, where $B$ is a subalgebra of $L$, and that $\mathrm{Asoc}(L)$ is precisely the largest nilpotent ideal of $L$. As $L$ is strongly solvable, we have $L^2 \leq \mathrm{Asoc}(L)$. This yields that both $L^2$ and $B$ are abelian.

(3)⇒(4): This is clear from Theorems 7.3 and 7.4 of [26] as above.

(4)⇒(5): Decompose $\mathrm{Asoc}(L) = Z(L) \oplus K$, where $K$ is an ideal of $L$. Put $E = K \dotplus B$. We have

$$C_E(K) \cap B \leq Z(E) \leq Z(L) \cap E = 0,$$

so that $B \lesssim \mathrm{gl}(K)$. Moreover, since $K \leq \mathrm{Asoc}(L)$ and $L = \mathrm{Asoc}(L) \rtimes B$, it follows that $K$ is completely reducible as a $B$-module. Hence $E$ is of type I.

(5)⇒(1): In Towers [27], it is proved that a direct sum of elementary Lie algebras is elementary. So this follows from Proposition 2.4. This completes the proof.

**Corollary 2.6** *An elementary solvable Lie algebra over a perfect field is almost algebraic.*

In [13], Gein and Varea showed that solvability was a subalgebra lattice property, provided that $L$ was at least three dimensional and the underlying field was perfect of characteristic different from $2, 3$. We now have that the same is true for strong solvability.

**Corollary 2.7** *Let L be a strongly solvable Lie algebra over a perfect field of characteristic different from $2, 3$, and let $L^*$ be a Lie algebra that is lattice isomorphic to L. Then either*

1. *$L^*$ is three-dimensional non-split simple, or*

2. *$L^*$ is strongly solvable and $\dim L = \dim L^*$.*

*Proof.* Simply combine Theorem 2.5 with Theorem 3.3 of [13].

# 3 The non-solvable case

If $C$ is a subalgebra of $L$ we denote by $R(C)$ the radical of $C$, and by $N(C)$ the nilradical of $C$.

**Proposition 3.1** *Let $F$ be perfect. Let $L$ be an elementary Lie algebra which is neither solvable nor semisimple. Then, $L = \text{Asoc}(L) \rtimes (B \dotplus S)$, where $B$ is abelian, $S$ is a semisimple subalgebra of $L$ and $[B, S] \leq B$. If $F$ has characteristic zero, then $[B, S] = 0$.*

*Proof.* As $L$ is $\phi$-free, $L = \text{Asoc}(L) \rtimes C$ for some subalgebra $C$ of $L$ and $N(L) = \text{Asoc}(L)$, by Theorem 7.3 of [26]. Let $B = R(C)$. Since $C$ is elementary, it splits over $B$, by [27, Theorem 2.4]. So, $C = B \dotplus S$, where $S$ is semisimple. It is clear that $R(L) = \text{Asoc}(L) \rtimes B$. By Proposition 2.1, we have that $R(L)$ is strongly solvable. So $B^2 \leq R(L)^2 \cap B \subseteq N(L) \cap B = \text{Asoc}(L) \cap B = 0$. If $F$ has characteristic zero, the final assertion follows from [25, Theorem 4]. $\blacksquare$

A Lie algebra all of whose proper subalgebras are abelian is called *semiabelian*.

**Example** Let $S$ be a simple semiabelian Lie algebra over a field of characteristic $p > 0$, let $B$ be a faithful finite-dimensional completely reducible $L$-module, and put $L = B \dotplus S$ where $B^2 = 0$ and $L$ acts on $B$ under the given $L$-module action. Then $L$ is elementary, but $S$ is not an ideal of $L$.

### Notes

- Since elementary Lie algebras are Lie $A$-algebras it follows from Proposition 2 of [21] that over a field of cohomological dimension $\leq 1$ every semisimple elementary Lie algebra is representable as a direct sum of simple ideals, each of which splits over some finite extension of the ground field into a direct sum of ideals isomorphic to sl$(2, F)$.

- Over a perfect field with non-trivial Brauer group there exists a finite-dimensional simple semiabelian Lie algebra (see Theorem 8.5 of [9]), and this is elementary.

- Let $G$ be the algebra constructed by Gein in Example 2 of [12]. This is a seven-dimensional Lie algebra over a certain perfect field $F$ of characteristic three. Every subalgebra of $G$ of dimension greater than one is simple. So, $G$ is elementary.

We finish this section by proving that the classification of elementary Lie algebras over an algebraically closed field of characteristic zero given by Towers in [27] remains true for any algebraically closed field of characteristic different from 2 or 3.

**Theorem 3.2** *Let $L$ be a Lie algebra over an algebraically closed field $F$ of characteristic $\neq 2$ or $3$. Then $L$ is elementary if and only if*

1. *$L$ is isomorphic to a direct sum of copies of $\mathrm{sl}(2, F)$, or*

2. *there is a basis $\{a_1, \ldots, a_m, b_1 \ldots, b_n\}$ for $L$ such that*

$$[a_i, b_j] = -[b_j, a_i] = \lambda_{ij} a_i \text{ for all } 1 \leq i \leq m, 1 \leq j \leq n$$

   *all other products being zero, or*

3. *$L \cong A \oplus B$, where $A$ is as in (1) and $B$ is as in (2).*

*Proof.* If $\mathrm{char}(F) = 0$, then the result is Theorem 3.2 of [27]. Assume $\mathrm{char}(F) = p > 3$. Let $L$ ($\neq 0$) be elementary. If $L$ is solvable, then $L = \mathrm{Asoc}(L) \rtimes B$, by Theorem 2.5. Decompose $\mathrm{Asoc}(L) = A_1 \oplus \cdots \oplus A_m$, where $A_i$ is a minimal ideal of $L$. As $F$ is algebraically closed, we have $\dim A_i = 1$. Hence $L$ is as in (2).

Now, let $L$ be semisimple. Since $L$ is an $A$-algebra, Proposition 2 of Premet and Semenov [21] applies and $L$ is as in (1).

Then suppose that $L$ is neither solvable nor semisimple. By Proposition 3.1 we have that $L = A \rtimes (B \dotplus S)$, where $0 \neq A = \mathrm{Asoc}(L)$, $B$ is abelian, $0 \neq S$ is a semisimple subalgebra of $L$ and $[B, S] \leq B$. From [21, Proposition 2] again it follows that $S = S_1 \oplus \cdots \oplus S_r$, where $S_i$ is an ideal of $S$ and $S_i \cong \mathrm{sl}(2, F)$ for every $1 \leq i \leq r$. Put $C_i = A \dotplus S_i$. We have that $C_i$ is a $\phi$-free Lie algebra and so $N(C_i) = \mathrm{Asoc}(C_i)$. This yields that $A = \mathrm{Asoc}(C_i)$ and therefore $A$ is a completely reducible $S_i$-module. Let $V$ be an irreducible $S_i$-submodule of $A$.

We claim that $\dim_F V = 1$. By general theory, there exists an element $e \in S_i$ such that $V$ is a cyclic $Fe$-module on which $e$ acts nilpotently. This can

be seen by looking at the representatives of the coadjoint orbits of SL(2) on $\mathrm{sl}(2, F)^*$ (see [11, §2]). Consequently, $M = V \dotplus Fe$ is a nilpotent subalgebra. This yields that $M$ is abelian, whence $[V, e] = 0$ and $\dim_F V = 1$.

Therefore, we have that $[A, S] = 0$. If $B = 0$, then we have that $L$ is as in (3). Suppose then $B \neq 0$. We have that $B \dotplus S$ is an elementary Lie algebra, whence $\mathrm{Asoc}(B \dotplus S) = N(B \dotplus S) = B$. As above, we obtain that $[B, S] = 0$, from which it follows that $L = (A \dotplus B) \oplus S$. Since $A \dotplus B$ is a solvable elementary Lie algebra, it is as in (2), and therefore $L$ is as in (3). This completes the proof in one direction. The converse is easily checked.

The above result does not hold in characteristic 2: over such a field, the three-dimensional simple Lie algebra with basis $e_1, e_2, e_3$ and products $[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$ is elementary. The exclusion of characteristic 3 is used in order to invoke the result of Semenov and Premet, which in turn relies on [20, Theorem 3]. This last result fails in characteristic 3, as is shown by the algebra $G$ ([12, Example 2]) referred to earlier. However, we know of no counter-example to the above result in characteristic 3: if we pass to the algebraic closure, then $G$ becomes $\mathrm{psl}(3)$, which is no longer elementary.

## 4  The characteristic zero case

A subalgebra $T$ of $L$ is said to be a *toral* subalgebra of $L$ if $T$ is abelian and $\mathrm{ad}_L t$ is semisimple for every $t \in T$. A Lie algebra $L$ is said to be *ad-semisimple* if $\mathrm{ad}x$ is semisimple for every $x \in L$.

**Proposition 4.1** *Let* $\mathrm{char}(F) = 0$. *For a solvable Lie algebra $L$ the following statements are equivalent:*

1. *$L$ is elementary,*

2. *$L$ is $\phi$-free and almost algebraic.*

*Proof.* Assume that $L$ is $\phi$-free and almost algebraic. Then $L = N(L) \dotplus T$, where $T$ is a toral subalgebra of $L$ (see [1] or [26, Theorem 7.5]). Moreover, we have $N(L) = \mathrm{Asoc}(L)$ by Theorem 7.4 of [26]. So, $L$ is elementary by Theorem 2.5. The converse follows from Corollary 2.6.

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**Proposition 4.2** *Let L be an ad-semisimple Lie algebra over a field of characteristic zero. Then L is elementary*

*Proof.* Let $L$ be a minimal counter-example. Then it suffices to show that $L$ is $\phi$-free. But $L = Z(L) \oplus S$, where $S$ is semisimple, by Levi's Theorem and Theorem 1 of [10]. It follows that $\phi(L) = 0$.

**Corollary 4.3** *Over the real field every compact semisimple Lie algebra is elementary.*

A Lie algebra $L$ is said to be *reductive* if its adjoint representation is completely reducible; equivalently, $L = S \oplus Z(L)$, where $S$ is a semisimple ideal of $L$ and $Z(L)$ is the centre of $L$ (see [15]).

**Proposition 4.4** *Let $\mathrm{char}(F) = 0$. Let $L$ be a Lie algebra such that its radical $R$ is elementary and $L/R$ is ad-semisimple. Then, $L$ is elementary and almost algebraic.*

*Proof.* Let $S(L)$ be the largest semisimple ideal of $L$. Since $S(L)$ is isomorphic to an ideal of $L/R$, by Proposition 4.2 it follows that $S(L)$ is elementary. Since $S(L)$ is a direct summand of $L$ and since $S(L)$ is almost algebraic, we may suppose without loss of generality that $S(L) = 0$. By Proposition 4.1 we have that $R$ is almost algebraic. Then $L$ is also almost algebraic by Corollary 3.1 of [1] (see also [18]).

Therefore $L = N(L) \rtimes (B \dot{+} S)$ where $B$ is a toral subalgebra of $L$, $S$ is a semisimple subalgebra of $L$ and $[B, S] = 0$. We have $N(R) = \mathrm{Asoc}(R)$ since $R$ is $\phi$-free. As $\mathrm{char}(F) = 0$, we have that $N(R)$ is a characteristic ideal of $R$ and so $N(R) \leq N(L)$. It follows that $N(L) = \mathrm{Asoc}(R)$ and so $N(L)$ is abelian. Put $A = N(L)$. Since every element of $B$ acts semisimply on $A$, we have that $A$ is completely reducible as a $(B \oplus S)$-module, see [15]. It follows that $A = \mathrm{Asoc}(L)$. This yields that $L$ is $\phi$-free. To prove that $L$ is elementary it suffices to show that every maximal subalgebra $M$ of $L$ is also $\phi$-free.

Let us first consider the case when $M$ does not contain $R$. Then $L = R + M$, $L/R \cong M/M \cap R$ and $M \cap R$ is the radical of $M$. We have that $M \cap R$ is elementary and $M/M \cap R$ is ad-semisimple. By the above, we obtain that $M$ is $\phi$-free.

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Now suppose that $R \leq M$. Since $M \cap S$ is ad-semisimple, by [10, Theorem 1] it follows that $M \cap S = Z(M \cap S) \oplus (M \cap S)^2$ and $(M \cap S)^2$ is semisimple. Put $B^* = B \oplus Z(M \cap S)$. We have that $B^* \oplus (M \cap S)^2$ is reductive. Moreover, since every element of $Z(M \cap S)$ acts semisimply on $S$, it acts also semisimply on $A$ (see [15], page 101). It follows that every element of $B^*$ acts semisimply on $A$. This yields that $A$ is completely reducible as a $(B^* \oplus (M \cap S)^2)$-module (see [15]) and therefore $A \leq \mathrm{Asoc}(M)$. On the other hand, we have $R(M) = A \dot{+} B^*$ and $A \leq N(M)$. We claim that $A = N(M)$. Let $x \in B^* \cap N(M)$. We have that $(\mathrm{ad}x)|_A$ is nilpotent and semisimple. So, $x \in C_L(A)$. Moreover, we have that $[x, B] \subseteq [S, B] = 0$. This yields that $x \in C_L(R)$. Since $C_L(R) \cap S$ is a semisimple ideal of $L$, we have $C_L(R) \cap S = 0$, whence $C_L(R) = Z(R)$. Decompose $x = b + z$, $b \in B$, $z \in Z(M \cap S)$. We have $z \in R \cap S = 0$. This yields that $x \in B \cap Z(R) \leq Z(L) = 0$ and therefore $A = N(M)$, as claimed. Hence $A = \mathrm{Asoc}(M)$. Since $M$ splits over $A$, it follows that $M$ is $\phi$-free. This completes the proof.

**Corollary 4.5** *Let* $\mathrm{char}(F) = 0$. *Let* $A$ *be a vector space of finite dimension. Let* $K$ *be a reductive subalgebra of* $\mathrm{gl}(A)$ *such that* $K^2$ *is non-zero and ad-semisimple and every non-zero element of* $Z(K)$ *is a semisimple transformation of* $A$. *Then, the semidirect product* $A \rtimes K$ *is an elementary almost algebraic Lie algebra.*

*Proof.* Put $L = A \rtimes K$. We have that $R(L) = A \dot{+} Z(K)$. By Proposition 2.4, it follows that $R(L)$ is an elementary Lie algebra. Also, we have $L/R(L) \cong K^2$. So that $L/R(L)$ is ad-semisimple. The result follows from Proposition 4.4.

A Lie algebra which can be constructed as in the above corollary will be called of *type II*.

**Theorem 4.6** *Let* $\mathrm{char}(F) = 0$. *A Lie algebra* $L$ *is elementary if and only if* $L \cong A \oplus B \oplus S$, *where* $A$ *is abelian,* $B$ *is a Lie algebra of type I or of type II and* $S$ *is an elementary semisimple Lie algebra.*

*Proof.* Let $L$ be elementary. Then $L$ splits over $Z(L)$, so $L = Z(L) \oplus \hat{L}$, where $\hat{L}$ is a centerless Lie algebra. Now let $S(\hat{L})$ be the largest semsimple ideal of $\hat{L}$. Then we have that $\hat{L} = K \oplus S(\hat{L})$, where $K$ is a centerless Lie

algebra which has no non-zero semisimple ideals. If $K$ is solvable, then we find that $K$ is an elementary Lie algebra of type I.

Then assume that $K$ is not solvable. By Proposition 3.1 it follows that $K = \operatorname{Asoc}(K) \rtimes (B \oplus S)$, where $B$ is abelian, $S$ is semisimple and $[B, S] = 0$. Let $0 \neq s \in S$ be such that $\operatorname{ad}_S s$ is nilpotent. Then we have that $\operatorname{Asoc}(K) + Fs$ is a nilpotent subalgebra of $L$. Hence $[s, \operatorname{Asoc}(L)] = 0$. This yields that $s \in C_K(R(K)) \cap S = 0$, which is a contradiction. Therefore $S$ has no non-zero ad-nilpotent elements. As $S$ is semisimple and $\operatorname{char}(F) = 0$, it follows that $S$ is ad-semisimple. Put $C = C_K(\operatorname{Asoc}(K)) \cap (B \oplus S)$. We have that $C$ is an ideal of the reductive Lie algebra $B \oplus S$, so $C = (C \cap B) \oplus (C \cap S)$. It follows that $C \cap B \leq Z(K) = 0$ and that $C \cap S = 0$ since it is a semisimple ideal of $K$. This yields that $B \oplus S \lesssim \operatorname{gl}(\operatorname{Asoc}(K))$ and therefore $K$ is of type II. This completes the proof in one direction.

The converse follows from Proposition 2.4 and Corollary 4.5.

**Corollary 4.7** *An elementary Lie algebra over a field of characteristic zero is almost algebraic.*

*Proof.* This follows from Theorem 4.6, Proposition 2.4 and Corollary 4.5.

**Corollary 4.8** *Let* $\operatorname{char}(F) = 0$. *For a Lie algebra* $L$ *without non-zero semisimple ideals, the following statements are equivalent:*

1. *$L$ is elementary,*

2. *$L$ is almost algebraic and an A-algebra,*

3. *$L$ is almost algebraic, $N(L)$ is abelian and $L/R(L)$ is ad-semisimple,*

4. *$L$ is almost algebraic, $\phi$-free and $L/R(L)$ is ad-semisimple.*

*Proof.* (1)$\Rightarrow$(2): This follows from Corollary 4.7.

From now on in this proof we assume that $L$ is almost algebraic. Then we have that $L = N(L) \dot{+} B \dot{+} S$, where $B$ is a toral subalgebra of $L$, $S$ is a semisimple subalgebra of $L$ and $[B, S] = 0$.

(2)$\Rightarrow$(3): Clearly, $N(L)$ is abelian. It remains to prove that $S$ is ad-semisimple. Let $s \in S$ such that $\operatorname{ad}_S s$ is nilpotent. Then $(\operatorname{ad} s)|_{N(L)}$ is nilpotent too. This yields that $N(L) + Fs$ is a nilpotent subalgebra of $L$ and therefore $[N(L), s] = 0$. Thus, $s \in C_L(R(L)) \cap S = 0$ because $L$ has no

non-zero semisimple ideals. Hence $S$ has no non-zero ad-nilpotent elements. As $\text{char}(F) = 0$, we have that $S$ is ad-semisimple.

(3)$\Rightarrow$(4): Since $N(L)$ is a completely reducible $(B \oplus S)$-module and since $N(L)$ is abelian, it follows that $N(L) = \text{Asoc}(L)$. Since $L$ splits on $N(L)$, we have that $L$ is $\phi$-free.

(4)$\Rightarrow$(1): By Theorem 7.4 of [26] we have $N(L) = \text{Asoc}(L)$. It then follows from Theorem 2.5 that $N(L) \dotplus B$ is elementary. Since $R(L) = N(L) \dotplus B$, Proposition 4.4 now gives that $L$ is elementary.

# 5  The real field case

A subalgebra $P$ of $L$ is called *parabolic* if $P \otimes_F \Omega$ contains a Borel subalgebra (that is, a maximal solvable subalgebra) of $L \otimes_F \Omega$, where $\Omega$ is the algebraic closure of $F$. Over a field of characteristic zero, all maximal subalgebras of a reductive Lie algebra are reductive or parabolic (see [4], [16], [19]); it follows that a reductive Lie algebra is elementary if and only if its parabolic subalgebras are $\phi$-free.

For results concerning Lie algebras over the real field we refer the reader to the books by Helgason ([14]) and Knapp ([17]).

**Theorem 5.1** *Let $F$ be the real field. For a simple Lie algebra $L$, the following statements are equivalent:*

1.  *$L$ is elementary,*

2.  *$L$ is an A-algebra,*

3.  *$L$ is compact or isomorphic to one of the following Lie algebras: $\text{sl}(2, \mathbb{R})$, $\text{sl}(2, \mathbb{C})^{\mathbb{R}}$, $\text{so}(n, 1)$ $(n > 3)$.*

*Proof.* (1)$\Rightarrow$(2): This is clear.

(2)$\Rightarrow$(3): Let $L$ be a non-compact A-algebra. Suppose first that $L = \bar{L}^{\mathbb{R}}$, the realisation of the complex simple Lie algebra $\bar{L}$. If $\bar{N}$ is a nilpotent subalgebra of $\bar{L}$ then $\bar{N}^{\mathbb{R}}$ is a nilpotent subalgebra of $L$, and hence abelian. It follows that $\bar{N}$ is abelian and thus that $\bar{L}$ is an A-algebra. The proof of Theorem 3.2 of [27] then shows that $\bar{L} \cong \text{sl}(2, \mathbb{C})$.

So assume now that $L$ is a non-compact real form of a complex simple Lie algebra. The only such algebras for which the nilpotent subalgebra $N$ in the Iwasawa decomposition of $L$ is abelian are $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{so}(n,1)$ $(n > 3)$.

(3)$\Rightarrow$(1): Suppose now that $L$ is one of the algebras described in (3). If $L$ is compact it is elementary by Corollary 4.3, and $\mathrm{sl}(2,\mathbb{R})$ is clearly elementary.

Next suppose that $L \cong \mathrm{sl}(2,\mathbb{C})^{\mathbb{R}}$, and let $S$ be a subalgebra of $L$. Then $S \otimes \mathbb{C}$ is a subalgebra of $L \otimes \mathbb{C}$, which is elementary. This yields $0 = \phi(S \otimes \mathbb{C}) = \phi(S) \otimes \mathbb{C}$, by [8], whence $\phi(S) = 0$ and $L$ is elementary.

Finally, let $L = \mathrm{so}(n,1)$ $(n > 3)$. We identify $L$ with

$$\left\{ \begin{pmatrix} B & u \\ u^T & 0 \end{pmatrix} : u \in \mathbb{R}^n, B \in M_{n \times n}(\mathbb{R}), B^T = -B \right\}$$

From the remarks at the beginning of this section it suffices to show that the parabolic subalgebras of $L$ are $\phi$-free. Now any such subalgebra is conjugate to a standard parabolic subalgebra $P$ with Langlands decomposition $P = (M \oplus A)\dot{+}N$, where $M \oplus A$ is reductive and $N$ is an ideal of $P$ contained in

$$\left\{ \begin{pmatrix} 0 & u & u \\ -u^T & 0 & 0 \\ u^T & 0 & 0 \end{pmatrix} : u \in \mathbb{R}^{n-1} \right\}.$$

Clearly $N$ is abelian and every element of $A$ acts semisimply on $N$, so $P$ is $\phi$-free.

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